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ASYMPTOTICALLY EXACT LOCAL DISCONTINUOUS GALERKIN ERROR ESTIMATES FOR THE LINEARIZED KORTEWEG-DE VRIES EQUATION IN ONE SPACE DIMENSION

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Abstract. We present and analyze *a posteriori* error estimates for the local discontinuous Galerkin (LDG) method for the linearized Korteweg-de Vries (KdV) equation in one space dimension. These estimates are computationally simple and are obtained by solving a local steady problem with no boundary condition on each element. We extend the work of Hufford and Xing [*J. Comput. Appl. Math.*, 255 (2014), pp. 441-455] to prove new superconvergence results towards particular projections of the exact solutions for the two auxiliary variables in the LDG method that approximate the first and second derivatives of the solution. The order of convergence is proved to be $k+3/2$, when polynomials of total degree not exceeding k are used. These results allow us to prove that the significant parts of the spatial discretization errors for the LDG solution and its spatial derivatives (up to second order) are proportional to $(k+1)$-degree Radau polynomials. We use these results to construct asymptotically exact *a posteriori* error estimates and prove that, for smooth solutions, these *a posteriori* LDG error estimates for the solution and its spatial derivatives, at a fixed time t, converge to the true errors at $\mathcal{O}(h^{k+3/2})$ rate in the L^2-norm. Finally, we prove that the global effectivity indices, for the solution and its spatial derivatives, converge to unity at $\mathcal{O}(h^{1/2})$ rate. Numerical results are presented to validate the theory.

Key words. Local discontinuous Galerkin method; KdV; superconvergence; Radau points; *a posteriori* error estimates.

1. Introduction

The famous nonlinear Korteweg-de Vries (KdV) equation

$$u_t + \alpha u_x + \gamma u u_x + \beta u_{xxx} = 0,$$

with constants α, β, and γ, is derived by Korteweg and de Vries in 1895. It describes the propagation of waves in a variety of nonlinear dispersive media. The KdV equation is a generic equation for the study of weakly nonlinear long waves and arises in many physical situations, such as surface water waves and plasma waves. It has been shown that the KdV equation describes a large class of solitons observed in various situations: acoustic waves on a crystal lattice, plasma waves, hydrodynamics internal or surface waves, elastic surface waves, and waves in optical fibers (see *e.g.*, [27]).

In this paper we develop and analyze an implicit residual-based *a posteriori* error estimates of the spatial errors for the semi-discrete local discontinuous Galerkin (LDG) method applied to the linearized KdV equation

$$u_t + \alpha u_x + \beta u_{xxx} = 0, \quad x \in [a,b],\ t \in [0,T], \tag{1.1a}$$

subject to the initial and periodic boundary conditions

$$u(x,0) = u_0(x), \quad x \in [a,b], \tag{1.1b}$$

$$u(a,t) = u(b,t), \quad u_x(a,t) = u_x(b,t), \quad u_{xx}(a,t) = u_{xx}(b,t), \quad t \in [0,T]. \tag{1.1c}$$

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We would like to emphasize that the assumption of periodic boundary conditions is for simplicity only and is not essential. In our analysis we select $u_0(x)$ such that the exact solution $u(x,t)$ is a smooth function on $[a,b] \times [0,T]$.

The LDG method we discuss in this paper is an extension of the discontinuous Galerkin (DG) method aimed at solving partial differential equations containing higher than first-order spatial derivatives. The DG method is a class of finite element methods, using discontinuous, piecewise polynomials as the numerical solution and the test functions. It was first developed by Reed and Hill [31] for solving hyperbolic conservation laws containing only first-order spatial derivatives in 1973. Consult [25] and the references cited therein for a detailed discussion of the history of DG method and a list of important citations on the DG method and its applications. The LDG method for solving convection-diffusion problems was first introduced by Cockburn and Shu in [26]. They further studied the stability and error estimates for the LDG method. Castillo *et al.* [19] presented the first *a priori* error analysis for the LDG method for a model elliptic problem. They considered arbitrary meshes with hanging nodes and elements of various shapes and studied general numerical fluxes. They showed that, for smooth solutions, the L^2 errors in ∇u and in u are of order k and $k+1/2$, respectively, when polynomials of total degree not exceeding k are used. Cockburn *et al.* [24] presented a superconvergence result for the LDG method for a model elliptic problem on Cartesian grids. They identified a special numerical flux for which the L^2-norms of the gradient and the potential are of orders $k+1/2$ and $k+1$, respectively, when tensor product polynomials of degree at most k are used.

Yan and Shu [35] developed the first LDG method for solving KdV type equations in one and two space dimensions. They proved L^2 stability and a cell entropy inequality for the square entropy for a class of nonlinear KdV equations in both one and multiple space dimensions. They also proved an optimal error estimate for the linear cases in the one-dimensional case. In [33], Xu and Shu proved L^2 error estimates for the semi-discrete LDG methods for the fully nonlinear KdV equation with smooth solution. The order of convergence is proved to be $k+1/2$, when k-degree piecewise polynomials with $k \geq 1$ are used. Later, Xu and Shu [34] proved optimal L^2 error estimates of the semi-discrete LDG methods for solving linear higher-order wave equations including the linearized KdV equation. More recently, Hufford and Xing [30] studied the superconvergence property of the LDG method for solving the linearized KdV equation. They selected a special projection of the initial condition and proved that the LDG solution is $\mathcal{O}(h^{k+3/2})$ super close to a particular projection of the exact solution, when the upwind flux is used for the convection term and the alternating flux is used for the dispersive term.

A posteriori error estimates lie in the heart of every adaptive finite element algorithm for differential equations. They are used to assess the quality of numerical solutions and guide the adaptive enrichment process where elements having high errors are enriched by h-refinement and/or p-refinement while elements with small errors are h- and/or p-coarsened. Furthermore, error estimates are used to stop the adaptive refinement process. For an introduction to the subject of *a posteriori* error estimation see the monograph of Ainsworth and Oden [6]. Several *a posteriori* DG error estimates are known for hyperbolic [22, 23, 28] and diffusive [29, 32] problems. Adjerid and Baccouch [3, 12, 10] investigated the global convergence of the implicit residual-based *a posteriori* error estimates of Adjerid *et al.* [5]. They proved that these *a posteriori* error estimates converge to the true spatial error in the L^2-norm under mesh refinement. Later, Adjerid and Baccouch [1, 2, 17] investigated DG methods on structured and unstructured triangular meshes with several finite element spaces to discover new superconvergence properties and compute accurate error estimates. In [11, 15], the author analyzed the superconvergence properties of the LDG formulation applied to transient convection-diffusion problems in one space dimension. We proved that the leading error term on each element for the solution is proportional to a $(k+1)$-degree right Radau polynomial while the leading error term for the solution's derivative is proportional to a $(k+1)$-degree left Radau polynomial. We further used these results to construct *a posteriori* error estimates and proved that these LDG error estimates are asymptotically exact under mesh refinement. In [8, 16], we presented new superconvergence results for the LDG method applied to the second-order scalar wave equation in one space dimension. Later, in [9], we investigated the global convergence of the *a posteriori* error estimates developed in [8]. We used the superconvergence results [16] and proved that, for smooth solutions, these *a posteriori* error estimates at a fixed time converge to the true spatial errors in the L^2-norm under mesh refinement. In [13, 14], we developed and analyzed a new superconvergent LDG method for approximating solutions to the fourth-order Euler-Bernoulli beam equation in one space dimension. We further constructed new *a posteriori* error estimates and proved that these error estimates converge to the true spatial errors in the L^2-norm under mesh refinement. Recently, Adjerid and Baccouch [18, 4] showed that LDG solutions are superconvergent at Radau points for two-dimensional convection-diffusion problems. They used these results to construct asymptotically correct *a posteriori* error estimates. More recently, the author [7] analyzed a superconvergent LDG method for the second-order wave equation on Cartesian grids. He further constructed efficient and accurate *a posteriori* error estimates.

In this paper, we apply the superconvergence results of Hufford and Xing [30] to prove that the $(k+3/2)$-th order superconvergence rate holds not only for the solution itself but also for the auxiliary variables in the LDG method approximating the various order derivatives of the solution. Our proofs are valid for arbitrary regular meshes and for P^k polynomials with arbitrary $k \geq 1$. Our new superconvergence results are needed to prove that the true errors can be divided into significant and less significant parts. The significant parts of the discretization errors for the LDG solution and its spatial derivatives are proportional to $(k+1)$-degree Radau polynomials. Superconvergence results are used to construct asymptotically exact *a posteriori* error estimates by solving a local steady problem on each element. We further prove that these error estimates converge to the true spatial errors at $\mathcal{O}(h^{k+3/2})$ rate. Finally, we prove that the global effectivity indices in the L^2-norm converge to unity at $\mathcal{O}(h^{1/2})$ rate. Our computational results indicate that the observed numerical convergence rates are higher than the theoretical rates. In our analysis time integration is assumed to be exact and thus we are only estimating the spatial errors of the semi-discrete LDG method.

This paper is organized as follows: In section 2 we present the LDG scheme for solving the linearized KdV equation and we introduce some notation and definitions. We also present few preliminary results which will be used in our error analysis. In section 3, we prove our main superconvergence results. In section 4, we develop our *a posteriori* error estimation procedure and prove that these error estimates converge to the true errors under mesh refinement in L^2-norm. In section 5, we present a numerical example to validate our theoretical results. We conclude and discuss our results in section 6.

2. The LDG method and preliminary results

2.1. The semi-discrete LDG scheme. In order to construct the LDG method, we first introduce two auxiliary variables $q = u_x$ and $p = q_x$ and rewrite our model problem (1.1a) as a first-order system in space

$$u_t + \alpha u_x + \beta p_x = 0, \quad p - q_x = 0, \quad q - u_x = 0. \tag{2.1}$$

We divide the computational domain $I = [a, b]$ into N subintervals $I_i = [x_{i-1}, x_i]$, $i = 1, \ldots, N$, where $a = x_0 < x_1 < \cdots < x_N = b$. We denote the length of I_i by $h_i = x_i - x_{i-1}$. We also denote $h = \max\limits_{1\le i\le N} h_i$ and $h_{min} = \min\limits_{1\le i\le N} h_i$ as the length of the largest and smallest subinterval, respectively. Here, we consider regular meshes, that is $h \le K h_{min}$, where $K \ge 1$ is a constant (independent of h) during mesh refinement.

Throughout this paper, $v\big|_i$ denotes the value of the function $v = v(x,t)$ at $x = x_i$. We also define $v^-\big|_i$ and $v^+\big|_i$ to be the left limit and the right limit of the function v at the discontinuity point x_i, *i.e.*,

$$v^-\big|_i = v^-(x_i,t) = \lim_{s\to 0^-} v(x_i+s,t), \qquad v^+\big|_i = v^+(x_i,t) = \lim_{s\to 0^+} v(x_i+s,t).$$

Let us multiply the three equations in (2.1) by test functions v, w, and z, respectively, integrate over an arbitrary subinterval I_i, and use integration by parts to write

$$\int_{I_i} u_t v dx - \int_{I_i} (\alpha u + \beta p) v_x dx + \alpha uv\big|_i - \alpha uv\big|_{i-1} + \beta pv\big|_i - \beta pv\big|_{i-1} = 0, \tag{2.2a}$$

$$\int_{I_i} pw dx + \int_{I_i} q w_x dx - qw\big|_i + qw\big|_{i-1} = 0, \tag{2.2b}$$

$$\int_{I_i} qz dx + \int_{I_i} u z_x dx - uz\big|_i + uz\big|_{i-1} = 0. \tag{2.2c}$$

We define the piecewise-polynomial space V_h^k as the space of polynomials of degree at most k in I_i, *i.e.*,

$$V_h^k = \{v : \ v|_{I_i} \in P^k(I_i), \ i = 1, \ldots, N\},$$

where $P^k(I_i)$ is the space of polynomials of degree at most k on I_i. Next, we approximate the exact solutions $u(.,t)$, $q(.,t)$, and $p(.,t)$ by piecewise polynomials $u_h(.,t) \in V_h^k$, $q_h(.,t) \in V_h^k$, and $p_h(.,t) \in V_h^k$, respectively, whose restriction to I_i are in $P^k(I_i)$. Here u_h, q_h, and p_h are not necessarily continuous at the endpoints of I_i since polynomials in the space V_h^k are allowed to have discontinuities across element boundaries.

The semi-discrete LDG method consists of finding u_h, q_h, p_h such that $\forall\ i = 1, \ldots, N$ and $\forall\ v, w, z \in V_h^k$,

$$\begin{aligned} &\int_{I_i} (u_h)_t v dx - \int_{I_i} (\alpha u_h + \beta p_h) v_x dx + \alpha \tilde{u}_h v^-\big|_i - \alpha \tilde{u}_h v^+\big|_{i-1} \\ &+ \beta \hat{p}_h v^-\big|_i - \beta \hat{p}_h v^+\big|_{i-1} = 0, \end{aligned} \tag{2.3a}$$

$$\int_{I_i} p_h w dx + \int_{I_i} q_h w_x dx - \hat{q}_h w^-\big|_i + \hat{q}_h w^+\big|_{i-1} = 0, \tag{2.3b}$$

$$\int_{I_i} q_h z dx + \int_{I_i} u_h z_x dx - \hat{u}_h z^-\big|_i + \hat{u}_h z^+\big|_{i-1} = 0, \tag{2.3c}$$

where the tilde and hatted terms, $\tilde{u}_h$, $\hat{u}_h$, $\hat{q}_h$, and $\hat{p}_h$ are the so-called numerical fluxes, which are yet to be determined. These numerical fluxes are nothing but discrete approximations to the traces of u, q and p on the boundary of the elements. They must be designed to guarantee stability and local solvability of all the auxiliary variables introduced to approximate the derivatives of the solution. The local solvability of all the auxiliary variables is why the method is called a "local" discontinuous Galerkin method in [26].

The initial condition $u_h(x,0) \in V_h^k$ is obtained using a special projection of the exact initial condition. This particular projection will be defined later.

In order to complete the definition of the semi-discrete LDG method we need to select the numerical fluxes $\tilde{u}_h,\ \hat{u}_h,\ \hat{q}_h$, and $\hat{p}_h$ on the boundaries of I_i. We begin by defining the numerical flux $\tilde{u}_h$ associated with the convection term. We pick the classical upwind flux which depends on the sign of α. If $\alpha > 0$ then $\tilde{u}_h$ should be picked as u_h^- and if $\alpha < 0$ then $\tilde{u}_h$ should be picked as u_h^+. Similarly, $\hat{q}_h$ should be picked as q_h^+ if $\beta > 0$ and q_h^- if $\beta < 0$. For the numerical fluxes $\hat{u}_h,\ \hat{p}_h$, it was shown in [30, 34, 35] that it is possible to obtain optimal error estimates and superconvergence results if these numerical fluxes are chosen as alternating fluxes *i.e.*, either $(\hat{u}_h = u_h^-,\ \hat{p}_h = p_h^+)$ or $(\hat{u}_h = u_h^+,\ \hat{p}_h = p_h^-)$. Without loss of generality, we assume $\alpha \geq 0$ and $\beta > 0$. Therefore, we can take the following numerical fluxes:

$$\tilde{u}_h = u_h^-, \quad \hat{u}_h = u_h^-, \quad \hat{q}_h = q_h^+, \quad \hat{p}_h = p_h^+. \tag{2.3d}$$

The fluxes (2.3d) guarantee stability and convergence; see [34]. We note that this choice is not unique. For instance the choice $\tilde{u}_h = u_h^-,\ \hat{u}_h = u_h^+,\ \hat{q}_h = q_h^+,\ \hat{p}_h = p_h^-$, is also fine.

2.2. Initial conditions for the LDG scheme. In order to prove superconvergence results, we need to use a suitable projection of the initial condition for the numerical scheme $u_h(x,0)$. We first need to define some projections, which are commonly used in the analysis of DG methods: For $k \geq 1$, we define two special projections $P_h^{\pm}$ into V_h^k as follows: For any smooth function u, $P_h^{\pm}u \in V_h^k$ and the restrictions of P_h^+u and P_h^-u to I_i are the unique polynomials in the finite element space V_h^k satisfying, for each $i = 1, \cdots, N$,

$$\int_{I_i} (P_h^- u - u) v dx = 0,\ \forall\ v \in P^{k-1}(I_i), \quad \text{and} \quad (P_h^- u)^-\big|_i = u\big|_i, \tag{2.4}$$

$$\int_{I_i} (P_h^+ u - u) v dx = 0,\ \forall\ v \in P^{k-1}(I_i), \quad \text{and} \quad (P_h^+ u)^+\big|_{i-1} = u\big|_{i-1}. \tag{2.5}$$

These special projections are used in the error estimates of the DG and LDG methods to derive optimal L^2 error bounds in the literature, *e.g.*, in [20]. They are mainly used to eliminate the jump terms at the element boundaries in the error estimates in order to prove the optimal L^2 error estimates.

Recently, Hufford and Xing [30] studied the superconvergence property for the LDG method for solving (1.1). They selected a special projection of the initial condition $u_h(x,0) = P_h^1 u(x,0) \in V_h^k$ and proved that the LDG solution is $\mathcal{O}(h^{k+3/2})$ super close to $P_h^- u$. The operator P_h^1 is designed to better control the error of the initial condition. It is defined as follows: for any function u we let $q = u_x$ and $p = q_x$, and suppose $q_h,\ p_h \in V_h^k$ are the unique solutions (with given $P_h^1 u$) to

$$\int_{I_i} p_h w dx + \int_{I_i} q_h w_x dx - q_h^+ w^-\big|_i + q_h^+ w^+\big|_{i-1} = 0, \quad \forall\ w \in V_h^k, \tag{2.6a}$$

$$\int_{I_i} q_h z dx + \int_{I_i} P_h^1 u\ z_x dx - (P_h^1 u)^- z^-\big|_i + (P_h^1 u)^- z^+\big|_{i-1} = 0, \quad \forall\ z \in V_h^k, \tag{2.6b}$$

then we require

(2.7a) $(P_h^- u - P_h^1 u)^- \big|_i = (P_h^+ q - q_h)^+ \big|_i - (P_h^+ p - p_h)^+ \big|_i,$

(2.7b) $\displaystyle\int_{I_i} (P_h^- u - P_h^1 u) v dx = \int_{I_i} \left((P_h^+ q - q_h) - (P_h^+ p - p_h) \right) v dx, \ \forall \ v \in P^{k-1}(I_i).$

Proof for the existence and uniqueness of $P_h^1 u$ is provided in [30], more precisely in its Lemma 2.1.

Lemma 2.1. *The operator P_h^1 exists and is unique. Moreover, we have the following estimate*

(2.8) $\left\| (P_h^- u - P_h^1 u)(.,0) \right\| + \left\| (P_h^+ q - q_h)(.,0) \right\| + \left\| (P_h^+ p - p_h)(.,0) \right\| \leq C \ h^{k+3/2},$

where $C = C(\alpha, \beta, \lambda, \|u\|_{k+3,I})$ is a constant and $\lambda = h/h_{min}$ is a constant during mesh refinements.

In our mathematical error analysis we will approximate the initial condition on each interval I_i as

$$u_h(x,0) = P_h^1 u(x,0), \ x \in I_i, \ i = 1, \cdots, N. \tag{2.9}$$

As discussed in [30], this operator is only needed for technical purposes in the proof of the error estimates. In our numerical experiments we used the special projection P_h^1, the projection P_h^-, and the standard L^2 projection and observed similar results.

2.3. Notations and preliminary results. We define the L^2 inner product of $u = u(x,t)$ and $v = v(x,t)$ on $I_i = [x_{i-1}, x_i]$ as $(u(.,t), v(.,t))_i = \int_{I_i} u(x,t) v(x,t) dx$. Denote $\|u(.,t)\|_{0,I_i} = ((u(.,t), u(.,t))_i)^{1/2}$ to be the standard L^2-norm of u on I_i. Let $H^s(I_i)$, where $s = 1, 2, \ldots$, denote the standard Sobolev space of square integrable functions on I_i with all derivatives $\partial_x^j u, \ j = 1, 2, \ldots, s$ being square integrable on I_i and equipped with the norm

$$\|u(.,t)\|_{s,I_i} = \left(\sum_{j=0}^{s} \left\| \partial_x^j u(.,t) \right\|_{0,I_i}^2 \right)^{1/2}.$$

We also define the norms on the whole computational domain I as follows:

$$\|u(.,t)\|_{0,I} = \left(\sum_{i=1}^{N} \|u(.,t)\|_{0,I_i}^2 \right)^{1/2}, \quad \|u(.,t)\|_{s,I} = \left(\sum_{i=1}^{N} \|u(.,t)\|_{s,I_i}^2 \right)^{1/2}.$$

For convenience, we use $\|u\|$ and $\|u\|_i$ to denote $\|u\|_{0,I}$ and $\|u\|_{0,I_i}$, respectively. Also, in the remainder of this paper, we will omit the notation $(.,t)$ used in norms unless needed for clarity. Thus, we use $\|u\|$ instead of $\|u(.,t)\|$ etc. We note that if $u \in H^s(I)$, $s = 1, 2, \ldots$, the norms $\|u(.,t)\|_{s,I}$ on the whole computational domain is the standard Sobolev norm $\left(\sum_{j=0}^{s} \left\| \partial_x^j u(.,t) \right\|_{0,I}^2 \right)^{1/2}$.

In our analysis we need the Legendre and Radau polynomials. Let us denote by $\tilde{L}_k$ the Legendre polynomial of degree k on $[-1,1]$. We define the $(k+1)$-degree right Radau polynomial on $[-1,1]$ as $\tilde{R}_{k+1}^+(\xi) = \tilde{L}_{k+1}(\xi) - \tilde{L}_k(\xi), \ -1 \leq \xi \leq 1$, which has $k+1$ real distinct roots, $-1 < \xi_0^+ < \cdots < \xi_k^+ = 1$. We also define the $(k+1)$-degree left Radau polynomial as $\tilde{R}_{k+1}^-(\xi) = \tilde{L}_{k+1}(\xi) + \tilde{L}_k(\xi), \ -1 \leq \xi \leq 1$, which has $k+1$ real distinct roots, $-1 = \xi_0^- < \cdots < \xi_k^- < 1$.

Mapping the physical element $I_i = [x_{i-1}, x_i]$ into a reference element $[-1,1]$ by the standard affine mapping $x(\xi, h_i) = \frac{x_i + x_{i-1}}{2} + \frac{h_i}{2}\xi$, we obtain the shifted Radau polynomials $R^{\pm}_{k+1,i}(x) = \tilde{R}^{\pm}_{k+1}\left(\frac{2x-x_i-x_{i-1}}{h_i}\right)$ on I_i. Next, we define the monic left and right Radau polynomials, $\psi^{\pm}_{k+1,i}(x)$, on I_i as $\psi^{\pm}_{k+1,i}(x) = \prod_{j=0}^{k}(x - x^{\pm}_{i,j})$, where $x^{\pm}_{i,j}$ are the roots of $\tilde{R}^{\pm}_{k+1,i}(\xi)$ shifted to I_i, *i.e.*,

$$x^{\pm}_{i,j} = \frac{x_i + x_{i-1}}{2} + \frac{h_i}{2}\xi^{\pm}_j, \ j = 0, 1, \dots, k. \tag{2.10}$$

Next, we recall some results from [11] (more precisely in its Lemma 2.1) which will be needed in our *a posteriori* error analysis.

Lemma 2.2. *The* $(k+1)$*-degree monic Radau polynomials on* I_i*,* $\psi^{\pm}_{k+1,i}(x)$*, satisfy*

$$\int_{I_i} {\psi^{+}_{k+1,i}}'\psi^{+}_{k+1,i}dx = -2c_k^2 h_i^{2k+2}, \quad \int_{I_i} {\psi^{-}_{k+1,i}}'\psi^{-}_{k+1,i}dx = 2c_k^2 h_i^{2k+2}, \tag{2.11a}$$

$$\int_{I_i} {\psi^{-}_{k+1,i}}'\psi^{+}_{k+1,i}dx = -2c_k^2 h_i^{2k+2}, \quad \left\|\psi^{+}_{k+1,i}\right\|_i^2 = \left\|\psi^{-}_{k+1,i}\right\|_i^2 = d_k h_i^{2k+3}, \tag{2.11b}$$

where

$$c_k = \frac{[(k+1)!]^2}{(2k+2)!}, \quad d_k = \frac{2(2k+2)}{(2k+1)(2k+3)}c_k^2. \tag{2.11c}$$

Throughout this paper, $e_u = u - u_h,\ e_q = q - q_h$, and $e_p = p - p_h$, respectively, denote the errors between the exact solutions of (2.1) and the numerical solutions defined in (2.3). Let the projection errors be defined as $\epsilon_u = u - P_h^- u,\ \epsilon_q = q - P_h^+ q,\ \epsilon_p = p - P_h^+ p$, and the errors between the numerical solutions and the projection of the exact solutions be defined as $\bar{e}_u = P_h^- u - u_h,\ \bar{e}_q = P_h^+ q - q_h,\ \bar{e}_p = P_h^+ p - p_h$. We note that the true errors can be split as

$$e_u = \epsilon_u + \bar{e}_u, \quad e_q = \epsilon_q + \bar{e}_q, \quad e_p = \epsilon_p + \bar{e}_p. \tag{2.12}$$

We also note that, by the definitions of the projections $P_h^{\pm}$, the following hold

$$\epsilon_u^-\big|_i = \epsilon_q^+\big|_{i-1} = \epsilon_p^+\big|_i = 0,$$

$$\int_{I_i}\epsilon_u v_x dx = \int_{I_i}\epsilon_q v_x dx = \int_{I_i}\epsilon_p v_x dx = 0, \quad \forall\ v \in\ P^k(I_i), \tag{2.13}$$

where we used the fact that $v \in\ P^k(I_i)$ and thus $v_x \in\ P^{k-1}(I_i)$.
We subtract (2.3) from (2.2) with $v,\ w,\ z \in V_h^k$ and we use the numerical fluxes (2.3d) to obtain the LDG orthogonality conditions for $e_u,\ e_q$, and e_p on I_i

$$\begin{aligned}&\int_{I_i}(e_u)_t v dx - \int_{I_i}(\alpha e_u + \beta e_p)v_x dx + \alpha e_u^- v^-\big|_i - \alpha e_u^- v^+\big|_{i-1}\\ &+\beta e_p^+ v^-\big|_i - \beta e_p^+ v^+\big|_{i-1} = 0,\end{aligned} \tag{2.14a}$$

$$\int_{I_i} e_p w dx + \int_{I_i} e_q w_x dx - e_q^+ w^-\big|_i + e_q^+ w^+\big|_{i-1} = 0, \tag{2.14b}$$

$$\int_{I_i} e_q z dx + \int_{I_i} e_u z_x dx - e_u^- z^-\big|_i + e_u^- z^+\big|_{i-1} = 0, \tag{2.14c}$$

which, after splitting the true errors as in (2.12) and applying (2.13), yields the following error equations

$$\int_{I_i}(e_u)_t v dx - \int_{I_i}(\alpha\bar{e}_u+\beta\bar{e}_p)v_x dx + \alpha\bar{e}_u^- v^-\big|_i - \alpha\bar{e}_u^- v^+\big|_{i-1}$$

$$+\beta\bar{e}_p^+ v^-\big|_i - \beta\bar{e}_p^+ v^+\big|_{i-1} = 0, \tag{2.15a}$$

$$\int_{I_i} e_p w dx + \int_{I_i}\bar{e}_q w_x dx - \bar{e}_q^+ w^-\big|_i + \bar{e}_q^+ w^+\big|_{i-1} = 0, \tag{2.15b}$$

$$\int_{I_i} e_q z dx + \int_{I_i}\bar{e}_u z_x dx - \bar{e}_u^- z^-\big|_i + \bar{e}_u^- z^+\big|_{i-1} = 0. \tag{2.15c}$$

Integrating by parts, the equations in (2.15) are equivalent to

$$\int_{I_i}(e_u)_t v dx + \int_{I_i}(\alpha(\bar{e}_u)_x + \beta(\bar{e}_p)_x) v dx + \alpha\left(\bar{e}_u^+ - \bar{e}_u^-\right) v^+\big|_{i-1}$$

$$+\beta\left(\bar{e}_p^+ - \bar{e}_p^-\right) v^-\big|_i = 0, \tag{2.16a}$$

$$\int_{I_i} e_p w dx - \int_{I_i}(\bar{e}_q)_x w dx + \left(\bar{e}_q^- w^- - \bar{e}_q^+\right) w^-\big|_i = 0, \tag{2.16b}$$

$$\int_{I_i} e_q z dx - \int_{I_i}(\bar{e}_u)_x z dx - \left(\bar{e}_u^+ - \bar{e}_u^-\right) z^+\big|_{i-1} = 0. \tag{2.16c}$$

Since $\bar{e}_u,\ \bar{e}_q,\ \bar{e}_p \in V_h^k$ are piecewise polynomials, we define $\bar{e}_u,\ \bar{e}_q$, and $\bar{e}_p$ on each element I_i, as

$$\bar{e}_u = \alpha_i + \frac{x-x_i}{h_i}c_i,\ \bar{e}_q = \beta_i + \frac{x-x_{i-1}}{h_i}d_i,\ \bar{e}_p = \gamma_i + \frac{x-x_{i-1}}{h_i}g_i, \tag{2.17a}$$

where $c_i(.,t),\ d_i(.,t),\ g_i(.,t) \in P^{k-1}(I_i)$ and

$$\alpha_i = \bar{e}_u^-\big|_i, \quad \beta_i = \bar{e}_q^+\big|_{i-1}, \quad \gamma_i = \bar{e}_p^+\big|_{i-1}. \tag{2.17b}$$

Throughout this paper, $c(.,t) \in V_h^{k-1}$, $d(.,t) \in V_h^{k-1}$, $g(.,t) \in V_h^{k-1}$, $\phi_1(x) \in V_h^1$, $\phi_2(x) \in V_h^1$, $\phi_3(x) \in V_h^1$, and $\phi_4(x) \in V_h^1$ denote piecewise polynomials which are defined as follows:

$$c(x,t) = c_i(x,t),\ d = d_i(x,t),\ g(x,t) = g_i(x,t),\ \text{on } I_i, \tag{2.18a}$$

$$\phi_1(x) = x - x_{i-1},\ \phi_2(x) = x - x_i,\ \phi_3(x) = \frac{x-x_{i-1}}{h_i},\ \phi_4(x) = \frac{x-x_i}{h_i},\ \text{on } I_i. \tag{2.18b}$$

Clearly, we have

$$\max_{x\in I}|\phi_1(x)| = \max_{x\in I}|\phi_2(x)| = h, \quad \max_{x\in I}|\phi_3(x)| = \max_{x\in I}|\phi_4(x)| = 1. \tag{2.18c}$$

In our analysis, we need the following well-known projection results [21]: For any smooth functions u, $q = u_x$, and $p = u_{xx}$, there exist positive constants $C_1 - C_3$ depend on the exact smooth solution u and its derivatives, but independent of the mesh size h, such that

$$\|\partial_t^s \epsilon_u\| \le C_1 h^{k+1}, \quad \|\partial_t^s \epsilon_q\| \le C_2 h^{k+1}, \quad \|\partial_t^s \epsilon_p\| \le C_3 h^{k+1}, \quad s = 0, 1, 2, \ldots. \tag{2.19}$$

Here $C_1 = \tilde{C}_1(\|\partial_t^s u\|_{k+1,I})$, $C_2 = \tilde{C}_2(\|\partial_t^s u\|_{k+2,I})$, $C_3 = \tilde{C}_3(\|\partial_t^s u\|_{k+3,I})$, where $\tilde{C}_1 - \tilde{C}_3$ are positive constants independent of h.
From now on, the notation $C,\ C_0,\ C_1,\ C^{\pm},$ etc. will be used to denote positive constants that are independent of the discretization parameters, but which may depend upon the exact smooth solution of the partial differential equation (1.1a)

and its derivatives. Furthermore, all the constants will be generic, *i.e.*, they may represent different constant quantities in different occurrences.
In the next lemma we recall the following results from [11] (more precisely in its Lemma 2.1) which will be needed in our analysis.

Lemma 2.3. *If $f(x) \in C^1$ on I_i, then*

$$\int_{I_i} \frac{x-x_{i-1}}{h_i} f(x) \frac{d}{dx}\left(\frac{x-x_i}{h_i} f(x)\right) dx = \frac{1}{2h_i} \int_{I_i} f^2(x) dx. \tag{2.20a}$$

$$\int_{I_i} \frac{x-x_i}{h_i} f(x) \frac{d}{dx}\left(\frac{x-x_{i-1}}{h_i} f(x)\right) dx = -\frac{1}{2h_i} \int_{I_i} f^2(x) dx. \tag{2.20b}$$

In the next theorem, we recall some results from [30] which will be needed in our error analysis.

Theorem 2.1. *Let $k \geq 1$ and $(u,\ q = u_x,\ p = q_x)$ and $(u_h,\ q_h,\ p_h)$ respectively, are solutions of (2.1) and (2.3) subject to $u_h(x,0) = P_h^1 u_0(x)$, then there exists a constant C independent of h such that*

$$\|e_u\| \leq C\ h^{k+1}. \tag{2.21a}$$

$$\|(e_u)_t\| \leq C\ h^{k+1}. \tag{2.21b}$$

$$\|e_q\| \leq C\ h^{k+1}. \tag{2.21c}$$

$$\|e_p\| \leq C\ h^{k+1}. \tag{2.21d}$$

$$\|\bar{e}_u\| \leq C\ h^{k+3/2}. \tag{2.21e}$$

Proof. Cf. Hufford and Xing [30]. More precisely, the estimates (2.21a)-(2.21d) can be found in its Lemma 2.2. The superconvergence result (2.21e) can be found in its Proposition 3.1. □

In order to prove that the $(k+3/2)$-th order superconvergence rate holds also for the two auxiliary variables in the LDG method, we state and prove the following additional results.

Theorem 2.2. *Under the same conditions as in Theorem 2.1, there exists a constant $C > 0$ such that*

$$\|(\bar{e}_q)_t(.,0)\| \leq C\ h^{k+1}. \tag{2.22a}$$

$$\|(\bar{e}_p)_t(.,0)\| \leq C\ h^{k+1}. \tag{2.22b}$$

$$\|(\bar{e}_u)_t(.,0)\| \leq C\ h^{k+3/2}. \tag{2.22c}$$

Proof. On the one hand, taking the first time derivative of (2.2b) and (2.2c) we obtain

$$\int_{I_i} p_t w dx + \int_{I_i} q_t w_x dx - q_t^+ w^-\big|_i + q_t^+ w^+\big|_{i-1} = 0, \quad \forall\ w \in V_h^k, \tag{2.23a}$$

$$\int_{I_i} q_t z dx + \int_{I_i} u_t z_x dx - u_t^- z^-\big|_i + u_t^- z^+\big|_{i-1} = 0, \quad \forall\ z \in V_h^k. \tag{2.23b}$$

Similarly, taking the first time derivative of (2.6a) and (2.6b) yields

$$\int_{I_i} (p_h)_t w dx + \int_{I_i} (q_h)_t w_x dx - (q_h)_t^+ w^-\big|_i + (q_h)_t^+ w^+\big|_{i-1} = 0, \tag{2.24a}$$

$$\int_{I_i} (q_h)_t z dx + \int_{I_i} (P_h^1 u)_t z_x dx - (P_h^1 u)_t^- z^-\big|_i + (P_h^1 u)_t^- z^+\big|_{i-1} = 0. \tag{2.24b}$$

Subtracting (2.24) from (2.23), we obtain

$$\int_{I_i}(e_p)_t w dx + \int_{I_i}(e_q)_t w_x dx - (e_q)_t^+ w^-\big|_i + (e_q)_t^+ w^+\big|_{i-1} = 0, \tag{2.25a}$$

$$\int_{I_i}(e_q)_t z dx + \int_{I_i}(u-P_h^1 u)_t z_x dx - (u-P_h^1 u)_t^- z^-\big|_i + (u-P_h^1 u)_t^- z^+\big|_{i-1} = 0, \tag{2.25b}$$

which, after splitting the errors as $u-P_h^1 u = u-P_h^- u+P_h^- u-P_h^1 u = \epsilon_u+P_h^- u-P_h^1 u$, $e_q = \epsilon_q + \bar{e}_q$, $e_p = \epsilon_p + \bar{e}_p$ and using the properties (2.13), is equivalent to

$$\int_{I_i}(e_p)_t w dx + \int_{I_i}(\bar{e}_q)_t w_x dx - (\bar{e}_q)_t^+ w^-\big|_i + (\bar{e}_q)_t^+ w^+\big|_{i-1} = 0, \tag{2.26}$$

$$\int_{I_i}(e_q)_t z dx + \int_{I_i}(P_h^- u-P_h^1 u)_t z_x dx - (P_h^- u-P_h^1 u)_t^- z^-\big|_i + (P_h^- u-P_h^1 u)_t^- z^+\big|_{i-1} = 0.$$

On the other hand, taking the first time derivative of (2.7) and using the properties (2.13) yields

$$(P_h^- u-P_h^1 u)_t^-\big|_i = (\bar{e}_q-\bar{e}_p)_t^+\big|_i,\ \int_{I_i}(P_h^- u-P_h^1 u)_t v dx = \int_{I_i}(\bar{e}_q-\bar{e}_p)_t v dx,\ \forall\, v\in P^{k-1}(I_i).$$

Combining these conditions with (2.26) we get

$$\int_{I_i}(e_p)_t w dx + \int_{I_i}(\bar{e}_q)_t w_x dx - (\bar{e}_q)_t^+ w^-\big|_i + (\bar{e}_q)_t^+ w^+\big|_{i-1} = 0, \tag{2.27a}$$

$$\int_{I_i}(e_q)_t z dx + \int_{I_i}(\bar{e}_q-\bar{e}_p)_t z_x dx - (\bar{e}_q-\bar{e}_p)_t^+ z^-\big|_i + (\bar{e}_q-\bar{e}_p)_t^+ z^+\big|_{i-1} = 0. \tag{2.27b}$$

Now, taking $w=(\bar{e}_q)_t$ and $z=(\bar{e}_q-\bar{e}_p)_t$ in (2.27), we get

$$\int_{I_i}(e_p)_t(\bar{e}_q)_t dx + \frac{1}{2}\left((\bar{e}_q)_t^+ - (\bar{e}_q)_t^-\right)^2\big|_i - \frac{1}{2}((\bar{e}_q)_t^+)^2\big|_i + \frac{1}{2}((\bar{e}_q)_t^+)^2\big|_{i-1} = 0,$$

$$\int_{I_i}(e_q)_t(\bar{e}_q-\bar{e}_p)_t dx + \frac{1}{2}\left((\bar{e}_q-\bar{e}_p)_t^+ - (\bar{e}_q-\bar{e}_p)_t^-\right)^2\big|_i - \frac{1}{2}((\bar{e}_q-\bar{e}_p)_t^+)^2\big|_i$$
$$+\frac{1}{2}((\bar{e}_q-\bar{e}_p)_t^+)^2\big|_{i-1} = 0.$$

Adding these two equations, using (2.12), summing over all elements, and using the periodic boundary conditions, we obtain, at $t=0$,

$$\|(\bar{e}_q)_t(.,0)\|^2 + \int_I\left((\epsilon_q)_t + (\epsilon_p)_t\right)(\bar{e}_q)_t dx - \int_I(\epsilon_q)_t(\bar{e}_p)_t dx + \frac{1}{2}\sum_{i=1}^N\left((\bar{e}_q)_t^+ - (\bar{e}_q)_t^-\right)^2\big|_i +$$
$$\frac{1}{2}\sum_{i=1}^N\left((\bar{e}_q-\bar{e}_p)_t^+ - (\bar{e}_q-\bar{e}_p)_t^-\right)^2\big|_i = 0.$$

Thus, at $t=0$, the following holds

$$\|(\bar{e}_q)_t(.,0)\|^2 \le -\int_I\left((\epsilon_q)_t + (\epsilon_p)_t\right)(\bar{e}_q)_t dx + \int_I(\epsilon_q)_t(\bar{e}_p)_t dx. \tag{2.28}$$

On the other hand, letting $w=(\bar{e}_q+\bar{e}_p)_t$ and $z=-(\bar{e}_p)_t$ in (2.27) leads to

$$\int_{I_i}(e_p)_t(\bar{e}_q+\bar{e}_p)_t dx + \int_{I_i}(\bar{e}_q)_t(\bar{e}_q+\bar{e}_p)_{xt} dx - (\bar{e}_q)_t^+(\bar{e}_q+\bar{e}_p)_t^-\big|_i + (\bar{e}_q)_t^+(\bar{e}_q+\bar{e}_p)_t^+\big|_{i-1} = 0,$$

$$-\int_{I_i}(e_q)_t(\bar{e}_p)_t dx - \int_{I_i}(\bar{e}_q-\bar{e}_p)_t(\bar{e}_p)_{xt} dx + (\bar{e}_q-\bar{e}_p)_t^+(\bar{e}_p)_t^-\big|_i - (\bar{e}_q-\bar{e}_p)_t^+(\bar{e}_p)_t^+\big|_{i-1} = 0.$$

Adding these two equations and using (2.12), we obtain

$$\begin{aligned}&\int_{I_i}((\bar e_p)_t)^2dx+\int_{I_i}((\epsilon_p)_t-(\epsilon_q)_t)\,(\bar e_p)_tdx+\int_{I_i}(\epsilon_p)_t(\bar e_q)_tdx+\int_{I_i}(\bar e_q)_t(\bar e_q)_{xt}dx\\&+\int_{I_i}(\bar e_p)_t(\bar e_p)_{xt}dx-(\bar e_q)_t^+(\bar e_q)_t^-\big|_i-(\bar e_p)_t^+(\bar e_p)_t^-\big|_i+((\bar e_q)_t^+)^2\big|_{i-1}+((\bar e_p)_t^+)^2\big|_{i-1}=0,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\int_{I_i}((\bar e_p)_t)^2dx+\int_{I_i}((\epsilon_p)_t-(\epsilon_q)_t)\,(\bar e_p)_tdx+\int_{I_i}(\epsilon_p)_t(\bar e_q)_tdx\\&+\frac12\left((\bar e_q)_t^+-(\bar e_q)_t^-\right)^2\big|_i-\frac12((\bar e_q)_t^+)^2\big|_i+\frac12((\bar e_q)_t^+)^2\big|_{i-1}+\frac12\left((\bar e_p)_t^+-(\bar e_p)_t^-\right)^2\big|_i\\&-\frac12((\bar e_p)_t^+)^2\big|_i+\frac12((\bar e_p)_t^+)^2\big|_{i-1}=0.\end{aligned}$$

Summing over all elements and using the periodic boundary conditions, we get, at $t=0$,

$$\begin{aligned}\|(\bar e_p)_t(.,0)\|^2+\frac12\sum_{i=1}^{N}\left[\left((\bar e_q)_t^+-(\bar e_q)_t^-\right)^2+\left((\bar e_p)_t^+-(\bar e_p)_t^-\right)^2\right]\Big|_i\\=\int_I((\epsilon_q)_t-(\epsilon_p)_t)\,(\bar e_p)_tdx-\int_I(\epsilon_p)_t(\bar e_q)_tdx.\end{aligned}$$

Hence, at $t=0$, the following holds

$$\|(\bar e_p)_t(.,0)\|^2\le\int_I((\epsilon_q)_t-(\epsilon_p)_t)\,(\bar e_p)_tdx-\int_I(\epsilon_p)_t(\bar e_q)_tdx. \tag{2.29}$$

Now, adding (2.28) and (2.29) yields

$$\|(\bar e_q)_t(.,0)\|^2+\|(\bar e_p)_t(.,0)\|^2\le-\int_I((\epsilon_q)_t+2(\epsilon_p)_t)\,(\bar e_q)_tdx+\int_I(2(\epsilon_q)_t-(\epsilon_p)_t)\,(\bar e_p)_tdx.$$

Applying Cauchy-Schwarz inequality and using the projection results (2.19) yields

$$\begin{aligned}\|(\bar e_q)_t(.,0)\|^2+\|(\bar e_p)_t(.,0)\|^2&\le\Big(\|(\epsilon_q)_t(.,0)\|+2\,\|(\epsilon_p)_t(.,0)\|\Big)\|(\bar e_q)_t(.,0)\|+\\&\quad\Big(2\,\|(\epsilon_q)_t(.,0)\|+\|(\epsilon_p)_t(.,0)\|\Big)\|(\bar e_p)_t(.,0)\|\\&\le C_1h^{k+1}\Big(\|(\bar e_q)_t(.,0)\|+\|(\bar e_p)_t(.,0)\|\Big).\end{aligned}$$

Using the inequalities $ab\le a^2+\frac14b^2$ and $(a+b)^2\le2(a^2+b^2)$, we obtain

$$\|(\bar e_q)_t(.,0)\|^2+\|(\bar e_p)_t(.,0)\|^2\le C_1^2h^{2k+2}+\frac12\Big(\|(\bar e_q)_t(.,0)\|^2+\|(\bar e_p)_t(.,0)\|^2\Big).$$

As a consequence, we obtain

$$\|(\bar e_q)_t(.,0)\|^2+\|(\bar e_p)_t(.,0)\|^2\le Ch^{2k+2},$$

which completes the proofs of (2.22a) and (2.22b).
Finally, we will prove (2.22c). Since $u_h(x,0)=P_h^1u(x,0)$, (2.7) can be written as

$$\bar e_u^-\big|_i=\bar e_q^+\big|_i-\bar e_p^+\big|_i,\quad\int_{I_i}\bar e_uvdx=\int_{I_i}(\bar e_q-\bar e_p)\,vdx,\ \forall\ v\in P^{k-1}(I_i).$$

Substituting these equations into (2.15a), we obtain, at $t = 0$

$$\int_{I_i} (e_u)_t v dx - (\alpha - \beta) \left(\int_{I_i} \bar{e}_u v_x dx - \bar{e}_u^- v^- \big|_i + \bar{e}_u^- v^+ \big|_{i-1} \right)$$
$$-\beta \left(\int_{I_i} \bar{e}_q v_x dx - \bar{e}_q^+ v^- \big|_i + \bar{e}_q^+ v^+ \big|_{i-1} \right) = 0. \quad (2.30)$$

We note that it follows from (2.15b) and (2.15c) that, at time $t = 0$,

$$\int_{I_i} e_p v dx + \int_{I_i} \bar{e}_q v_x dx - \bar{e}_q^+ v^- \big|_i + \bar{e}_q^+ v^+ \big|_{i-1} = 0, \quad \forall\, v \in P^k(I_i) \quad (2.31a)$$
$$\int_{I_i} e_q v dx + \int_{I_i} \bar{e}_u v_x dx - \bar{e}_u^- v^- \big|_i + \bar{e}_u^- v^+ \big|_{i-1} = 0, \quad \forall\, v \in P^k(I_i). \quad (2.31b)$$

Combining (2.30) and (2.31), we get, at $t = 0$

$$\int_{I_i} (e_u)_t v dx + \beta \int_{I_i} e_q v dx + (\alpha - \beta) \int_{I_i} e_p v dx = 0. \quad (2.32)$$

Taking $v = (\bar{e}_u)_t(x, 0)$, summing the above equality over all elements, and using the fact that $e_u = \bar{e}_u + \epsilon_u$, we obtain, at time $t = 0$,

$$\|(\bar{e}_u)_t(., 0)\|^2 = \Theta + \Lambda, \quad (2.33)$$

where

$$\Theta = -\int_I \left((\epsilon_u)_t + \beta\epsilon_q + (\alpha - \beta)\epsilon_p\right) (\bar{e}_u)_t dx, \quad \Lambda = -\int_I \left(\beta\bar{e}_q + (\alpha - \beta)\bar{e}_p\right) (\bar{e}_u)_t dx.$$

On the one hand, applying Cauchy-Schwarz inequality and the inequalities $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ and $(a + b)^2 \leq 2(a^2 + b^2)$, we get

$$\begin{aligned}
\Lambda &\leq \Big(\beta \|\bar{e}_q(., 0)\| + |\alpha - \beta| \|\bar{e}_p(., 0)\| \Big) \|(\bar{e}_u)_t(., 0)\| \\
&\leq \frac{1}{2}\Big(\beta \|\bar{e}_q(., 0)\| + |\alpha - \beta| \|\bar{e}_p(., 0)\| \Big)^2 + \frac{1}{2} \|(\bar{e}_u)_t(., 0)\|^2 \\
&\leq \beta^2 \|\bar{e}_q(., 0)\|^2 + (\alpha - \beta)^2 \|\bar{e}_p(., 0)\|^2 + \frac{1}{2} \|(\bar{e}_u)_t(., 0)\|^2,
\end{aligned}$$

which, after using the estimate (2.8), yields

$$\Lambda \leq C_1 h^{2k+3} + \frac{1}{2} \|(\bar{e}_u)_t(., 0)\|^2. \quad (2.34)$$

On the other hand, substituting the definition of $\bar{e}_u$ given in (2.17) into Θ and using the fact that $(\epsilon_u)_t$, ϵ_q, and ϵ_p are orthogonal to any piecewise constant functions (this is by the property of the projections $P_h^{\pm}$ (2.13)), we get, at $t = 0$,

$$\Theta = -\sum_{i=1}^{N} \int_{I_i} \left((\epsilon_u)_t + \beta\epsilon_q + (\alpha - \beta)\epsilon_p\right) \frac{x - x_i}{h_i} (c_i)_t dx,$$

which, after applying Cauchy-Schwarz inequality, using (2.18), and the projection result (2.19), yields

$$\begin{aligned}
\Theta &\leq \left(\max_{x\in I} |\phi_4(x)| \right) \Big(\|(\epsilon_u)_t(., 0)\| + \beta \|\epsilon_q(., 0)\| + |\alpha - \beta| \|\epsilon_p(., 0)\| \Big) \|c_t(., 0)\| \\
&\leq C_2 h^{k+1} \|c_t(., 0)\|. \quad (2.35)
\end{aligned}$$

In order to estimate $\|c_t(.,0)\|$, we take the first time derivative of (2.16c), we substitute the definition of $\bar{e}_u$ given in (2.17), and we choose the test function as $z = \frac{x-x_{i-1}}{h_i}(c_i)_t(x,0)$ to obtain, at $t=0$,

$$\int_{I_i}(e_q)_t\frac{x-x_{i-1}}{h_i}(c_i)_t dx - \int_{I_i}\left(\frac{x-x_i}{h_i}(c_i)_t\right)_x\frac{x-x_{i-1}}{h_i}(c_i)_t dx = 0,$$

which, after using (2.20a), is equivalent to

$$\int_{I_i}(e_q)_t\frac{x-x_{i-1}}{h_i}(c_i)_t dx - \frac{1}{2h_i}\int_{I_i}(c_i)_t^2 dx = 0.$$

Hence, we obtain

$$\int_{I_i}(c_i)_t^2 dx \quad = \quad 2\int_{I_i}(e_q)_t(x-x_{i-1})(c_i)_t dx.$$

Summing over all element, applying Cauchy-Schwarz inequality, and using (2.18), we obtain

$$\|c_t(.,0)\|^2 \quad \leq \quad 2\left(\max_{x\in I}|\phi_1(x)|\right)\|(e_q)_t(.,0)\|\,\|c_t(.,0)\| \leq 2h\,\|(e_q)_t(.,0)\|\,\|c_t(.,0)\|.$$

Applying the estimates (2.22a), we get

$$\|c_t(.,0)\| \leq C_3 h^{k+2}. \tag{2.36}$$

Combining (2.33), (2.34), (2.35), and (2.36), we obtain

$$\|(\bar{e}_u)_t(.,0)\|^2 \leq C_4 h^{2k+3} + \frac{1}{2}\|(\bar{e}_u)_t(.,0)\|^2,$$

which completes the proof of (2.22c) $\square$

3. Superconvergence error analysis

Before we proof the main superconvergence results, we state and prove the following results which will be needed in our analysis.

Lemma 3.1. *Suppose that the conditions in Theorem 2.1 are satisfied. If c, d and g are the functions defined in (2.18a) then there exists a positive constant C independent of h such that, $\forall\ t\in[0,T]$,*

$$\|c\| \leq Ch^{k+2}, \quad \|d\| \leq Ch^{k+2}, \quad \|g\| \leq Ch^{k+2}. \tag{3.1}$$

Proof. Adding (2.16a) and (2.16c) with $z=\alpha v$, where $v\in P^k(I_i)$, we obtain

$$\int_{I_i}((e_u)_t+\alpha e_q)\,v dx + \beta\int_{I_i}(\bar{e}_p)_x v dx - \beta\bar{e}_p^- v^-\big|_i + \beta\bar{e}_p^+ v^-\big|_i = 0. \tag{3.2}$$

Substituting (2.17) into (3.2), (2.16b), and (2.16c), choosing the test functions as $v=\frac{x-x_i}{h_i}g_i$, $w=\frac{x-x_i}{h_i}d_i$, $z=\frac{x-x_{i-1}}{h_i}c_i$, and using (2.20a) and (2.20b), we obtain

$$\int_{I_i}((e_u)_t+\alpha e_q)\frac{x-x_i}{h_i}g_i dx - \frac{\beta}{2h_i}\int_{I_i}g_i^2 dx = 0,$$
$$\int_{I_i}e_p\frac{x-x_i}{h_i}d_i dx + \frac{1}{2h_i}\int_{I_i}d_i^2 dx = 0, \quad \int_{I_i}e_q\frac{x-x_{i-1}}{h_i}c_i dx - \frac{1}{2h_i}\int_{I_i}c_i^2 dx = 0,$$

since $v^-\big|_i = w^-\big|_i = z^+\big|_{i-1} = 0$. Thus,

$$\int_{I_i}g_i^2 dx = \frac{2}{\beta}\int_{I_i}((e_u)_t+\alpha e_q)(x-x_i)g_i dx,$$
$$\int_{I_i}d_i^2 dx = -2\int_{I_i}e_p(x-x_i)g_i dx, \quad \int_{I_i}c_i^2 dx = 2\int_{I_i}e_q(x-x_{i-1})c_i dx.$$

Summing over all element, applying Cauchy-Schwarz inequality, and using (2.18), we obtain

$$\begin{aligned}\|g\|^2 &\le \frac{2}{\beta}\left(\max_{x\in I}|\phi_2(x)|\right)\|(e_u)_t+\alpha e_q\|\,\|g\| \le \frac{2}{\beta}h\left(\|(e_u)_t\|+\alpha\|e_q\|\right)\|g\|,\\ \|d\|^2 &\le 2\left(\max_{x\in I}|\phi_2(x)|\right)\|e_p\|\,\|d\| \le 2h\|e_p\|\,\|d\|,\\ \|c\|^2 &\le 2\left(\max_{x\in I}|\phi_1(x)|\right)\|e_q\|\,\|c\| \le 2h\|e_q\|\,\|c\|.\end{aligned}$$

We complete the proof of the lemma by applying the estimates (2.21b), (2.21c), and (2.21d). □

Now, we are ready to prove that q_h and p_h are $\mathcal{O}(h^{k+3/2})$ super close to $P_h^+ q$ and $P_h^+ p$, respectively.

Theorem 3.1. *Under the same conditions as in Theorem 2.1, there exists a constant $C>0$ such that, $\forall\ t\in[0,T]$,*

$$\|(\bar{e}_u)_t\| \le Ch^{k+3/2}. \tag{3.3a}$$

$$\|\bar{e}_q\| \le Ch^{k+3/2}. \tag{3.3b}$$

$$\|\bar{e}_p\| \le Ch^{k+3/2}. \tag{3.3c}$$

Proof. Taking the first time derivation of (2.15a), (2.16b), (2.16c), and then choosing the test functions as $v=(\bar{e}_u)_t$, $w=\beta(\bar{e}_q)_t$, and $z=-\beta(\bar{e}_p)_t$ we get

$$\begin{aligned}&\int_{I_i}(e_u)_{tt}(\bar{e}_u)_t dx-\alpha\int_{I_i}(\bar{e}_u)_t(\bar{e}_u)_{xt}dx+\alpha((\bar{e}_u)_t^-)^2\big|_i-\alpha(\bar{e}_u)_t^-(\bar{e}_u)_t^+\big|_{i-1}\\ &-\beta\int_{I_i}(\bar{e}_p)_t(\bar{e}_u)_{xt}dx+\beta(\bar{e}_u)_t^-(\bar{e}_p)_t^+\big|_i-\beta(\bar{e}_u)_t^+(\bar{e}_p)_t^+\big|_{i-1}=0,\\ &\beta\int_{I_i}(e_p)_t(\bar{e}_q)_t dx-\beta\int_{I_i}(\bar{e}_q)_{xt}(\bar{e}_q)_t dx+\beta((\bar{e}_q)_t^-)^2\big|_i-\beta(\bar{e}_q)_t^+(\bar{e}_q)_t^-\big|_i=0,\\ &-\beta\int_{I_i}(e_q)_t(\bar{e}_p)_t dx+\beta\int_{I_i}(\bar{e}_u)_{xt}(\bar{e}_p)_t dx+\beta(\bar{e}_u)_t^+(\bar{e}_p)_t^+\big|_{i-1}-\beta(\bar{e}_u)_t^-(\bar{e}_p)_t^+\big|_{i-1}=0.\end{aligned}$$

Adding these three equations, splitting the errors as in (2.12), and using the fact that $\int_{I_i}(\bar{e}_u)_t(\bar{e}_u)_{xt}dx=\frac12((\bar{e}_u)_t^-)^2\big|_i-\frac12((\bar{e}_u)_t^+)^2\big|_{i-1}$ and $\int_{I_i}(\bar{e}_q)_{xt}(\bar{e}_q)_t dx=\frac12((\bar{e}_q)_t^-)^2\big|_i-\frac12((\bar{e}_q)_t^+)^2\big|_{i-1}$, we get

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{I_i}(\bar{e}_u)_t^2dx+\int_{I_i}(\epsilon_u)_{tt}(\bar{e}_u)_t dx+\beta\int_{I_i}(\epsilon_p)_t(\bar{e}_q)_t dx-\beta\int_{I_i}(\epsilon_q)_t(\bar{e}_p)_t dx\\ &+\frac{\alpha}{2}((\bar{e}_u)_t^-)^2\big|_i+\frac{\alpha}{2}((\bar{e}_u)_t^+)^2\big|_{i-1}-\alpha(\bar{e}_u)_t^-(\bar{e}_u)_t^+\big|_{i-1}+\frac{\beta}{2}((\bar{e}_q)_t^-)^2\big|_i+\frac{\beta}{2}((\bar{e}_q)_t^+)^2\big|_{i-1}\\ &-\beta(\bar{e}_q)_t^+(\bar{e}_q)_t^-\big|_i+\beta(\bar{e}_u)_t^-(\bar{e}_p)_t^+\big|_i-\beta(\bar{e}_u)_t^-(\bar{e}_p)_t^+\big|_{i-1}=0,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{I_i}(\bar{e}_u)_t^2dx+\int_{I_i}(\epsilon_u)_{tt}(\bar{e}_u)_t dx+\beta\int_{I_i}(\epsilon_p)_t(\bar{e}_q)_t dx-\beta\int_{I_i}(\epsilon_q)_t(\bar{e}_p)_t dx\\ &+\frac{\alpha}{2}\left((\bar{e}_u)_t^+-(\bar{e}_u)_t^-\right)^2\big|_{i-1}+\frac{\alpha}{2}((\bar{e}_u)_t^-)^2\big|_i-\frac{\alpha}{2}((\bar{e}_u)_t^-)^2\big|_{i-1}+\frac{\beta}{2}\left((\bar{e}_q)_t^+-(\bar{e}_q)_t^-\right)^2\big|_i\\ &+\frac{\beta}{2}((\bar{e}_q)_t^+)^2\big|_{i-1}-\frac{\beta}{2}((\bar{e}_q)_t^+)^2\big|_i+\beta(\bar{e}_u)_t^-(\bar{e}_p)_t^+\big|_i-\beta(\bar{e}_u)_t^-(\bar{e}_p)_t^+\big|_{i-1}=0.\end{aligned}$$

Summing the above equation over i and using the periodic boundary condition, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|(\bar{e}_u)_t\|^2 + \int_I (\epsilon_u)_{tt}(\bar{e}_u)_t dx + \beta \int_I (\epsilon_p)_t(\bar{e}_q)_t dx - \beta \int_I (\epsilon_q)_t(\bar{e}_p)_t dx$$
$$+\frac{\alpha}{2}\sum_{i=1}^{N}\left((\bar{e}_u)_t^+ - (\bar{e}_u)_t^-\right)^2\big|_{i-1} + \frac{\beta}{2}\sum_{i=1}^{N}\left((\bar{e}_q)_t^+ - (\bar{e}_q)_t^-\right)^2\big|_{i} = 0.$$

Thus, the following holds

$$\frac{1}{2}\frac{d}{dt}\|(\bar{e}_u)_t\|^2 \le -\int_I (\epsilon_u)_{tt}(\bar{e}_u)_t dx - \beta \int_I (\epsilon_p)_t(\bar{e}_q)_t dx + \beta \int_I (\epsilon_q)_t(\bar{e}_p)_t dx.$$

Integrating the above inequality with respect to time and using integration by parts with respect to time, we get

$$\begin{aligned} \|(\bar{e}_u)_t\|^2 \quad &\le \quad \|(\bar{e}_u)_t(.,0)\|^2 - 2\int_I \Big((\epsilon_u)_{tt}\bar{e}_u + \beta(\epsilon_p)_t\bar{e}_q - \beta(\epsilon_q)_t\bar{e}_p \Big)(x,0)dx + \\ &\qquad 2\int_0^t \left(\int_I (\epsilon_u)_{ttt}\bar{e}_u dx + \beta\int_I (\epsilon_p)_{tt}\bar{e}_q dx - \beta \int_I (\epsilon_q)_{tt}\bar{e}_p dx \right) dt. \end{aligned} \tag{3.4}$$

Using the bound for initial error (2.22c), substituting the definitions of $\bar{e}_u, \bar{e}_q$ and $\bar{e}_p$ given in (2.17) into the right-hand side of (3.4), and using the fact that $(\epsilon_u)_{tt}$, $(\epsilon_u)_{ttt}$, $(\epsilon_q)_t$, $(\epsilon_q)_{tt}$, $(\epsilon_p)_t$, and $(\epsilon_p)_{tt}$ are orthogonal to any piecewise constant functions (due to the properties given in (2.13)), we get

$$\begin{aligned} &\|(\bar{e}_u)_t\|^2 \le C_0 h^{2k+3} \\ &-2\sum_{i=1}^{N}\int_{I_i}\left((\epsilon_u)_{tt}\frac{x-x_i}{h_i}c_i + \beta(\epsilon_p)_t\frac{x-x_{i-1}}{h_i}d_i - \beta(\epsilon_q)_t\frac{x-x_{i-1}}{h_i}g_i \right)(x,0)dx \\ &+2\int_0^t \left(\sum_{i=1}^{N}\int_{I_i}\left((\epsilon_u)_{ttt}\frac{x-x_i}{h_i}c_i + \beta(\epsilon_p)_{tt}\frac{x-x_{i-1}}{h_i}d_i - \beta(\epsilon_q)_{tt}\frac{x-x_{i-1}}{h_i}g_i \right) dx \right) dt, \end{aligned}$$

which, after applying Cauchy-Schwarz inequality, leads to

$$\begin{aligned} &\|(\bar{e}_u)_t\|^2 \le C_0 h^{2k+3} + 2\max_{x\in I}|\phi_4(x)|\,\|(\epsilon_u)_{tt}(.,0)\|\,\|c(.,0)\| \\ &+2\beta\max_{x\in I}|\phi_3(x)|\Big(\|(\epsilon_p)_t(.,0)\|\,\|d(.,0)\| + \|(\epsilon_q)_t(.,0)\|\,\|g(.,0)\| \Big) + \\ &\int_0^t \left(2\max_{x\in I}|\phi_4(x)|\,\|(\epsilon_u)_{ttt}\|\,\|c\| + 2\beta\max_{x\in I}|\phi_3(x)|\Big(\|(\epsilon_p)_{tt}\|\,\|d\| + \|(\epsilon_q)_{tt}\|\,\|g\| \Big) \right) dt. \end{aligned}$$

Using (2.18) and the projection results (2.19), we arrive at

$$\begin{aligned} &\|(\bar{e}_u)_t\|^2 \le C_0 h^{2k+3} \\ &+C_1 h^{k+1}\left(\|c(.,0)\| + \|d(.,0)\| + \|g(.,0)\|\right) + C_2 h^{k+1}\int_0^t \left(\|c\| + \|d\| + \|g\|\right) dt \\ &\le C_0 h^{2k+3} + C_1 h^{k+1}\left(\|c(.,0)\| + \|d(.,0)\| + \|g(.,0)\|\right) + C_2 T h^{k+1}\left(\|c\| + \|d\| + \|g\|\right). \end{aligned}$$

Combining this with the estimates in (3.1), we obtain

$$\|(\bar{e}_u)_t\|^2 \quad \le \quad C h^{2k+3}, \quad \forall\, t \in [0,T],$$

which completes the proof of (3.3a).

Next, we will prove (3.3b). Taking the test functions in (2.15a) and (2.16b) as $v = -\bar{e}_p$ and $w = (\bar{e}_u)_t$, respectively, and then taking the first time derivative of (2.15c) and letting $z = \bar{e}_q$, we obtain

$$
\begin{aligned}
&-\int_{I_i}(e_u)_t\bar{e}_p dx + \alpha\int_{I_i}\bar{e}_u(\bar{e}_p)_x dx - \alpha\bar{e}_u^-\bar{e}_p^-\big|_i + \alpha\bar{e}_u^-\bar{e}_p^+\big|_{i-1} + \beta\int_{I_i}\bar{e}_p(\bar{e}_p)_x dx\\
&-\beta\bar{e}_p^+\bar{e}_p^-\big|_i + \beta(\bar{e}_p^+)^2\big|_{i-1} = 0,\\
&\int_{I_i} e_p(\bar{e}_u)_t dx - \int_{I_i}(\bar{e}_q)_x(\bar{e}_u)_t dx + \bar{e}_q^-(\bar{e}_u)_t^-\big|_i - \bar{e}_q^+(\bar{e}_u)_t^-\big|_i = 0,\\
&\int_{I_i}(e_q)_t\bar{e}_q dx + \int_{I_i}(\bar{e}_u)_t(\bar{e}_q)_x dx - (\bar{e}_u)_t^-\bar{e}_q^-\big|_i + (\bar{e}_u)_t^-\bar{e}_q^+\big|_{i-1} = 0.
\end{aligned}
$$

Adding these three equations, applying (2.12), and using the fact that $\int_{I_i}\bar{e}_p(\bar{e}_p)_x dx = \frac{1}{2}(\bar{e}_p^-)^2\big|_i - \frac{1}{2}(\bar{e}_p^+)^2\big|_{i-1}$, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{I_i}\bar{e}_q^2 dx + \int_{I_i}(\epsilon_q)_t\bar{e}_q dx - \int_{I_i}(\epsilon_u)_t\bar{e}_p dx + \int_{I_i}\epsilon_p(\bar{e}_u)_t dx + \alpha\int_{I_i}\bar{e}_u(\bar{e}_p)_x dx\\
&-\alpha\bar{e}_u^-\bar{e}_p^-\big|_i + \alpha\bar{e}_u^-\bar{e}_p^+\big|_{i-1} + \frac{\beta}{2}(\bar{e}_p^-)^2\big|_i + \frac{\beta}{2}(\bar{e}_p^+)^2\big|_{i-1} - \beta\bar{e}_p^+\bar{e}_p^-\big|_i\\
&-(\bar{e}_u)_t^-\bar{e}_q^+\big|_i + (\bar{e}_u)_t^-\bar{e}_q^+\big|_{i-1} = 0,
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{I_i}\bar{e}_q^2 dx + \int_{I_i}(\epsilon_q)_t\bar{e}_q dx - \int_{I_i}(\epsilon_u)_t\bar{e}_p dx + \int_{I_i}\epsilon_p(\bar{e}_u)_t dx + \alpha\int_{I_i}\bar{e}_u(\bar{e}_p)_x dx\\
&-\alpha\bar{e}_u^-\bar{e}_p^-\big|_i + \alpha\bar{e}_u^-\bar{e}_p^+\big|_{i-1} + \frac{\beta}{2}\left(\bar{e}_p^+ - \bar{e}_p^-\right)^2\big|_i + \frac{\beta}{2}(\bar{e}_p^+)^2\big|_{i-1} - \frac{\beta}{2}(\bar{e}_p^+)^2\big|_i\\
&-(\bar{e}_u)_t^-\bar{e}_q^+\big|_i + (\bar{e}_u)_t^-\bar{e}_q^+\big|_{i-1} = 0.
\end{aligned} \tag{3.5}
$$

On the other hand, letting $w = \alpha\bar{e}_q$ and $z = -\alpha\bar{e}_p$ in (2.15b) and (2.15c), respectively, yields

$$\alpha\int_{I_i} e_p\bar{e}_q dx + \alpha\int_{I_i}\bar{e}_q(\bar{e}_q)_x dx - \alpha\bar{e}_q^+\bar{e}_q^-\big|_i + \alpha(\bar{e}_q^+)^2\big|_{i-1} = 0, \tag{3.6a}$$

$$-\alpha\int_{I_i} e_q\bar{e}_p dx - \alpha\int_{I_i}\bar{e}_u(\bar{e}_p)_x dx + \alpha\bar{e}_u^-\bar{e}_p^-\big|_i - \alpha\bar{e}_u^-\bar{e}_p^+\big|_{i-1} = 0. \tag{3.6b}$$

Adding the two equations in (3.6) and using (2.12), we get

$$
\begin{aligned}
&\alpha\int_{I_i}\epsilon_p\bar{e}_q dx - \alpha\int_{I_i}\epsilon_q\bar{e}_p dx - \alpha\int_{I_i}\bar{e}_u(\bar{e}_p)_x dx + \frac{\alpha}{2}\left(\bar{e}_q^+ - \bar{e}_q^-\right)^2\big|_i\\
&+\frac{\alpha}{2}(\bar{e}_q^+)^2\big|_{i-1} - \frac{\alpha}{2}(\bar{e}_q^+)^2\big|_i + \alpha\bar{e}_u^-\bar{e}_p^-\big|_i - \alpha\bar{e}_u^-\bar{e}_p^+\big|_{i-1} = 0.
\end{aligned} \tag{3.7}
$$

Now, adding (3.5) and (3.7) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{I_i}\bar{e}_q^2 dx + \int_{I_i}\epsilon_p(\bar{e}_u)_t dx + \int_{I_i}\left((\epsilon_q)_t + \alpha\epsilon_p\right)\bar{e}_q dx - \int_{I_i}\left((\epsilon_u)_t + \alpha\epsilon_q\right)\bar{e}_p dx\\
&+\frac{\beta}{2}\left(\bar{e}_p^+ - \bar{e}_p^-\right)^2\big|_i + \frac{\alpha}{2}\left(\bar{e}_q^+ - \bar{e}_q^-\right)^2\big|_i + \frac{\beta}{2}(\bar{e}_p^+)^2\big|_{i-1} - \frac{\beta}{2}(\bar{e}_p^+)^2\big|_i\\
&+\frac{\alpha}{2}(\bar{e}_q^+)^2\big|_{i-1} - \frac{\alpha}{2}(\bar{e}_q^+)^2\big|_i - (\bar{e}_u)_t^-\bar{e}_q^+\big|_i + (\bar{e}_u)_t^-\bar{e}_q^+\big|_{i-1} = 0,
\end{aligned}
$$

which, after summing over all elements and applying the periodic boundary conditions, yields

$$\frac{1}{2}\frac{d}{dt}\|\bar{e}_q\|^2 + \int_I \epsilon_p(\bar{e}_u)_t dx + \int_I ((\epsilon_q)_t + \alpha\epsilon_p)\,\bar{e}_q dx - \int_I ((\epsilon_u)_t + \alpha\epsilon_q)\,\bar{e}_p dx$$
$$+\frac{\beta}{2}\sum_{i=1}^{N}\left(\bar{e}_p^+ - \bar{e}_p^-\right)^2\big|_i + \frac{\alpha}{2}\sum_{i=1}^{N}\left(\bar{e}_q^+ - \bar{e}_q^-\right)^2\big|_i = 0.$$

This is clearly implies

$$\frac{1}{2}\frac{d}{dt}\|\bar{e}_q\|^2 \le -\int_I \epsilon_p(\bar{e}_u)_t dx - \int_I ((\epsilon_q)_t + \alpha\epsilon_p)\,\bar{e}_q dx + \int_I ((\epsilon_u)_t + \alpha\epsilon_q)\,\bar{e}_p dx,$$

which, after integrating over the interval $[0,t]$, gives

$$\begin{aligned}\|\bar{e}_q\|^2 \quad &\le \quad \|\bar{e}_q(.,0)\|^2 - 2\int_0^t\left(\int_I \epsilon_p(\bar{e}_u)_t dx\right)dt - 2\int_0^t\left(\int_I ((\epsilon_q)_t + \alpha\epsilon_p)\,\bar{e}_q dx\right)dt \\ &\quad +2\int_0^t\left(\int_I ((\epsilon_u)_t + \alpha\epsilon_q)\,\bar{e}_p dx\right)dt.\end{aligned}$$

Using the estimate (2.8) and a simple integration by parts with respect to time, we get

$$\begin{aligned}\|\bar{e}_q\|^2 \quad &\le \quad C_0h^{2k+3} - 2\int_I \epsilon_p(x,0)\bar{e}_u(x,0)dx + 2\int_0^t\left(\int_I (\epsilon_p)_t\bar{e}_u dx\right)dt \\ &\quad -2\int_0^t\left(\int_I ((\epsilon_q)_t + \alpha\epsilon_p)\,\bar{e}_q dx\right)dt + 2\int_0^t\left(\int_I ((\epsilon_u)_t + \alpha\epsilon_q)\,\bar{e}_p dx\right)dt.\end{aligned} \tag{3.8}$$

Substituting the definitions of $\bar{e}_u, \bar{e}_q$ and $\bar{e}_p$ given in (2.17) into the the right-hand side of (3.8) and using the fact that $(\epsilon_u)_t$, ϵ_q, $(\epsilon_q)_t$, and ϵ_p are orthogonal to any piecewise constant functions, which are due to the property (2.13), we get

$$\begin{aligned}\|\bar{e}_q\|^2 \quad &\le \quad C_0h^{2k+3} - 2\sum_{i=1}^{N}\int_{I_i}\epsilon_p(x,0)\frac{x-x_i}{h_i}c_i(x,0)dx + 2\int_0^t\left(\sum_{i=1}^{N}\int_{I_i}(\epsilon_p)_t\frac{x-x_i}{h_i}c_i dx\right)dt \\ &\quad -2\int_0^t\left(\sum_{i=1}^{N}\int_{I_i}((\epsilon_q)_t + \alpha\epsilon_p)\,\frac{x-x_{i-1}}{h_i}d_i dx\right)dt \\ &\quad +2\int_0^t\left(\sum_{i=1}^{N}\int_{I_i}((\epsilon_u)_t + \alpha\epsilon_q)\,\frac{x-x_{i-1}}{h_i}g_i dx\right)dt,\end{aligned}$$

which, after applying Cauchy-Schwarz inequality, leads to

$$\begin{aligned}\|\bar{e}_q\|^2 \quad &\le \quad C_0h^{2k+3} + 2\max_{x\in I}|\phi_4(x)|\left(\|\epsilon_p(.,0)\|\,\|c(.,0)\| + \int_0^t \|(\epsilon_p)_t\|\,\|c\|\,dt\right) + \\ &\quad 2\max_{x\in I}|\phi_3(x)|\left[\int_0^t (\|(\epsilon_q)_t\| + \alpha\,\|\epsilon_p\|)\,\|d\|\,dt + \int_0^t (\|(\epsilon_u)_t\| + \alpha\,\|\epsilon_q\|)\,\|g\|\,dt\right].\end{aligned}$$

Using (2.18), the projection results (2.19), and the estimates in (3.1), we conclude that $\forall\ t \in [0,T]$,

$$\begin{aligned}\|\bar{e}_q\|^2 \quad &\le \quad C_0h^{2k+3} + C_1h^{k+1}\|c(.,0)\| + C_2h^{k+1}\int_0^t (\|c\| + \|d\| + \|g\|)\,dt \\ &\le \quad C_3h^{2k+3} + C_2h^{k+1}\int_0^t \left(C_4h^{k+2} + C_5h^{k+2} + C_6h^{k+2}\right)dt \\ &\le \quad C_3h^{2k+3} + C_2h^{k+1}T\left(C_4h^{k+2} + C_5h^{k+2} + C_6h^{k+2}\right)dt \le Ch^{2k+3},\end{aligned}$$

which completes the proof of (3.3b).

Finally, we will prove the estimate (3.3c). Since the proof is long, we will first derive three important inequalities which will be needed in our proof.

Inequality 1: Taking $w = \beta \bar{e}_q \in P^k(I_i)$ in (2.16b) we get

$$\beta \int_{I_i} e_p \bar{e}_q dx - \beta \int_{I_i} (\bar{e}_q)_x \bar{e}_q dx + \beta (\bar{e}_q^-)^2\big|_i - \beta \bar{e}_q^+ \bar{e}_q^-\big|_i = 0.$$

Since $\int_{I_i} (\bar{e}_q)_x \bar{e}_q dx = \frac{1}{2}(\bar{e}_q^-)^2\big|_i - \frac{1}{2}(\bar{e}_q^+)^2\big|_{i-1}$, we have therefore the error equation

$$\beta \int_{I_i} e_p \bar{e}_q dx + \frac{\beta}{2}\left(\bar{e}_q^+ - \bar{e}_q^-\right)^2\big|_i + \frac{\beta}{2}(\bar{e}_q^+)^2\big|_{i-1} - \frac{\beta}{2}(\bar{e}_q^+)^2\big|_i = 0.$$

Summing the above equation over i and using the periodic boundary condition, we arrive at

$$\beta \int_I e_p \bar{e}_q dx + \frac{\beta}{2}\sum_{i=1}^{N}\left(\bar{e}_q^+ - \bar{e}_q^-\right)^2\big|_i = 0.$$

Using $e_p = \bar{e}_p + \epsilon_p$, applying Cauchy-Schwarz inequality, and using the estimate (3.3b), we obtain

$$\begin{aligned}\frac{\beta}{2}\sum_{i=1}^{N}\left(\bar{e}_q^+ - \bar{e}_q^-\right)^2\big|_i = -\beta\int_I \bar{e}_p\bar{e}_q dx - \beta\int_I \epsilon_p \bar{e}_q dx &\leq \beta\left\|\bar{e}_q\right\|\left\|\bar{e}_p\right\| - \beta\int_I \epsilon_p\bar{e}_q dx \\ &\leq C_1 h^{k+3/2}\left\|\bar{e}_p\right\| - \beta\int_I \epsilon_p \bar{e}_q dx. \qquad (3.9)\end{aligned}$$

Inequality 2: Adding (2.16a) and (2.16c) with $z = \alpha v$, where $v \in P^k(I_i)$, we obtain

$$\int_{I_i}\left((e_u)_t + \alpha e_q\right) v dx + \beta\int_{I_i}(\bar{e}_p)_x v dx - \beta \bar{e}_p^- v^-\big|_i + \beta \bar{e}_p^+ v^-\big|_i = 0.$$

Taking the test function as $v = \bar{e}_p$ and using the fact that $\int_{I_i}\bar{e}_p(\bar{e}_p)_x dx = \frac{1}{2}(\bar{e}_p^-)^2\big|_i - \frac{1}{2}(\bar{e}_p^+)^2\big|_{i-1}$, we get

$$\int_{I_i}\left((e_u)_t + \alpha e_q\right)\bar{e}_p dx - \frac{\beta}{2}\left(\bar{e}_p^+ - \bar{e}_p^-\right)^2\big|_i + \frac{\beta}{2}(\bar{e}_p^+)^2\big|_i - \frac{\beta}{2}(\bar{e}_p^+)^2\big|_{i-1} = 0,$$

which, after summing over all elements and applying the periodic boundary conditions, yields

$$\frac{\beta}{2}\sum_{i=1}^{N}\left(\bar{e}_p^+ - \bar{e}_p^-\right)^2\big|_i = \int_I\left((e_u)_t + \alpha e_q\right)\bar{e}_p dx.$$

Using (2.12), applying Cauchy-Schwarz inequality, and using the estimates (3.3a) and (3.3b), we obtain

$$\begin{aligned}\frac{\beta}{2}\sum_{i=1}^{N}\left(\bar{e}_p^+ - \bar{e}_p^-\right)^2\big|_i &\leq \Big(\left\|(\bar{e}_u)_t\right\| + \alpha\left\|\bar{e}_q\right\|\Big)\left\|\bar{e}_p\right\| + \int_I\left((\epsilon_u)_t + \alpha\epsilon_q\right)\bar{e}_p dx \\ &\leq C_2 h^{k+3/2}\left\|\bar{e}_p\right\| + \int_I\left((\epsilon_u)_t + \alpha\epsilon_q\right)\bar{e}_p dx. \qquad (3.10)\end{aligned}$$

Inequality 3: We take $v=-\bar{e}_q$ in (2.16a), $w=\beta\bar{e}_p$ in (2.16b), and $z=-\alpha\bar{e}_q$ in (2.16c) to obtain

$$\begin{aligned}
&-\int_{I_i}(e_u)_t\bar{e}_q dx-\alpha\int_{I_i}(\bar{e}_u)_x\bar{e}_q dx-\alpha\bar{e}_u^+\bar{e}_q^+\big|_{i-1}+\alpha\bar{e}_u^-\bar{e}_q^+\big|_{i-1}\\
&-\beta\int_{I_i}(\bar{e}_p)_x\bar{e}_q dx+\beta\bar{e}_p^-\bar{e}_q^-\big|_i-\beta\bar{e}_p^+\bar{e}_q^-\big|_i=0,\\
&\beta\int_{I_i}e_p\bar{e}_p dx-\beta\int_{I_i}(\bar{e}_q)_x\bar{e}_p dx+\beta\bar{e}_q^-\bar{e}_p^-\big|_i-\beta\bar{e}_q^+\bar{e}_p^-\big|_i=0,\\
&-\alpha\int_{I_i}e_q\bar{e}_q dx+\alpha\int_{I_i}(\bar{e}_u)_x\bar{e}_q dx+\alpha\bar{e}_u^+\bar{e}_q^+\big|_{i-1}-\alpha\bar{e}_u^-\bar{e}_q^+\big|_{i-1}=0.
\end{aligned}$$

Adding these three equations and integrating $\int_{I_i}(\bar{e}_p)_x\bar{e}_q dx$ by parts, we get

$$\beta\int_{I_i}e_p\bar{e}_p dx-\int_{I_i}((e_u)_t+\alpha e_q)\bar{e}_q dx-\beta\bar{e}_p^+\bar{e}_q^-\big|_i+\beta\bar{e}_p^+\bar{e}_q^+\big|_{i-1}+\beta\bar{e}_q^-\bar{e}_p^-\big|_i-\beta\bar{e}_q^+\bar{e}_p^-\big|_i=0,$$

or equivalently,

$$\begin{aligned}
&\beta\int_{I_i}\bar{e}_p^2 dx-\alpha\int_{I_i}\bar{e}_q^2 dx-\int_{I_i}(\bar{e}_u)_t\bar{e}_q dx-\int_{I_i}((\epsilon_u)_t+\alpha\epsilon_q)\bar{e}_q dx+\beta\int_{I_i}\epsilon_p\bar{e}_p dx+\\
&\beta\left(\bar{e}_q^+-\bar{e}_q^-\right)\left(\bar{e}_p^+-\bar{e}_p^-\right)\big|_i-\beta\bar{e}_p^+\bar{e}_q^+\big|_i+\beta\bar{e}_p^+\bar{e}_q^+\big|_{i-1}=0.
\end{aligned}$$

Summing over all elements and using the periodic boundary conditions, we get

$$\begin{aligned}
\beta\left\|\bar{e}_p\right\|^2+\beta\sum_{i=1}^{N}\left(\bar{e}_q^+-\bar{e}_q^-\right)\left(\bar{e}_p^+-\bar{e}_p^-\right)\big|_i &= \alpha\left\|\bar{e}_q\right\|^2+\int_I(\bar{e}_u)_t\bar{e}_q dx\\
&\quad+\int_I((\epsilon_u)_t+\alpha\epsilon_q)\bar{e}_q dx-\beta\int_I\epsilon_p\bar{e}_p dx,
\end{aligned}$$

which, after using the inequality $-\frac{1}{2}a^2-\frac{1}{2}b^2\le ab$ with $a=\left(\bar{e}_q^+-\bar{e}_q^-\right)\big|_i$ and $b=\left(\bar{e}_p^+-\bar{e}_p^-\right)\big|_i$, yields

$$\begin{aligned}
\beta\left\|\bar{e}_p\right\|^2-\frac{\beta}{2}\sum_{i=1}^{N}\left((\bar{e}_q^+-\bar{e}_q^-)^2+(\bar{e}_p^+-\bar{e}_p^-)^2\right)\big|_i &\le \alpha\left\|\bar{e}_q\right\|^2+\int_I(\bar{e}_u)_t\bar{e}_q dx\\
&\quad+\int_I((\epsilon_u)_t+\alpha\epsilon_q)\bar{e}_q dx-\beta\int_I\epsilon_p\bar{e}_p dx.
\end{aligned}$$

Applying Cauchy-Schwarz inequality and using the estimates (3.3a) and (3.3b), we obtain

$$\begin{aligned}
&\beta\left\|\bar{e}_p\right\|^2-\frac{\beta}{2}\sum_{i=1}^{N}\left((\bar{e}_q^+-\bar{e}_q^-)^2+(\bar{e}_p^+-\bar{e}_p^-)^2\right)\big|_i\le\alpha\left\|\bar{e}_q\right\|^2+\left\|(\bar{e}_u)_t\right\|\left\|\bar{e}_q\right\|\\
&+\int_I((\epsilon_u)_t+\alpha\epsilon_q)\bar{e}_q dx-\beta\int_I\epsilon_p\bar{e}_p dx\\
&\le C_3h^{2k+3}+\int_I((\epsilon_u)_t+\alpha\epsilon_q)\bar{e}_q dx-\beta\int_I\epsilon_p\bar{e}_p dx. \qquad (3.11)
\end{aligned}$$

Now we combine (3.9), (3.10), and (3.11) to obtain

$$\beta\left\|\bar{e}_p\right\|^2\le C_3h^{2k+3}+C_4h^{k+3/2}\left\|\bar{e}_p\right\|+\int_I((\epsilon_u)_t+\alpha\epsilon_q-\beta\epsilon_p)\left(\bar{e}_q+\bar{e}_p\right)dx. \qquad (3.12)$$

Substituting (2.17) into the last term of the right-hand side of (3.12) and using the fact that $(\epsilon_u)_t$, ϵ_q, and ϵ_p are orthogonal to any piecewise constant functions, which are due to the properties in (2.13), we get

$$\beta \|\bar{e}_p\|^2 \leq C_3 h^{2k+3} + C_4 h^{k+3/2} \|\bar{e}_p\| + \sum_{i=1}^{N} \int_{I_i} ((\epsilon_u)_t + \alpha\epsilon_q - \beta\epsilon_p) \frac{x - x_{i-1}}{h_i} (d_i + g_i)\, dx.$$

Applying Cauchy-Schwarz inequality, using (2.18), (3.1), and the projection results (2.19), we obtain

$$\begin{aligned} \beta \|\bar{e}_p\|^2 &\leq C_3 h^{2k+3} + C_4 h^{k+3/2} \|\bar{e}_p\| \\ &\quad + \left(\max_{x \in I} |\phi_3(x)| \right) \Big(\|(\epsilon_u)_t\| + \alpha \|\epsilon_q\| + \beta \|\epsilon_p\| \Big) \Big(\|d\| + \|g\| \Big) \\ &\leq C_4 h^{k+3/2} \|\bar{e}_p\| + C_5 h^{2k+3}. \end{aligned}$$

Dividing by β and using the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ with $a = \frac{C_4}{\beta} h^{k+3/2}$ and $b = \|\bar{e}_p\|$ yields

$$\|\bar{e}_p\|^2 \leq \frac{C_4^2}{2\beta^2} h^{2k+3} + \frac{1}{2} \|\bar{e}_p\|^2 + \frac{C_5}{\beta} h^{2k+3}.$$

Thus,

$$\|\bar{e}_p\|^2 \leq \frac{C_4^2}{\beta^2} h^{2k+3} + \frac{2C_5}{\beta} h^{2k+3},$$

which completes the proof of (3.3c). □

Before we state the superconvergence results at Radau points, we recall some results from [11] which will be needed in our error analysis. We first define four interpolation operators $\pi^{\pm}$ and $\hat{\pi}^{\pm}$. The projection π^{+} is defined as follows: For any function u, $\pi^{+} u\big|_{I_i} \in P^k(I_i)$ and interpolates u at the roots of the $(k+1)$-degree right Radau polynomial shifted to I_i, $x_{i,j}^{+},\ j = 0, 1, \ldots, k$, defined in (2.10). Similarly, $\pi^{-} u\big|_{I_i} \in P^k(I_i)$ and interpolates u at $x_{i,j}^{-},\ j = 0, 1, \ldots, k$. Next, the interpolation operators $\hat{\pi}^{\pm}$ are such that $\hat{\pi}^{\pm} u\big|_{I_i} \in P^{k+1}(I_i)$ and are defined as follows: $\hat{\pi}^{+} u\big|_{I_i}$ interpolates u at $x_{i,j}^{+},\ j = 0, 1, \cdots, k$, and at an additional point $\bar{x}_1$ in I_i with $\bar{x}_1 \neq x_{i,j}^{+},\ j = 0, 1, \cdots, k$. Similarly, $\hat{\pi}^{-} u\big|_{I_i}$ interpolates u at $x_{i,j}^{-},\ j = 0, 1, \cdots, k$, and at an additional point $\bar{x}_2$ in I_i with $\bar{x}_2 \neq x_{i,j}^{-},\ j = 0, 1, \cdots, k$.
In the next lemma, we recall some properties of the operators $P_h^{\pm}$ and $\pi^{\pm}$ needed in our analysis [11]. In particular, we show that the interpolation errors can be divided into significant parts and less significant parts.

Lemma 3.2. *Let u, $q = u_x$, $p = u_{xx} \in H^{k+2}$, and $P_h^{\pm}$ and $\pi^{\pm}$ as defined above. If*

$$\psi_{k+1,i}^{-}(x) = \prod_{j=0}^{k} (x - x_{i,j}^{-}), \quad \psi_{k+1,i}^{+}(x) = \prod_{j=0}^{k} (x - x_{i,j}^{+}), \tag{3.13}$$

then the interpolation errors can be split as:

$$u - \pi^{+} u = \phi_1 + \gamma_1, \quad \phi_1(x,t) = \alpha_1(t) \psi_{k+1,i}^{+}(x), \quad \gamma_1 = u - \hat{\pi}^{+} u, \quad \text{on } I_i, \tag{3.14a}$$

$$q - \pi^{-} q = \phi_2 + \gamma_2, \quad \phi_2(x,t) = \alpha_2(t) \psi_{k+1,i}^{-}(x), \quad \gamma_2 = q - \hat{\pi}^{-} q, \quad \text{on } I_i, \tag{3.14b}$$

$$p - \pi^{-} p = \phi_3 + \gamma_3, \quad \phi_3(x,t) = \alpha_3(t) \psi_{k+1,i}^{-}(x), \quad \gamma_3 = p - \hat{\pi}^{-} p, \quad \text{on } I_i, \tag{3.14c}$$

where $\alpha_1(t)$, $\alpha_2(t)$, *and* $\alpha_3(t)$ *are the coefficients of* x^{k+1} *in the* $(k+1)$*-degree polynomials* $\hat{\pi}^+u$, $\hat{\pi}^-q$, *and* $\hat{\pi}^-p$, *respectively, and*

(3.14d)
$$\|\phi_1\|_{s,I_i} \le Ch_i^{k+1-s}\|u\|_{k+1,I_i},\ 0\le s\le k,\quad \|\gamma_1\|_{s,I_i} \le Ch_i^{k+2-s}\|u\|_{k+2,I_i},\ 0\le s\le k+1.$$

(3.14e)
$$\|\phi_2\|_{s,I_i} \le Ch_i^{k+1-s}\|q\|_{k+1,I_i},\ 0\le s\le k,\quad \|\gamma_2\|_{s,I_i} \le Ch_i^{k+2-s}\|q\|_{k+2,I_i},\ 0\le s\le k+1.$$

(3.14f)
$$\|\phi_3\|_{s,I_i} \le Ch_i^{k+1-s}\|p\|_{k+1,I_i},\ 0\le s\le k,\quad \|\gamma_3\|_{s,I_i} \le Ch_i^{k+2-s}\|p\|_{k+2,I_i},\ 0\le s\le k+1.$$

Moreover,

$$\left\|\pi^+u - P_h^-u\right\|_i \le Ch_i^{k+2}\|u\|_{k+2,I_i}, \tag{3.15a}$$

$$\left\|\pi^-q - P_h^+q\right\|_i \le Ch_i^{k+2}\|q\|_{k+2,I_i}, \tag{3.15b}$$

$$\left\|\pi^-p - P_h^+p\right\|_i \le Ch_i^{k+2}\|p\|_{k+2,I_i}. \tag{3.15c}$$

Proof. The proofs of (3.14a), (3.14b), (3.14d), (3.14e), (3.15a), and (3.15b) can be found in [11]. The proofs of the other results are similar and are omitted. □

Now we are ready to prove our main superconvergence results. In particular, we show that the significant parts of the discretization error e_u is proportional to the $(k+1)$-degree right Radau polynomial and the significant parts of the discretization errors e_q and e_p are proportional to $(k+1)$-degree left Radau polynomials.

Theorem 3.2. *Under the assumptions of theorem 3.1, there exists a positive constant* C *such that* $\forall\ t\in[0,T]$,

$$\left\|u_h - \pi^+u\right\| \le Ch^{k+3/2},\ \left\|q_h - \pi^-q\right\| \le Ch^{k+3/2},\ \left\|p_h - \pi^-p\right\| \le Ch^{k+3/2}, \tag{3.16}$$

and on I_i,

$$e_u = \alpha_1\psi_{k+1,i}^+ + \omega_1,\ e_q = \alpha_2\psi_{k+1,i}^- + \omega_2,\ e_p = \alpha_3\psi_{k+1,i}^- + \omega_3, \tag{3.17a}$$

where

$$\omega_1 = \gamma_1 + \pi^+u - u_h,\quad \omega_2 = \gamma_2 + \pi^-q - q_h,\quad \omega_3 = \gamma_3 + \pi^-p - p_h, \tag{3.17b}$$

and

$$\sum_{i=1}^{N}\|\partial_x^s\omega_j\|_i^2 \le C\ h^{2(k-s)+3},\quad j=1,2,3,\quad s=0,1. \tag{3.18}$$

Finally,

$$\|e_u\|_{1,I}^2 \le Ch^{2k},\quad \|e_q\|_{1,I}^2 \le Ch^{2k},\quad \|e_p\|_{1,I}^2 \le Ch^{2k}. \tag{3.19}$$

Proof. Adding and subtracting P_h^-u, P_h^+q, and P_h^+p to $u_h-\pi^+u$, $q_h-\pi^-q$, and $p_h-\pi^-p$, respectively, we write

$$u_h - \pi^+u = -\bar{e}_u + P_h^-u - \pi^+u,\quad q_h - \pi^-q = -\bar{e}_q + P_h^+q - \pi^-q,$$
$$p_h - \pi^-p = -\bar{e}_p + P_h^+p - \pi^-p.$$

Taking the L^2-norm and applying the triangle inequality, we get

$$\left\|u_h - \pi^+u\right\| \le \|\bar{e}_u\| + \left\|P_h^-u - \pi^+u\right\|,\quad \left\|q_h - \pi^-q\right\| \le \|\bar{e}_q\| + \left\|P_h^+q - \pi^-q\right\|,$$
$$\left\|p_h - \pi^-p\right\| \le \|\bar{e}_p\| + \left\|P_h^+p - \pi^-p\right\|.$$

Using the estimates (2.21e), (3.3b), (3.3c), and (3.15a) we establish (3.16).

Adding and subtracting π^+u, π^-q, and π^-p to e_u, e_q, and e_p, respectively, we get

$$e_u = u - \pi^+u + \pi^+u - u_h, \quad e_q = q - \pi^-q + \pi^-q - q_h, \quad e_p = p - \pi^-p + \pi^-p - p_h.$$

Furthermore, one can split the interpolation errors $u - \pi^+u$, $q - \pi^-q$, and $p - \pi^-p$ on I_i as in (3.14a)-(3.14c) to obtain

(3.20a) $\quad e_u = \phi_1 + \gamma_1 + \pi^+u - u_h = \phi_1 + \omega_1, \quad$ where $\omega_1 = \gamma_1 + \pi^+u - u_h$,

(3.20b) $\quad e_q = \phi_2 + \gamma_2 + \pi^-q - q_h = \phi_2 + \omega_2, \quad$ where $\omega_2 = \gamma_2 + \pi^-q - q_h$,

(3.20c) $\quad e_p = \phi_3 + \gamma_3 + \pi^-p - p_h = \phi_3 + \omega_3, \quad$ where $\omega_3 = \gamma_3 + \pi^-p - p_h$.

Next, we use Cauchy-Schwarz inequality and the inequality $|ab| \le \frac{1}{2}(a^2 + b^2)$ to write

$$\begin{aligned}
\|\omega_1\|_i^2 &= \left(\gamma_1 + \pi^+u - u_h, \gamma_1 + \pi^+u - u_h\right)_i = \|\gamma_1\|_i^2 + 2\left(\gamma_1, \pi^+u - u_h\right)_i + \left\|\pi^+u - u_h\right\|_i^2 \\
&\le 2\left(\|\gamma_1\|_i^2 + \left\|\pi^+u - u_h\right\|_i^2\right), \\
\|\omega_2\|_i^2 &= \left(\gamma_2 + \pi^-q - q_h, \gamma_2 + \pi^-q - q_h\right)_i = \|\gamma_2\|_i^2 + 2\left(\gamma_2, \pi^-q - q_h\right)_i + \left\|\pi^-q - q_h\right\|_i^2 \\
&\le 2\left(\|\gamma_2\|_i^2 + \left\|\pi^-q - q_h\right\|_i^2\right) \\
\|\omega_3\|_i^2 &= \left(\gamma_3 + \pi^-p - p_h, \gamma_3 + \pi^-p - p_h\right)_i = \|\gamma_3\|_i^2 + 2\left(\gamma_3, \pi^-p - p_h\right)_i + \left\|\pi^-p - p_h\right\|_i^2 \\
&\le 2\left(\|\gamma_3\|_i^2 + \left\|\pi^-p - p_h\right\|_i^2\right).
\end{aligned}$$

Summing over all elements and applying (3.14d)-(3.14d) and (3.16) yields

$$\sum_{i=1}^{N} \|\omega_j\|_i^2 \le C_1 h^{2k+4} + C_2 h^{2k+3} \le C h^{2k+3}, \quad j = 1, 2, 3,$$

which completes the proof of (3.18) for $s = 0$.
In order to prove the estimates (3.18) for $s = 1$, we use Cauchy-Schwarz inequality and the inequality $|ab| \le \frac{1}{2}(a^2 + b^2)$ to get

$$\begin{aligned}
\|(\omega_1)_x\|_i^2 &= \left(\left(\gamma_1 + \pi^+u - u_h\right)_x, \left(\gamma_1 + \pi^+u - u_h\right)_x\right)_i \le 2\left[\|(\gamma_1)_x\|_i^2 + \left\|(\pi^+u - u_h)_x\right\|_i^2\right], \\
\|(\omega_2)_x\|_i^2 &= \left(\left(\gamma_2 + \pi^-q - q_h\right)_x, \left(\gamma_2 + \pi^-q - q_h\right)_x\right)_i \le 2\left[\|(\gamma_2)_x\|_i^2 + \left\|(\pi^-q - q_h)_x\right\|_i^2\right], \\
\|(\omega_3)_x\|_i^2 &= \left(\left(\gamma_3 + \pi^-p - p_h\right)_x, \left(\gamma_3 + \pi^-p - p_h\right)_x\right)_i \le 2\left[\|(\gamma_3)_x\|_i^2 + \left\|(\pi^-p - p_h)_x\right\|_i^2\right].
\end{aligned}$$

Using the inverse inequalities $\|(\pi^+u - u_h)_x\|_i \le Ch^{-1}\|\pi^+u - u_h\|_i$, $\|(\pi^-q - q_h)_x\|_i \le Ch^{-1}\|\pi^-q - q_h\|_i$, and $\|(\pi^-p - p_h)_x\|_i \le Ch^{-1}\|\pi^-p - p_h\|_i$, we obtain the estimates

$$\begin{aligned}
\|(\omega_1)_x\|_i^2 &\le C\left[\|(\gamma_1)_x\|_i^2 + h^{-2}\left\|\pi^+u - u_h\right\|_i^2\right], \\
\|(\omega_2)_x\|_i^2 &\le C\left[\|(\gamma_2)_x\|_i^2 + h^{-2}\left\|\pi^-q - q_h\right\|_i^2\right], \\
\|(\omega_3)_x\|_i^2 &\le C\left[\|(\gamma_3)_x\|_i^2 + h^{-2}\left\|\pi^-p - p_h\right\|_i^2\right].
\end{aligned}$$

Summing over all elements and applying the standard error estimates (3.14d)-(3.14f) and the estimate (3.16) we establish (3.18) for $s = 1$.

In order to show (3.19), we note that

$$\|e_u\|_{1,I}^2 = \|e_u\|^2 + \sum_{i=1}^{N} \|(e_u)_x\|_i^2\,, \quad \|e_q\|_{1,I}^2 = \|e_q\|^2 + \sum_{i=1}^{N} \|(e_q)_x\|_i^2\,,$$
$$\|e_p\|_{1,I}^2 = \|e_p\|^2 + \sum_{i=1}^{N} \|(e_p)_x\|_i^2\,. \tag{3.21}$$

Differentiating (3.20) with respect to x, taking the L^2-norm, and using Cauchy-Schwarz inequality and the inequality $|ab| \le \frac{1}{2}(a^2+b^2)$ leads to

$$\|(e_u)_x\|_i^2 = ((\phi_1)_x + (\omega_1)_x, (\phi_1)_x + (\omega_1)_x)_i \le 2\left[\|(\phi_1)_x\|_i^2 + \|(\omega_1)_x\|_i^2\right],$$
$$\|(e_q)_x\|_i^2 = ((\phi_2)_x + (\omega_2)_x, (\phi_2)_x + (\omega_2)_x)_i \le 2\left[\|(\phi_2)_x\|_i^2 + \|(\omega_2)_x\|_i^2\right],$$
$$\|(e_p)_x\|_i^2 = ((\phi_3)_x + (\omega_3)_x, (\phi_3)_x + (\omega_3)_x)_i \le 2\left[\|(\phi_3)_x\|_i^2 + \|(\omega_3)_x\|_i^2\right].$$

Summing over all elements and applying (3.14d)-(3.14f), and (3.18) we obtain

$$\sum_{i=1}^{N} \|(e_u)_x\|_i^2 \le Ch^{2k}, \quad \sum_{i=1}^{N} \|(e_q)_x\|_i^2 \le Ch^{2k}, \quad \sum_{i=1}^{N} \|(e_p)_x\|_i^2 \le Ch^{2k}. \tag{3.22}$$

Finally, substituting (2.21a), (2.21c), (2.21d), and (3.22) into (3.21) establishes (3.19). □

4. *A posteriori* error estimation

In this section, we present a technique to compute asymptotically correct *a posteriori* estimates of the LDG errors. These estimates are computed by solving a local steady problem with no boundary conditions on each element. We further prove that the LDG discretization error estimates converge to the true spatial errors in the L^2-norm as $h \to 0$.

We first present the weak finite element formulations to compute *a posteriori* error estimates for the linearized KdV equation (1.1). Multiplying the three equations in (2.1) by test functions v, w, and z, respectively, integrating over an arbitrary element I_i, and replacing u by $u_h + e_u$, q by $q_h + e_q$, and p by $p_h + e_p$ we get

$$\int_{I_i} (\alpha(e_u)_x + \beta(e_p)_x)\, v dx = \int_{I_i} (R_{h,1} - (e_u)_t)\, v dx, \quad x \in [a,b],\ t \in [0,T], \tag{4.1a}$$
$$-\int_{I_i} (e_q)_x w dx = \int_{I_i} (R_{h,2} - e_p)\, w dx, \quad x \in [a,b],\ t \in [0,T], \tag{4.1b}$$
$$-\int_{I_i} (e_u)_x z dx = \int_{I_i} (R_{h,3} - e_q)\, z dx, \quad x \in [a,b],\ t \in [0,T], \tag{4.1c}$$

where

$$R_{h,1} = -(u_h)_t - \alpha(u_h)_x - \beta(p_h)_x, \quad R_{h,2} = (q_h)_x - p_h, \quad R_{h,3} = (u_h)_x - q_h. \tag{4.1d}$$

Substituting (3.17) into the left-hand sides of (4.1a)-(4.1c) and choosing $v = z = \psi^+_{k+1,i}(x)$, $w = \psi^-_{k+1,i}(x)$, we obtain

$$\int_{I_i} \left(\alpha\alpha_1 {\psi^+_{k+1,i}}' + \beta\alpha_3 {\psi^-_{k+1,i}}'\right) \psi^+_{k+1,i} dx = \int_{I_i} \left(R_{h,1} - (e_u)_t - \alpha(\omega_1)_x - \beta(\omega_3)_x\right) \psi^+_{k+1,i} dx,$$

$$-\alpha_2 \int_{I_i} {\psi^-_{k+1,i}}' \psi^-_{k+1,i} dx = \int_{I_i} \left(R_{h,2} - e_p + (\omega_2)_x\right) \psi^-_{k+1,i} dx,$$

$$-\alpha_1 \int_{I_i} {\psi^+_{k+1,i}}' \psi^+_{k+1,i} dx = \int_{I_i} \left(R_{h,3} - e_q + (\omega_1)_x\right) \psi^+_{k+1,i} dx. \tag{4.2}$$

Using the properties in (2.11) and solving for $\alpha_i,\ i = 1-3$, we obtain

$$\alpha_1(t) = \frac{1}{2c_k^2 h_i^{2k+2}} \int_{I_i} \left(R_{h,3} - e_q + (\omega_1)_x\right) \psi^+_{k+1,i} dx, \tag{4.3a}$$

$$\alpha_2(t) = -\frac{1}{2c_k^2 h_i^{2k+2}} \int_{I_i} \left(R_{h,2} - e_p + (\omega_2)_x\right) \psi^-_{k+1,i} dx, \tag{4.3b}$$

$$\alpha_3(t) = -\frac{1}{2\beta c_k^2 h_i^{2k+2}} \int_{I_i} \left(R_{h,1} + \alpha R_{h,3} - (e_u)_t - \alpha e_q - \beta(\omega_3)_x\right) \psi^+_{k+1,i} dx. \tag{4.3c}$$

Our error estimate procedure consists of approximating the true errors on each element I_i by the leading terms as $e_u \approx E_u$, $e_q \approx E_q$, and $e_p \approx E_p$, where

$$E_u = a_1(t)\psi^+_{k+1,i}(x), \quad E_q = a_2(t)\psi^-_{k+1,i}(x), \quad E_p = a_3(t)\psi^-_{k+1,i}(x), \quad x \in I_i, \tag{4.4}$$

where the coefficients of the leading terms of the errors, $a_i,\ i = 1-3$, are obtained from the coefficients $\alpha_i,\ i = 1-3$ defined in (4.3) by neglecting the terms ω_i, e_q, e_p, and the time change $(e_u)_t$, *i.e.*,

$$a_1(t) = \frac{1}{2c_k^2 h_i^{2k+2}} \int_{I_i} R_{h,3}\psi^+_{k+1,i} dx, \quad a_2(t) = -\frac{1}{2c_k^2 h_i^{2k+2}} \int_{I_i} R_{h,2}\psi^-_{k+1,i} dx,$$

$$a_3(t) = -\frac{1}{2\beta c_k^2 h_i^{2k+2}} \int_{I_i} \left(R_{h,1} + \alpha R_{h,3}\right) \psi^+_{k+1,i} dx. \tag{4.5}$$

An accepted efficiency measure of *a posteriori* error estimates is the effectivity index. In this paper, we use the global effectivity indices

$$\theta_u(t) = \frac{\|E_u\|}{\|e_u\|}, \quad \theta_q(t) = \frac{\|E_q\|}{\|e_q\|}, \quad \theta_p(t) = \frac{\|E_p\|}{\|e_p\|}.$$

Ideally, the global effectivity indices should stay close to one and should converge to one under mesh refinement.

Next, we will show that the error estimates E_u, E_q, and E_p converge to the exact errors e_u, e_q, and e_p, respectively, in the L^2-norm as $h \to 0$. Furthermore we will prove the convergence to unity of the global effectivity indices $\theta_u(t), \theta_q(t)$, and $\theta_p(t)$ under mesh refinement.

Before stating our main result we state and prove the following preliminary results.

Theorem 4.1. *Suppose that* $(u,\ q,\ p)$ *and* $(u_h,\ q_h,\ p_h)$*, respectively, are solutions of (2.1) and (2.3). If* $\alpha_j,\ j = 1-3$ *and* $a_j,\ j = 1-3$ *are given by (4.3) and (4.5), respectively, then there exists a positive constant* C *independent of* h *such that, at any fixed time* $t \in [0, T]$,

$$\sum_{i=1}^{N} (a_1 - \alpha_1)^2 \left\|\psi^+_{k+1,i}\right\|_i^2 \le Ch^{2k+3}, \quad \sum_{i=1}^{N} (a_2 - \alpha_2)^2 \left\|\psi^-_{k+1,i}\right\|_i^2 \le Ch^{2k+3},$$

$$\sum_{i=1}^{N}(a_3-\alpha_3)^2\left\|\psi^+_{k+1,i}\right\|_i^2 \le Ch^{2k+3}. \tag{4.6}$$

Proof. Subtracting (4.3) from (4.5) we obtain

$$a_1-\alpha_1 = \frac{1}{2c_k^2h_i^{2k+2}}\int_{I_i}(e_q-(\omega_1)_x)\,\psi^+_{k+1,i}dx, \tag{4.7a}$$

$$a_2-\alpha_2 = -\frac{1}{2c_k^2h_i^{2k+2}}\int_{I_i}(e_p-(\omega_2)_x)\,\psi^-_{k+1,i}dx, \tag{4.7b}$$

$$a_3-\alpha_3 = -\frac{1}{2\beta c_k^2h_i^{2k+2}}\int_{I_i}((e_u)_t+\alpha e_q+\beta(\omega_3)_x)\,\psi^+_{k+1,i}dx. \tag{4.7c}$$

Using the inequalities $(a+b)^2 \le 2(a^2+b^2)$, $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, and applying Cauchy-Schwarz inequality yields

$$(a_1-\alpha_1)^2 \le \frac{1}{2c_k^4h_i^{4k+4}}\left\|\psi^+_{k+1,i}\right\|_i^2\left(\|e_q\|_i^2+\|(\omega_1)_x\|_i^2\right), \tag{4.8a}$$

$$(a_2-\alpha_2)^2 \le \frac{1}{2c_k^4h_i^{4k+4}}\left\|\psi^-_{k+1,i}\right\|_i^2\left(\|e_p\|_i^2+\|(\omega_2)_x\|_i^2\right), \tag{4.8b}$$

$$(a_3-\alpha_3)^2 \le \frac{3}{4\beta^2c_k^4h_i^{4k+4}}\left\|\psi^+_{k+1,i}\right\|_i^2\left(\|(e_u)_t\|_i^2+\alpha^2\|e_q\|_i^2+\beta^2\|(\omega_3)_x\|_i^2\right). \tag{4.8c}$$

Multiplying (4.8a) by $\left\|\psi^+_{k+1,i}\right\|_i^2$ and (4.8b) and (4.8c) by $\left\|\psi^-_{k+1,i}\right\|_i^2$ and using (2.11b) and (2.11c), we get

$$(a_1-\alpha_1)^2\left\|\psi^+_{k+1,i}\right\|_i^2 \le \tilde{c}_kh_i^2\left(\|e_q\|_i^2+\|(\omega_1)_x\|_i^2\right), \tag{4.9a}$$

$$(a_2-\alpha_2)^2\left\|\psi^-_{k+1,i}\right\|_i^2 \le \tilde{c}_kh_i^2\left(\|e_p\|_i^2+\|(\omega_2)_x\|_i^2\right), \tag{4.9b}$$

$$(a_3-\alpha_3)^2\left\|\psi^-_{k+1,i}\right\|_i^2 \le \frac{3\tilde{c}_k}{2\beta^2}h_i^2\left(\|(e_u)_t\|_i^2+\alpha^2\|e_q\|_i^2+\beta^2\|(\omega_3)_x\|_i^2\right), \tag{4.9c}$$

where $\tilde{c}_k$ is a constant given by

$$\tilde{c}_k = \frac{d_k^2}{2c_k^4} = \frac{2(2k+2)^2}{(2k+1)^2(2k+3)^2}. \tag{4.9d}$$

Finally, summing over all elements and using the fact that $h = \max_{1\le i\le N} h_i$, we write

$$\sum_{i=1}^{N}(a_1-\alpha_1)^2\left\|\psi^+_{k+1,i}\right\|_i^2 \le \tilde{c}_kh^2\left(\|e_q\|^2+\sum_{i=1}^{N}\|(\omega_1)_x\|_i^2\right), \tag{4.10a}$$

$$\sum_{i=1}^{N}(a_2-\alpha_2)^2\left\|\psi^-_{k+1,i}\right\|_i^2 \le \tilde{c}_kh^2\left(\|e_p\|^2+\sum_{i=1}^{N}\|(\omega_2)_x\|_i^2\right), \tag{4.10b}$$

$$\sum_{i=1}^{N}(a_3-\alpha_3)^2\left\|\psi^-_{k+1,i}\right\|_i^2 \le \frac{3\tilde{c}_k}{2\beta^2}h^2\left(\|(e_u)_t\|^2+\alpha^2\|e_q\|^2+\beta^2\sum_{i=1}^{N}\|(\omega_3)_x\|_i^2\right). \tag{4.10c}$$

Combining this estimate with (2.21b), (2.21c), (2.21d), and (3.18) we establish (4.6). $\square$

Theorem 4.2. *Under the assumptions of theorem 4.1, there exists a positive constant C independent of h such that, at any fixed t, we have*

$$\sum_{i=1}^{N}(a_1+\alpha_1)^2\left\|\psi_{k+1,i}^{+}\right\|_i^2 \le Ch^{2k+2}, \quad \sum_{i=1}^{N}(a_2+\alpha_2)^2\left\|\psi_{k+1,i}^{-}\right\|_i^2 \le Ch^{2k+2},$$

$$\sum_{i=1}^{N}(a_3+\alpha_3)^2\left\|\psi_{k+1,i}^{-}\right\|_i^2 \le Ch^{2k+2}. \tag{4.11}$$

Proof. Using (2.1), we rewrite (4.1d) as

$$R_{h,1}=(e_u)_t+\alpha(e_u)_x+\beta(e_p)_x, \quad R_{h,2}=e_p-(e_q)_x, \quad R_{h,3}=e_q-(e_u)_x. \tag{4.12}$$

Combining (4.12) and (4.5) we obtain

$$a_1^2 = \frac{1}{4c_k^4h_i^{4k+4}}\left[\int_{I_i}\left(e_q-(e_u)_x\right)\psi_{k+1,i}^{+}dx\right]^2, \tag{4.13a}$$

$$a_2^2 = \frac{1}{4c_k^4h_i^{4k+4}}\left[\int_{I_i}\left(e_p-(e_q)_x\right)\psi_{k+1,i}^{-}dx\right]^2, \tag{4.13b}$$

$$a_3^2 = \frac{1}{4\beta^2c_k^4h_i^{4k+4}}\left[\int_{I_i}\left((e_u)_t+\alpha e_q+\beta(e_p)_x\right)\psi_{k+1,i}^{+}dx\right]^2. \tag{4.13c}$$

Applying the inequalities $(a+b)^2\le 2(a^2+b^2)$, $(a+b+c)^2\le 3(a^2+b^2+c^2)$, and Cauchy-Schwarz inequality yields

$$a_1^2 \le \frac{1}{2c_k^4h_i^{4k+4}}\left\|\psi_{k+1,i}^{+}\right\|_i^2\left(\|e_q\|_i^2+\|(e_u)_x\|_i^2\right), \tag{4.14a}$$

$$a_2^2 \le \frac{1}{2c_k^4h_i^{4k+4}}\left\|\psi_{k+1,i}^{-}\right\|_i^2\left(\|e_p\|_i^2+\|(e_q)_x\|_i^2\right), \tag{4.14b}$$

$$a_3^2 \le \frac{3}{4\beta^2c_k^4h_i^{4k+4}}\left\|\psi_{k+1,i}^{+}\right\|_i^2\left(\|(e_u)_t\|_i^2+\alpha^2\|e_q\|_i^2+\beta^2\|(e_p)_x\|_i^2\right). \tag{4.14c}$$

Multiplying (4.14a) by $\left\|\psi_{k+1,i}^{+}\right\|_i^2$ and (4.14b) and (4.14c) by $\left\|\psi_{k+1,i}^{-}\right\|_i^2$, using (2.11b), $h=\max_{1\le i\le N}h_i$, and summing over all elements, we obtain

$$\begin{aligned}
\sum_{i=1}^{N}a_1^2\left\|\psi_{k+1,i}^{+}\right\|_i^2 &\le \tilde{c}_kh^2\left(\|e_q\|^2+\sum_{i=1}^{N}\|(e_u)_x\|_i^2\right)\le \tilde{c}_kh^2\left(\|e_q\|^2+\|e_u\|_{1,I}^2\right),\\
\sum_{i=1}^{N}a_2^2\left\|\psi_{k+1,i}^{-}\right\|_i^2 &\le \tilde{c}_kh^2\left(\|e_p\|^2+\sum_{i=1}^{N}\|(e_q)_x\|_i^2\right)\le \tilde{c}_kh^2\left(\|e_p\|^2+\|e_q\|_{1,I}^2\right),\\
\sum_{i=1}^{N}a_3^2\left\|\psi_{k+1,i}^{-}\right\|_i^2 &\le \frac{3\tilde{c}_k}{2\beta^2}h^2\left(\|(e_u)_t\|^2+\alpha^2\|e_q\|^2+\beta^2\|e_p\|_{1,I}^2\right),
\end{aligned}$$

where $\tilde{c}_k$ is the same constant defined in (4.9d).
Using the estimates (2.21b), (2.21c), (2.21d), and (3.19), we obtain

$$\sum_{i=1}^{N}a_1^2\left\|\psi_{k+1,i}^{+}\right\|_i^2 \le Ch^{2k+2}, \ \sum_{i=1}^{N}a_2^2\left\|\psi_{k+1,i}^{-}\right\|_i^2 \le Ch^{2k+2}, \ \sum_{i=1}^{N}a_3^2\left\|\psi_{k+1,i}^{-}\right\|_i^2 \le Ch^{2k+2}. \tag{4.15}$$

Taking the L^2 inner product of $\psi^{\pm}_{k+1,i}$ and $\phi_j,\ j=1-3$, defined in (3.14a)-(3.14c), and applying Cauchy-Schwarz inequality, we get

$$|\alpha_1|\left\|\psi^{+}_{k+1,i}\right\|_i^2=\left|(\phi_1,\psi^{+}_{k+1,i})_i\right|\leq\left\|\psi^{+}_{k+1,i}\right\|_i\|\phi_1\|_i\,,$$
$$|\alpha_2|\left\|\psi^{-}_{k+1,i}\right\|_i^2=\left|(\phi_2,\psi^{-}_{k+1,i})_i\right|\leq\left\|\psi^{-}_{k+1,i}\right\|_i\|\phi_2\|_i\,,$$
$$|\alpha_3|\left\|\psi^{-}_{k+1,i}\right\|_i^2=\left|(\phi_3,\psi^{-}_{k+1,i})_i\right|\leq\left\|\psi^{-}_{k+1,i}\right\|_i\|\phi_3\|_i\,.$$

Hence, we have

$$\alpha_1^2\left\|\psi^{+}_{k+1,i}\right\|_i^2\leq\|\phi_1\|_i^2\,,\quad \alpha_2^2\left\|\psi^{-}_{k+1,i}\right\|_i^2\leq\|\phi_2\|_i^2\,,\quad \alpha_3^2\left\|\psi^{-}_{k+1,i}\right\|_i^2\leq\|\phi_3\|_i^2\,.$$

Summing over all elements and applying (3.14a)-(3.14c) we get

$$\sum_{i=1}^{N}\alpha_1^2\left\|\psi^{+}_{k+1,i}\right\|_i^2\leq\sum_{i=1}^{N}\|\phi_1\|_i^2\leq Ch^{2k+2},\quad \sum_{i=1}^{N}\alpha_2^2\left\|\psi^{-}_{k+1,i}\right\|_i^2\leq\sum_{i=1}^{N}\|\phi_2\|_i^2\leq Ch^{2k+2},$$

$$\sum_{i=1}^{N}\alpha_3^2\left\|\psi^{-}_{k+1,i}\right\|_i^2\leq\sum_{i=1}^{N}\|\phi_3\|_i^2\leq Ch^{2k+2}. \tag{4.16}$$

Adding (4.15) and (4.16) and using the inequality $(a+b)^2\leq 2(a^2+b^2)$ yields (4.11). □

The main results of this section are stated in the following theorem. In particular we state and prove asymptotic results of our *a posteriori* error estimates.

Theorem 4.3. *Let $k\geq 1$ and (u,q,p) and (u_h,q_h,p_h), respectively, are solutions of (2.1) and (2.3) subject to the approximated initial condition $u_h(x,0)=P_h^1u(x,0)$. If E_u, E_q, and E_p are given in (4.4), where $a_j,\ j=1-3$ are defined in (4.5), then there exists a positive constant C independent of h such that*

$$\|e_u-E_u\|^2\leq C\ h^{2k+3},\quad \|e_q-E_q\|^2\leq C\ h^{2k+3},\quad \|e_p-E_p\|^2\leq C\ h^{2k+3}. \tag{4.17}$$

Furthermore, there exist positive constants C_1-C_3 independent of h such that

$$\|e_u\|^2=\sum_{i=1}^{N}a_1^2\left\|\psi^{+}_{k+1,i}\right\|_i^2+\hat{\epsilon}_1,\quad \textit{where } |\hat{\epsilon}_1|\leq C_1h^{2k+5/2}, \tag{4.18a}$$
$$\|e_q\|^2=\sum_{i=1}^{N}a_2^2\left\|\psi^{-}_{k+1,i}\right\|_i^2+\hat{\epsilon}_2,\quad \textit{where } |\hat{\epsilon}_2|\leq C_2h^{2k+5/2}, \tag{4.18b}$$
$$\|e_p\|^2=\sum_{i=1}^{N}a_3^2\left\|\psi^{-}_{k+1,i}\right\|_i^2+\hat{\epsilon}_3,\quad \textit{where } |\hat{\epsilon}_3|\leq C_3h^{2k+5/2}. \tag{4.18c}$$

and, as $h\to 0$ with t kept fixed,

$$\frac{\|E_u\|^2}{\|e_u\|^2}=1+\mathcal{O}(h^{1/2}),\quad \frac{\|E_q\|^2}{\|e_q\|^2}=1+\mathcal{O}(h^{1/2}),\quad \frac{\|E_p\|^2}{\|e_p\|^2}=1+\mathcal{O}(h^{1/2}). \tag{4.19}$$

Proof. First, we will prove (4.17). Subtracting (4.4) from (3.17a) and using $(a+b)^2 \le 2a^2 + 2b^2$, we obtain

$$\begin{aligned}
\|e_u - E_u\|_i^2 &= \left\|(\alpha_1 - a_1)\psi^+_{k+1,i} + \omega_1\right\|_i^2 \le 2(\alpha_1 - a_1)^2 \left\|\psi^+_{k+1,i}\right\|_i^2 + 2\|\omega_1\|_i^2, \\
\|e_q - E_q\|_i^2 &= \left\|(\alpha_2 - a_2)\psi^-_{k+1,i} + \omega_2\right\|_i^2 \le 2(\alpha_2 - a_2)^2 \left\|\psi^-_{k+1,i}\right\|_i^2 + 2\|\omega_2\|_i^2, \\
\|e_p - E_p\|_i^2 &= \left\|(\alpha_3 - a_3)\psi^-_{k+1,i} + \omega_3\right\|_i^2 \le 2(\alpha_3 - a_3)^2 \left\|\psi^-_{k+1,i}\right\|_i^2 + 2\|\omega_3\|_i^2,
\end{aligned}$$

Summing over all elements and applying the estimates (3.18) and (4.6) yields

$$\begin{aligned}
\|e_u - E_u\|^2 &= \sum_{i=1}^N \|e_u - E_u\|_i^2 \le 2\sum_{i=1}^N (\alpha_1 - a_1)^2 \left\|\psi^+_{k+1,i}\right\|_i^2 + 2\sum_{i=1}^N \|\omega_1\|_i^2 \le Ch^{2k+3}, \\
\|e_q - E_q\|^2 &= \sum_{i=1}^N \|e_q - E_q\|_i^2 \le 2\sum_{i=1}^N (\alpha_2 - a_2)^2 \left\|\psi^-_{k+1,i}\right\|_i^2 + 2\sum_{i=1}^N \|\omega_2\|_i^2 \le Ch^{2k+3}, \\
\|e_p - E_p\|^2 &= \sum_{i=1}^N \|e_p - E_p\|_i^2 \le 2\sum_{i=1}^N (\alpha_3 - a_3)^2 \left\|\psi^-_{k+1,i}\right\|_i^2 + 2\sum_{i=1}^N \|\omega_3\|_i^2 \le Ch^{2k+3}.
\end{aligned}$$

Next, we will prove (4.18). From (3.20) the LDG errors can be split as

$$e_u = \phi_1 + \omega_1, \quad e_q = \phi_2 + \omega_2, \quad e_p = \phi_3 + \omega_3, \quad x \in I_i.$$

Taking the L^2 norm of the LDG errors, we obtain

$$\|e_u\|_i^2 = \|\phi_1\|_i^2 + 2(\phi_1, \omega_1)_i + \|\omega_1\|_i^2 = \alpha_1^2 \left\|\psi^+_{k+1,i}\right\|_i^2 + \epsilon_1, \tag{4.20a}$$

$$\|e_q\|_i^2 = \|\phi_2\|_i^2 + 2(\phi_2, \omega_2)_i + \|\omega_2\|_i^2 = \alpha_2^2 \left\|\psi^-_{k+1,i}\right\|_i^2 + \epsilon_2, \tag{4.20b}$$

$$\|e_p\|_i^2 = \|\phi_3\|_i^2 + 2(\phi_3, \omega_3)_i + \|\omega_3\|_i^2 = \alpha_3^2 \left\|\psi^-_{k+1,i}\right\|_i^2 + \epsilon_3, \tag{4.20c}$$

where

$$\epsilon_j = 2(\phi_j, \omega_j)_i + \|\omega_j\|_i^2, \quad j = 1, 2, 3. \tag{4.20d}$$

Summing over all elements we obtain

$$\|e_u\|^2 = \sum_{i=1}^N \alpha_1^2 \left\|\psi^+_{k+1,i}\right\|_i^2 + \tilde\epsilon_1, \ \|e_q\|^2 = \sum_{i=1}^N \alpha_2^2 \left\|\psi^-_{k+1,i}\right\|_i^2 + \tilde\epsilon_2, \ \|e_p\|^2 = \sum_{i=1}^N \alpha_3^2 \left\|\psi^-_{k+1,i}\right\|_i^2 + \tilde\epsilon_3,$$

where

$$\tilde\epsilon_1 = \sum_{i=1}^N \epsilon_1, \quad \tilde\epsilon_2 = \sum_{i=1}^N \epsilon_2, \quad \tilde\epsilon_3 = \sum_{i=1}^N \epsilon_3.$$

Next, we write the LDG errors as

$$\|e_u\|^2 = \sum_{i=1}^N a_1^2 \left\|\psi^+_{k+1,i}\right\|_i^2 + \hat\epsilon_1, \quad \hat\epsilon_1 = \sum_{i=1}^N (\alpha_1^2 - a_1^2) \left\|\psi^+_{k+1,i}\right\|_i^2 + \tilde\epsilon_1, \tag{4.21a}$$

$$\|e_q\|^2 = \sum_{i=1}^N a_2^2 \left\|\psi^-_{k+1,i}\right\|_i^2 + \hat\epsilon_2, \quad \hat\epsilon_2 = \sum_{i=1}^N (\alpha_2^2 - a_2^2) \left\|\psi^-_{k+1,i}\right\|_i^2 + \tilde\epsilon_2, \tag{4.21b}$$

$$\|e_p\|^2 = \sum_{i=1}^N a_3^2 \left\|\psi^-_{k+1,i}\right\|_i^2 + \hat\epsilon_3, \quad \hat\epsilon_3 = \sum_{i=1}^N (\alpha_3^2 - a_3^2) \left\|\psi^-_{k+1,i}\right\|_i^2 + \tilde\epsilon_3. \tag{4.21c}$$

Applying Cauchy-Schwarz inequality to (4.20d) yields

$$|\epsilon_j| \le 2\left\|\phi_j\right\|_i \left\|\omega_j\right\|_i + \left\|\omega_j\right\|_i^2, \quad j=1,2,3.$$

Summing over all elements and applying the Cauchy-Schwarz inequality with the estimates (3.14d), (3.14e), (3.14f) and (3.18), we get

$$|\tilde{\epsilon}_j| \le \sum_{i=1}^{N} |\epsilon_j| \le C\ h^{2k+5/2}, \quad j=1,2,3. \tag{4.22}$$

Next, we bound the first term in $\hat{\epsilon}_j,\ j=1-3$. Using Cauchy-Schwarz inequality, the inequality $(a+b)^2 \le 2a^2+2b^2$, and applying Theorems 4.1 and 4.2, we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}(\alpha_1^2-a_1^2)\left\|\psi_{k+1,i}^{+}\right\|_i^2 = \sum_{i=1}^{N}(\alpha_1-a_1)\left\|\psi_{k+1,i}^{+}\right\|_i(\alpha_1+a_1)\left\|\psi_{k+1,i}^{+}\right\|_i\\
&\le \left(\sum_{i=1}^{N}(\alpha_1-a_1)^2\left\|\psi_{k+1,i}^{+}\right\|_i^2\right)^{1/2}\left(\sum_{i=1}^{N}(\alpha_1+a_1)^2\left\|\psi_{k+1,i}^{+}\right\|_i^2\right)^{1/2} \le Ch^{2k+5/2},
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{N}(\alpha_2^2-a_2^2)\left\|\psi_{k+1,i}^{-}\right\|_i^2 = \sum_{i=1}^{N}(\alpha_2-a_2)\left\|\psi_{k+1,i}^{-}\right\|_i(\alpha_2+a_2)\left\|\psi_{k+1,i}^{-}\right\|_i\\
&\le \left(\sum_{i=1}^{N}(\alpha_2-a_2)^2\left\|\psi_{k+1,i}^{-}\right\|_i^2\right)^{1/2}\left(\sum_{i=1}^{N}(\alpha_2+a_2)^2\left\|\psi_{k+1,i}^{-}\right\|_i^2\right)^{1/2} \le Ch^{2k+5/2},
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{N}(\alpha_3^2-a_3^2)\left\|\psi_{k+1,i}^{-}\right\|_i^2 = \sum_{i=1}^{N}(\alpha_3-a_3)\left\|\psi_{k+1,i}^{-}\right\|_i(\alpha_3+a_3)\left\|\psi_{k+1,i}^{-}\right\|_i\\
&\le \left(\sum_{i=1}^{N}(\alpha_3-a_3)^2\left\|\psi_{k+1,i}^{-}\right\|_i^2\right)^{1/2}\left(\sum_{i=1}^{N}(\alpha_3+a_3)^2\left\|\psi_{k+1,i}^{-}\right\|_i^2\right)^{1/2} \le Ch^{2k+5/2}.
\end{aligned}$$

Finally, combining these estimates with (4.21) and (4.22) completes the proof of (4.18).

In order to prove (4.19) we use (4.18) to write

$$\left\|e_u\right\|^2 = \left\|E_u\right\|^2 + \hat{\epsilon}_1, \quad \left\|e_q\right\|^2 = \left\|E_q\right\|^2 + \hat{\epsilon}_2, \quad \left\|e_p\right\|^2 = \left\|E_p\right\|^2 + \hat{\epsilon}_3.$$

Using the fact that $\left\|e_u\right\|^2 = \mathcal{O}(h^{2k+2})$, $\left\|e_q\right\|^2 = \mathcal{O}(h^{2k+2})$, $\left\|e_p\right\|^2 = \mathcal{O}(h^{2k+2})$, and $\hat{\epsilon}_j = \mathcal{O}(h^{2k+5/2}),\ j=1-3$, we complete the proof of (4.19). □

In the previous theorem, we proved that the residual-based *a posteriori* error estimates converge to the true spatial errors at $\mathcal{O}(h^{k+3/2})$ rate. We also proved that the global effectivity indices in the L^2-norm converge to unity at $\mathcal{O}(h^{1/2})$ rate. We note that $e_u - E_u = u - (u_h + E_u)$, $e_q - E_q = q - (q_h + E_q)$, and $e_p - E_p = p - (p_h + E_p)$. Hence the computable quantities $u_h + E_u$, $q_h + E_q$, and $p_h + E_p$, respectively, converge to the exact solutions u, $q = u_x$, and $p = u_{xx}$ at $\mathcal{O}(h^{k+3/2})$ rate. We emphasize that this accuracy enhancement is achieved by adding the error estimates to the approximate solutions only once at the end of the computation *i.e.*, at $t = T$. This leads to very efficient computations of the post-processed approximations u_h+E_u, q_h+E_q, and p_h+E_p. Additionally, E_u, E_q, and E_p are computationally efficient because our LDG error estimates are obtained by solving a local steady problem with no boundary conditions on each element.

5. Numerical experiments

In this section, we present numerical experiments to validate our theoretical results. We consider the following linear one-dimensional KdV equation

$$u_t + u_x + 2u_{xxx} = 0, \quad x \in [0, 2\pi],\ t \in [0, 1],$$

with an initial condition $u(x, 0) = \sin(x)$ and periodic boundary conditions. The exact solution is given by $u(x, t) = \sin(x + t)$. We solve this problem using the LDG method on uniform meshes having $N = 5,\ 10,\ 20,\ 40$ elements and using the spaces P^k with $k = 1,\ 2$ and 3. The initial condition is determined by the standard L^2 projection $u_h(x, 0) = P_h u(x, 0)$. In fact, we have used the special projection P_h^1, the projection P_h^-, and the standard L^2 projection as the initial condition and observed similar results. To save space, we only report the results when the standard L^2 projection is used as the initial condition. Temporal integration is performed by the fourth-order classical explicit Runge-Kutta method. A time step Δt is chosen so that temporal errors are small relative to spatial errors. We do not discuss the influence of the time discretization error in this paper.

In Figure 1, we present the L^2 errors between the numerical solutions and the projection of the exact solutions. The errors are plotted in log scale just for easy visualization. For each P^k space, we fit, in a least-squares sense, the data sets with a linear function and then calculate from the fitting result the slopes of the fitting lines. For each k, the slope of the fitting line is shown. These results indicate that the LDG solution u_h and its spatial derivatives q_h, and p_h are $\mathcal{O}(h^{k+2})$ super close to the projections $P_h^- u,\ P_h^+ q$ and $P_h^+ p$, respectively. Although the superconvergence rate is proved to be of order $k + 3/2$, our computational results indicate that the observed numerical convergence rate is higher than the theoretical rate.

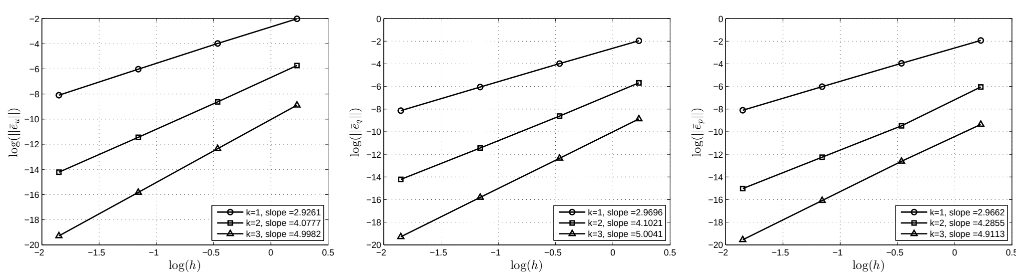

Figure 1: Log-log plots of $||\bar{e}_u||$, $||\bar{e}_q||$ and $||\bar{e}_p||$ at time $t = 1$ versus mesh sizes h on uniform meshes having $N = 5,\ 10,\ 20,\ 40$ elements using P^k, $k = 1$ to 3 (from left to right).

We compute the maximum LDG errors $\|e_u\|^*$ at shifted roots of $(k+1)$-degree right-Radau polynomial on each element I_i and then take the maximum over all elements $I_i,\ i = 1, \cdots, N$. Similarly, the maximum LDG errors $\|e_q\|^*$ and $\|e_p\|^*$ are computed at shifted roots of $(k+1)$-degree left-Radau polynomial on each element and by taking the maximum over all elements *i.e.*,

$$\|e_u\|^* = \max_{1\le i\le N}\left(\max_{1\le j\le k+1}|e_u(x_{j,i}^+, t)|\right), \quad \|e_q\|^* = \max_{1\le i\le N}\left(\max_{1\le j\le k+1}|e_q(x_{j,i}^-, t)|\right),$$

$$\|e_p\|^* = \max_{1\le i\le N}\left(\max_{1\le j\le k+1}|e_p(x_{k,i}^-, t)|\right),$$

where $x_{j,i}^{\pm}$ are the shifted roots of $R_{k+1,i}^{\pm}$ on I_i.

The maximum errors at the superconvergence points as well as their order of convergence shown in Figure 2 indicate that the LDG errors e_u, e_q, and e_p at time $t = 1$ are $\mathcal{O}(h^{k+2})$ superconvergent at Radau points.

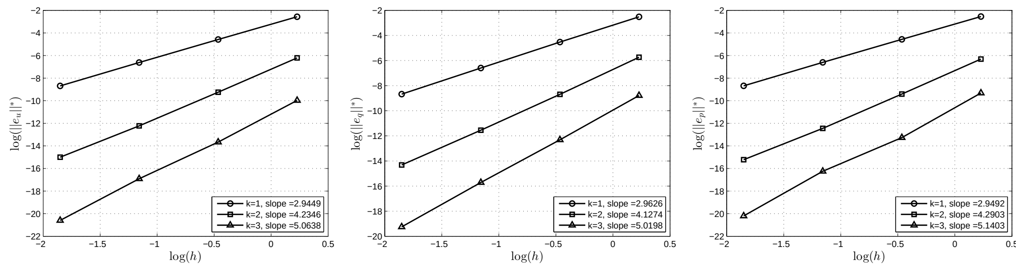

Figure 2: Log-log plots of $||e_u||^*$, $||e_q||^*$ and $||e_p||^*$ at time $t = 1$ versus mesh sizes h on uniform meshes having $N = 5,\ 10,\ 20,\ 40$ elements using P^k, $k = 1$ to 3.

On each element we apply the error estimation procedure (4.4)-(4.5) to compute error estimates for the LDG solution and its derivatives up to second order. Let $\delta e_u,\ \delta e_q,\ \delta e_p$ and $\delta\theta_u,\ \delta\theta_q,\ \delta\theta_p$ be defined as

$$\delta e_u(t) = \|e_u - E_u\|, \quad \delta e_q(t) = \|e_q - E_q\|, \quad \delta e_p(t) = \|e_p - E_p\|,$$
$$\delta\theta_u(t) = \left|\theta_u(t) - 1\right|, \quad \delta\theta_q(t) = \left|\theta_q(t) - 1\right|, \quad \delta\theta_p(t) = \left|\theta_p(t) - 1\right|.$$

The results shown in Figure 3 indicate that the numerical convergence rate at $t = 1$ for $\delta e_u,\ \delta e_q$ and δe_p is $\mathcal{O}(h^{k+3})$. The convergence rate is higher than the theoretical rate which is proved to be of order $k + 3/2$. Finally, the errors $\delta\theta_u$, $\delta\theta_q$, and $\delta\theta_p$ as

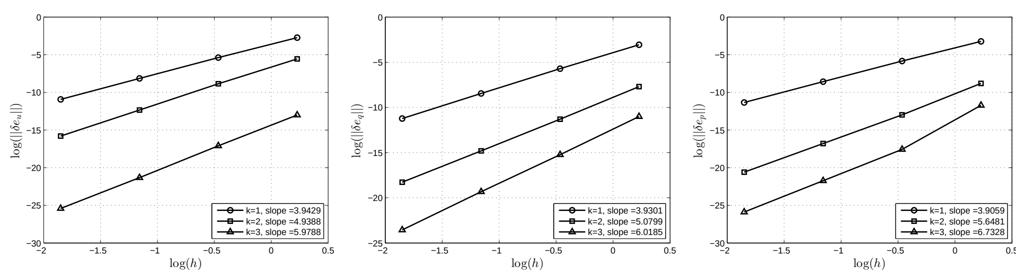

Figure 3: Convergence rates at $t = 1$ for δe_u, δe_q and δe_p (left to right) on uniform meshes having $N = 5,\ 10,\ 20,\ 40$ elements using P^k, $k = 1$ to 3.

well as their order of convergence shown in Figure 4 suggest that the convergence rate at $t = 1$ for $\delta\theta_u$, $\delta\theta_q$, and $\delta\theta_p$ is $\mathcal{O}(h^2)$ under mesh refinement. These results indicate that the observed numerical convergence rate is higher than the theoretical rate which is proved to be $\mathcal{O}(h^{1/2})$. We note that the effectivity indices stay close to unity for all times and converge under h- and p-refinements. Numerical results further indicate that the error estimates converge to the true error with decreasing mesh size and increasing polynomial degree k.

6. Concluding remarks

In this paper we constructed and analyzed *a posteriori* error estimates for the LDG method for the linearized KdV equation in one space dimension. These error estimates are computationally simple and are computed by solving a local steady

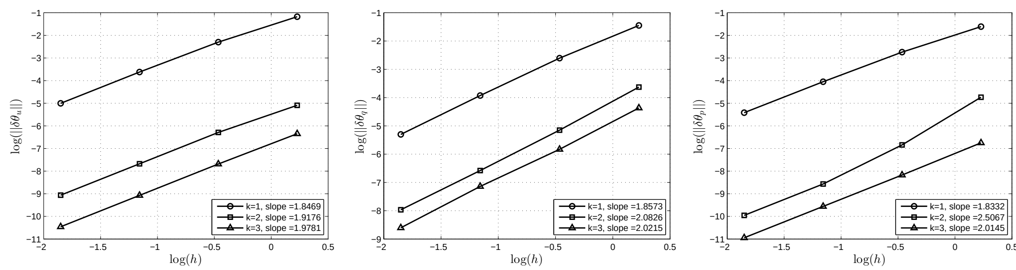

Figure 4: Convergence rates at $t = 1$ for $\delta\theta_u$, $\delta\theta_q$, and $\delta\theta_p$ (left to right) on uniform meshes having $N = 5$, 10, 20, 40 elements using P^k, $k = 1$ to 3.

problem with no boundary conditions on each element. We first extended the work of Hufford and Xing [30] to prove new superconvergence results for the auxiliary variables in the LDG method that approximate the first and second derivatives of the solution. More precisely, we proved that the $(k+3/2)$-th order superconvergence rate holds also for the two auxiliary variables. We applied these superconvergence results to show that the significant parts of the discretization errors for the k-degree LDG solution and its spatial derivatives up to second order are proportional to $(k+1)$-degree Radau polynomials. Superconvergence at Radau points were used to construct residual-based *a posteriori* error estimates of the spatial errors. We proved that, for smooth solutions, these *a posteriori* LDG error estimates for the solution and its spatial derivatives at a fixed time t converge to the true errors at $\mathcal{O}(h^{k+3/2})$ rate. Finally, we proved that the global effectivity indices, for the solution and its derivatives in the L^2-norm converge to unity at $\mathcal{O}(h^{1/2})$ rate. Our computational results indicate that the observed numerical convergence rates are higher than the theoretical rates. In our analysis time integration is assumed to be exact and thus we are only estimating the spatial errors of the semi-discrete LDG method. The extension of this proof for variable coefficient problems is straightforward. We are currently investigating the superconvergence properties of the LDG method applied to two-dimensional problems on rectangular and triangular meshes. Extending our *a posteriori* error analysis to problems on tetrahedral meshes will be investigated in the future. Finally, because we observed superconvergence of order $k + 2$ in our numerical examples, future work will include investigating how to improve our proofs to obtain optimal superconvergence results.

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