5-2018

Simple Approximations to the Renewal Function

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SIMPLE APPROXIMATIONS TO THE RENEWAL FUNCTION

University Honors Program Thesis

University of Nebraska at Omaha

Submitted by

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April, 2018

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UNIVERSITY OF NEBRASKA AT OMAHA

HONORS THESIS/PROJECT/CREATIVE ACTIVITY ABSTRACT

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STATE:	YES
PROGRAM SIZE:	450+
THESIS:	REQUIRED
THESIS ISSUED:	PROGRAM

ABSTRACT:

In reliability theory, a renewal process is a stochastic model for arrival times or events occurring in a certain system. For a renewal process, it is of interest to be able to estimate the number of events that will occur in the time interval $(0, t]$. The renewal function, $M(t)$, is the expected value of renewals to occur within the system from $(0, t]$. It is a solution of the renewal equation. Since closed-form solutions of the renewal equation are mostly non-existent, approximation methods are used. Simpler approximation methods than those currently available are presented and are applied to data. The approximations presented are especially useful for time instances less than the mean of the underlying distribution of the process, which suggests application to studies like warranty analysis.

1 Introduction

Renewal processes are significant models for areas of applied statistics such as reliability and inventory control. They describe the way that arrivals or failures occur within in a certain process where the underlying distribution of time between event occurrences is similar. The renewal function partially answers the question of how many arrivals or failures are expected to happen within a certain time period. To motivate the importance of this application, consider the following example about failures within a system:

There is a room filled with many lighting fixtures, and each fixture must have a functioning light bulb present to keep the room fully lit. The problem in question for this process is determining what number of light bulbs on average is needed to keep the room fully lit for t continuous hours. This is ascertained by determining how many light bulbs will burn out on average in t hours and how many light bulbs are needed to replace them. Assuming that the expected life of each light bulb is similar, the renewal function satisfies the problem of determining the average number of light bulbs needed to keep the room lit for the period of time $(0, t]$. Because this result is dependent on t (a variable of time), we call the expected value of needed light bulbs $M(t)$.

The theoretical calculation of $M(t)$ requires n-dimensional integration, but unfortunately, closed-form solutions of it usually do not exist. To address this issue, approximation methods are used to estimate $M(t)$. There are existing approximation techniques (some of which are discussed here); however, it is desirable to find even simpler and less calculation-heavy methods of approximation than those present. While forfeiting some accuracy, the approximations presented here serve as quick and simple estimations of the renewal function. In particular, they work especially well for when the time instance observed is significantly less than the underlying mean of the renewal process in question. This prompts application to warranty analysis, where it is necessary to estimate the amount of product failures that will occur at some time before the mean failure time.

In order to prime the reader on basic but necessary knowledge of renewal theory, a brief introduction to renewal processes and the renewal function is provided. Examples of the existing approximation methods are discussed, and then the new approximation techniques are provided with their numerical results. Finally, these methods are applied to a sample data set.

2 Renewal Processes and the Renewal Function

A system where any failed component is instantaneously replaced by an identical component or repaired to its original state can be modeled by a renewal process. A renewal process is a counting process where the time between failures is distributed the same way. A renewal process can be thought of as a generalization of a Poisson process. The generalization here is that instead of the time between occurrences being independent and identically distributed (IID) exponential random variables, the inter-arrival times are IID random variables from any arbitrary distribution (Ross, 2014).

Let $X_1, X_2, X_3, \ldots$ be non-negative IID random variables sharing a common cumulative distribution function (CDF) and probability density function (PDF), $F(x)$ and $f(x)$, respectively. These random variables represent the distribution of time between successive failures within a renewal process. So X_n is the time between the $(n-1)$th and the nth renewal. As mentioned, these times can represent any failure distribution such as the time for a light bulb to burn out or a component of some product to fail. Let S_i denote the time of the ith renewal in the system. We have that,

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, \quad n > 0.$$

The time for the first renewal is represented as $S_1 = X_1$; the time until the second renewal is represented as $S_2 = X_1 + X_2$, and so on. Thus, the time until the nth renewal is $S_n = X_1 + X_2 + \cdots + X_n$ (Ross, 2014).

Let $t \geq 0$. We define the random variable N as the amount of renewals that occur by time t. Here,

$$N = N(t) = \sup\{n \geq 1 : X_1 + X_2 + \cdots + X_n \leq t\} = \sup\{n \geq 1 : S_n \leq t\}.$$

The expected value of N is denoted as $M(t)$ and represents the expected number of renewals to occur by time t. Here, $M(t) = E[N]$ is the renewal function. It can be shown that,

$$M(t) = \sum_{n=1}^{\infty} P(S_n \leq t);$$

however, the calculation of $P(S_n \leq t)$ requires n dimensional integration, and is not easily attainable. In theory, we are able to find this, but we look toward other ways of calculating $M(t)$.

To find analytical solutions for $M(t)$, it is necessary to note the certain integral equations that $M(t)$ satisfies. The general renewal equation for a given time t summarized from (Xie, 1989) is,

$$M(t) = F(t) + \int_0^t M(t-x)dF(x), \tag{1}$$

and since $F(t)$ has the PDF $f(x)$ this is,

$$M(t) = F(t) + \int_0^t M(t-x)f(x)dx. \tag{2}$$

Since $F(t) = 0$ for all $t < 0$, another form of (1) is,

$$M(t) = F(t) + \int_0^t F(t-s)dM(s). \tag{3}$$

Also, these are all equivalent to

$$M(t) = \sum_{n=1}^{\infty} F^{(n)}(t), \tag{4}$$

where $F^{(n)}(t)$ is the n-fold convolution of $F(t)$. These renewal equations form the basis of some existing approximation methods as well as the ones presented here.

Some final results of renewal processes are limiting theorems which provide preliminary rough approximations for the renewal function $M(t)$. Let $\mu = E[X_i]$ and

$\sigma^2 = Var[X_i]$ for $i = 1, 2, \ldots, n$. It is known that,

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\mu}, \text{ if } \mu > 0.$$

The approximation of $M(t)$ this provides is,

$$M(t) \approx \frac{t}{\mu}. \tag{5}$$

We also have,

$$\lim_{t\to\infty} \left[\frac{M(t)}{t} - \frac{1}{\mu}\right] = \frac{\mu^2 - \sigma^2}{2\mu^2}$$

Thus, another approximation of $M(t)$ is,

$$M(t) \approx \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2}. \tag{6}$$

(5) and (6) will provide better approximations when t is large, but they usually will not be as useful in an interval such as $t < \mu$. (6) will provide provide results less than 0 if t is too small and the variance is less than μ^2. Some of the approximation methods presented are as computationally simple to attain as these estimates, but they will provide better results for smaller values of t that are less than the mean.

3 Existing Approximation Methods

Before exploring and developing the ideas surrounding the suggested simple approximations, it is of value to observe existing approximation techniques. While evaluating the new approximations, these existing techniques served as substitutes for the theoretical values of $M(t)$ because of their accuracy.

3.1 RS-Method

Xie presents an approximation technique for solving renewal-type integrals that "is easy to program, quite understandable, and has good convergence properties" (1989). It provides good results for evaluating the renewal function in most cases, which is useful for applying it to any distribution. The RS-method is called such because of its application of numerical Riemann-Stieltjes integration.

The theory of this application, as explained by Xie, begins with the fact that an integral in the form $\int_a^b f(x)dg(x)$ is equal to the limit of the following sums,

$$\begin{aligned} J_+ &= \sup_n \sum_{i=1}^{n} m_i * [g(x_i) - g(x_{i-1})] \\ J_- &= \inf_n \sum_{i=1}^{n} M_i * [g(x_i) - g(x_{i-1})] \end{aligned} \tag{7}$$

where,

$$M_i = \sup f(x) \quad \text{given} \quad x_{i-1} \leq x \leq x_i,$$
$$m_i = \inf f(x) \quad \text{given} \quad x_{i-1} \leq x \leq x_i,$$
$$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b,$$

by Riemann-Stieltjes integration. This can be used to discretize the integrand of the renewal equation in the form of (3). Xie states that the simplest way to approximate the results of (7) is to use the midpoint method to approximate $f(t)$. He writes,

$$\int_a^b f(x)dg(x) \approx \sum_{i=1}^{n} f(x_{i-1/2})(g(x_i) - g(x_{i-1})) \tag{8}$$

where,

$$x_{i-1/2} = \frac{(x_i + x_{i-1})}{2}.$$

By applying this to the renewal equation integral in (3) where for a given t, the partition $\{t_i, 0 \leq i \leq n\}$ satisfies $0 = t_0 < t_1 ... < t_n = t$, it follows from (8) that

$$M(t_i) \approx F(t_i) + \sum_{j=1}^{i} F(t_i - t_{j-1/2})(M(t_j) - M(t_{j-1})),$$

which can be solved recursively by the following equations:

$$M(t_i) = \frac{F(t_i) + S_i - F(t_i - t_{j-1/2})M(t_{i-1})}{1 - F(t_i - t_{i-1/2}}$$
$$S_i = \sum_{j=1}^{i-1} F(t_i - t_{j-1/2})(M(t_j) - M(t_{j-1})). \quad (9)$$

The suggested way to program this method for numerical results (which I used) is to use two vectors. The first vector, F of size N, stores calculations of $F(t_i)$ and the other vector, M of the same size, stores calculations of $M(t_i)$. N is determined by the amount of steps and step length size used in the interval $(0, t]$. Clearly, increasing the amount of sub-intervals between $(0, t]$ increases the accuracy of the discretization of the integral in the renewal equation, but $N = 500$ works well for the purposes of this technique.

3.2 Approximations for Renewal Function of Large Weibull Shape Parameter

Cui and Xie present approximations for Weibull renewal functions based on Normal approximations of the Weibull distribution. A random variable X having the Weibull distribution, has the CDF $F(t) = 1 - \exp(-(t)^{\beta})$, with $t \geq 0$, $\beta > 0$ and mean and variance, $\mu = \Gamma(\frac{1}{\beta+1})$ and $\sigma^2 = \Gamma(\frac{2}{\beta}+1)-\mu^2$. The approximation technique they present is to be used with Weibull renewal functions where $\beta > 3$. They note that their result is less accurate than a previous approximation method, but is good because of its computational convenience and simplicity (Cui & Xie, 2003).

Their approximation is reliant on equating the mean and variance of the Weibull distribution with the mean and variance of a normal distribution. They note the fact that when β is about 3-4 for a Weibull distribution, the distribution can be compared to a normal distribution, and for even larger β, a normal distribution is still pretty good for approximating the Weibull. A table in their article details the small error between the two. Using this result, the approximation becomes more focused on the properties of the normal distribution.

They recall the renewal equation in the form of (4), where $F^{(n)}(t) = P\{X_1 + X_2 + \cdots + X_n \leq x\}$ and $\{X_i\}$ for $i = 1, 2, \ldots$ are IID Weibull random variables. Using the result mentioned, each $X_i \sim N(\mu, \sigma^2)$. So $F^{(n)}(t)$ becomes the probability of the sum of normal random variables, which will be distributed again as a normal random variable. This probability can be approximated by $\phi(\frac{x-n\mu}{\sqrt{n}\sigma})$. $\phi(x)$ here is the standard normal cumulative distribution. Applying this to the renewal equation (4) provides,

$$M(t) \approx \sum_{n=1}^{\infty} \phi\left(\frac{t-n\mu}{\sqrt{n}\sigma}\right) \tag{10}$$

Only a number of the terms from this sum are necessary to evaluate and the rest can be truncated. Cui and Xie remind us of the fact that for a standard normal distribution $\phi(-6) \leq 10^{-7}$ and $1-\phi(6) \leq 10^{-7}$. So the error of omitting any $\phi(x)$ for $x < 6$ or $x > 6$ is minimal. Then for (10), it is only necessary to observe $-6 \leq \frac{t-n\mu}{\sqrt{n}\sigma} \leq 6$, and the n's to consider are,

$$\left[\frac{\sqrt{36\sigma^2+4\mu t}-6\sigma}{2\mu}\right]^2 \leq n \leq \left[\frac{\sqrt{36\sigma^2+4\mu t}+6\sigma}{2\mu}\right]^2. \tag{11}$$

Applying this truncation to (10), Cui and Xie find the following:

$$\begin{aligned} M(t) &\approx \sum_{n=1}^{\infty} \phi\left(\frac{t-n\mu}{\sqrt{n}\sigma}\right) \approx n_1 - 1 + \sum_{n=n_1}^{n_1+k} \phi\left(\frac{t-n\mu}{\sqrt{n}\sigma}\right) \\ \text{where } n_1 &= \left[\frac{\sqrt{36\sigma^2+4\mu t}-6\sigma}{2\mu}\right]^2 + 1 \\ k &= \left[\frac{\sqrt{36\sigma^2+4\mu t}+6\sigma}{2\mu}\right]^2 - n_1. \end{aligned} \tag{12}$$

Cui and Xie also outline three approximation methods that can be used for the calculation of the standard normal function $\phi(x)$ in the equation above.

4 Approximations of the Renewal Function

The existing approximations mentioned work well when they are necessary; however, we want to circumvent calculation-heavy approaches like Xie's RS-Method. Cui and Xie's Weibull approximation is pretty simple computationally, but unfortunately its use is limited to Weibull renewal processes with $\beta > 3$. The approximations here work in the general case for any distribution, and they are simpler. The drawback of the simplicity is that these methods become quick and dirty estimates of the renewal function, and the results are mostly only acceptable for time instances that are less than the mean.

The basis of all the simple approximations developed is the renewal equation in the form:

$$M(t) = F(t) + \int_0^t M(t-x)f(x)dx. \qquad \text{((2) revisited)}$$

Of the following approximations explained, Approximations A, B, C, G, and H are based on manipulations of the renewal equation (2). On the other hand, results for Approximations D, E, F, I, and J are found by substituting the other approximations in for $M(t-x)$ in the renewal equation (2). We have in the following derivations that $N(t)$ and $M(t)$ represent the number of renewals and the renewal function respectively. Also, $F(t)$ and $f(t)$ are the cumulative distribution function and the probability density function respectively.

4.1 Approximation A

To begin to derive Approximation A we note an inequality that can be assumed from the renewal equation. We have,

$$m(t) \geq F(t), \tag{13}$$

which clearly holds from (2). Now that we have $F(t)$ as a lower bound for $M(t)$, we will try to find an upper bound for $M(t)$ as well. It is known that the number of renewals to occur in a time interval, $N(t)$, is a non-decreasing function of t. Using this fact we have,

$$N(t-x) \leq N(t) \; \forall t \geq 0, x \geq 0, \text{ and } x \leq t.$$

By taking the expected value of this inequality, we are given the following relationship:

$$M(t-x) \leq M(t) \implies M(t-x)f(x) \leq M(t)f(x).$$

If we integrate both sides of this inequality from 0 to t we are are provided the following result in conjunction with results of (2):

$$\begin{aligned} M(t) &\leq F(t) + \int_0^t M(t-x)f(x)dx \\ &= F(t) + M(t) \cdot \int_0^t f(x)dx \\ &= F(t) + M(t)F(t). \end{aligned}$$

From this inequality the following relationship can be deduced:

$$M(t) \leq F(t) + M(t)F(t) \implies M(t)(1 - F(t)) \leq F(t) \implies M(t) \leq \frac{F(t)}{1 - F((t))}. \quad (14)$$

We now have (13) and (14) as upper and lower bounds of the renewal function. It is important to note the form that these two bounds maintain to develop this approximation. Both (13) and (14) follow the form:

$$\frac{F(t)}{1 - c \cdot F(t)}, \quad (15)$$

where c is some constant. For the lower bound (13), $c = 0$ and for the upper bound (14), $c = 1$. This suggests an approximation for $M(t)$ based on manipulations of this constant. If we let the value of c be the average of the two values of c from our bounds ($c = \frac{1}{2}$), we can approximate the renewal function by the following equation:

$$M(t) \approx \frac{F(t)}{1 - \frac{1}{2}F(t)}. \quad \text{(Approximation A)}$$

4.2 Approximation B

To derive Approximation B we look towards the bounds (13) and (14) developed from Approximation A. For this approximation we assume that the value of the renewal equation is approximately the average of the values given by the lower and upper bounds of the renewal equation rather than approximating c for an equation in the form of (15).

Here we take the arithmetic mean of the values of the bounds. This approximation is simply,

$$M(t) \approx \frac{1}{2}\left(F(t) + \frac{F(t)}{1 - F(t)}\right). \qquad \text{(Approximation B)}$$

4.3 Approximation C

To derive Approximation C we again rely on the bounds (13) and (14) developed from Approximation A. This approximation is similar to Approximation B where we are interested in using the actual values of the bounds to find an approximate value of the renewal function. The difference with this approximation is that we will use the geometric mean for the values of the bounds rather than the arithmetic mean from Approximation B. This approximation is simply,

$$M(t) \approx \sqrt{F(t) \cdot \frac{F(t)}{1 - F(t)}} = \frac{F(t)}{\sqrt{1 - F(t)}}. \qquad \text{(Approximation C)}$$

It is known that the geometric mean is less than the arithmetic mean, therefore it is expected for the result from Approximation B to be greater than or equal to the result of Approximation C for a given time t.

4.4 Approximations D-F

To derive Approximations D, E, and F we go back to the original renewal equation (2) that we began with. Within this renewal equation, the term $\int_0^t M(t-x)f(x)dx$ is where complications arise for finding a closed-form solution. It requires recursive integration for us to find the preceding values of $M(t)$. We turn to our previous approximations of $M(t)$ as a substitute instead of pursuing the recursion. Approximations A, B, and C defined at $t = t - x$ are substituted for the term $M(t - x)$ in the renewal equation for Approximations D, E, and F. This allows us to integrate directly to approximate the value of the renewal function for a specific time instance. Using the results of Approximations A, B, and C we have the following:

$$M(t) \approx F(t) + \int_0^t \frac{F(t-x)}{1-\frac{1}{2}F(t-x)} \cdot f(x)dx \qquad \text{(Approximation D)}$$

$$M(t) \approx F(t) + \int_0^t \frac{1}{2}\left(F(t-x) + \frac{F(t-x)}{1-F(t-x)}\right) \cdot f(x)dx \qquad \text{(Approximation E)}$$

$$M(t) \approx F(t) + \int_0^t \frac{F(t-x)}{\sqrt{1-F(t-x)}} \cdot f(x)dx \qquad \text{(Approximation F)}$$

We note that these approximations are essentially the CDF at time t added to the integral term which is a small adjustment. Because of this, we can expect Approximations D, E, and F to perform better than Approximations A, B, C due to how close the value of $F(t)$ is to $M(t)$ for small values of $t < \mu$. We will make use of this in the next approximations.

4.5 Approximation G

To derive Approximation G we again manipulate the renewal equation (2) by making some assumptions about the value of $M(t-x)$. For the renewal function $M(t)$ we are interested in values on the interval $(0, t]$, and thus, we consider values of $M(t-x)$ within the same interval. If we let $x = 0$ then $M(t-x) = M(t)$. Similarly, if we let $x = t$ then $M(t-x) = M(0) = 0$. Clearly, the value of $M(t-x)$ will be contained in between these two values so taking the average of the two will suffice as an approximate value of $M(t-x)$. We have,

$$M(t-x) \approx \frac{1}{2}(M(t) + M(0)) = \frac{1}{2}M(t). \qquad (16)$$

We will now use (16) in the renewal equation for the value of $M(t-x)$. With this substitution, the renewal equation becomes,

$$M(t) \approx F(t) + \int_0^t \frac{1}{2}M(t) \cdot f(x)dx.$$

We note that the value of $\frac{1}{2} \cdot M(t)$ will be a constant so it is moved outside of the integral, and $\int_0^t f(x)dx$ is merely the cumulative distribution function, so it is replaced with $F(t)$:

$$M(t) \approx F(t) + \frac{1}{2}M(t) \cdot F(t).$$

As mentioned before, $M(t) \approx F(t)$ for the values of t we are concerned with. Replacing $M(t)$ with $F(t)$ in our approximation provides the following:

$$M(t) \approx F(t) + \frac{1}{2}F(t)^2. \qquad \text{(Approximation G)}$$

4.6 Approximation H

Approximation H is found similarly to Approximation G. The distinction between the two is that the approximation of $M(t-x)$ is adjusted. The approximated value of $M(t-x)$ from G will actually be close to the value of evaluating $M(t)$ at $t = \frac{t}{2}$. Thus,

$$M(t-x) \approx \frac{1}{2}M(t) \approx M\left(\frac{t}{2}\right)$$

We use this result almost identically to how (16) was used in approximation G:

$$\begin{aligned} M(t) &\approx F(t) + \int_0^t M\left(\frac{t}{2}\right) \cdot f(x)dx \\ &= F(t) + M\left(\frac{t}{2}\right) \cdot \int_0^t f(x)dx \\ &= F(t) + M\left(\frac{t}{2}\right) \cdot F(t) \\ &\approx F(t) + F\left(\frac{t}{2}\right) \cdot F(t) \end{aligned}$$

And this approximation becomes:

$$M(t) \approx F(t)\left(1 + F\left(\frac{t}{2}\right)\right) \qquad \text{(Approximation H)}$$

4.7 Approximations I and J

Approximations I and J are derived similarly to Approximations D, E, and F. In this case, we recall the results of Approximations G and H which can be used to approximate the value of $M(t-x)$ in the renewal equation (2). We substitute these approximations defined at $t = t - x$ to produce the following:

$$M(t) \approx F(t) + \int_0^t \left[F(t) + \frac{1}{2}F(t)^2\right] \cdot f(x)dx \qquad \text{(Approximation I)}$$

$$M(t) \approx F(t) + \int_0^t F(t-x) \cdot \left(1 + F\left(\frac{t-x}{2}\right)\right) \cdot f(x)dx \qquad \text{(Approximation J)}$$

5 Numerical Results

Approximations A-J were tested by application to four well known distributions, which were the Gamma, Linear Failure Rate, Pareto, and Weibull distributions. The Gamma distribution is defined by the PDF:

$$f(x) = F'(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x) \text{ for } x \geq 0,\ \alpha > 0,\ \text{and } \beta > 0,$$

but for our purposes, the parameter β was limited to $\beta = 1$. The Linear Failure Rate distribution is defined by the CDF:

$$F(x) = 1 - \exp(-\theta_1 x - \theta_2 x^2) \text{ for } x \geq 0,\ \theta_1 > 0,\ \text{and } \theta_2 > 0.$$

The Pareto distribution CDF is in the form:

$$F(x) = \left(\frac{x}{x+\theta_1}\right)^{\theta_2} \text{ for } x \geq 0,\ \theta_1 > 0,\ \text{and } \theta_2 > 0,$$

and the Weibull distribution has CDF:

$$F(x) = 1 - \exp\left(-\frac{x}{\lambda}\right)^{\theta} \text{ for } x \geq 0,\ \lambda > 0,\ \text{and } \theta > 0.$$

For our purposes, λ is restricted to $\lambda = 1$.

The approximations were applied to these four distributions with varying parameters. Because of its accuracy and versatility, Xie's RS-Method Approximation was used to generate accepted results for comparison to the true value of the renewal function. The results for Xie's Approximation as well as Approximations A-J were obtained by programs written in Maple. The time instances observed were mostly $t < \mu$; however, a few results past the mean are included as well. Charts of these results are presented on subsequent pages:

5.1 Gamma Distributions

Table 1: Gamma Distribution ($\alpha = \mu = \sigma^2 = 0.5$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
0.05	0.30650	0.28333	0.28913	0.28621	0.30274	0.30357	0.30316	0.27896	0.24817	0.30210	0.30429
0.10	0.46860	0.41733	0.43632	0.42672	0.45709	0.46085	0.45895	0.40489	0.34528	0.45452	0.46062
0.15	0.60855	0.52544	0.56440	0.54457	0.58609	0.59537	0.59066	0.50269	0.41612	0.58037	0.59140
0.20	0.73761	0.61936	0.68506	0.65138	0.70129	0.71911	0.71001	0.58473	0.47291	0.69126	0.70800
0.25	0.86007	0.70362	0.80300	0.75167	0.80710	0.83684	0.82156	0.65596	0.52050	0.79173	0.81476
0.30	0.97806	0.78052	0.92076	0.84775	0.90577	0.95119	0.92770	0.71902	0.56142	0.88411	0.91392
0.35	1.09284	0.85147	1.03997	0.94101	0.99863	1.06381	1.02986	0.77555	0.59722	0.96984	1.00681
0.40	1.20520	0.91738	1.16182	1.03239	1.08653	1.17590	1.12900	0.82667	0.62891	1.04985	1.09431
0.45	1.31568	0.97889	1.28726	1.12254	1.17006	1.28838	1.22582	0.87319	0.65722	1.12485	1.17704
0.50	1.42466	1.03649	1.41709	1.21194	1.24967	1.40201	1.32084	0.91572	0.68269	1.19535	1.25549
0.55	1.53242	1.09055	1.55201	1.30098	1.32567	1.51747	1.41445	0.95476	0.70573	1.26178	1.33002
0.60	1.63919	1.14139	1.69269	1.38997	1.39833	1.63535	1.50700	0.99071	0.72668	1.32447	1.40093
0.65	1.74511	1.18925	1.83975	1.47916	1.46787	1.75621	1.59877	1.02389	0.74579	1.38371	1.46849
0.70	1.85034	1.23435	1.99381	1.56878	1.53446	1.88058	1.68999	1.05457	0.76328	1.43975	1.53290
0.75	1.95496	1.27689	2.15548	1.65900	1.59827	2.00896	1.78087	1.08301	0.77933	1.49281	1.59437

Table 2: Gamma Distribution ($\alpha = \mu = \sigma^2 = 1$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
0.1	0.1	0.09992	0.10017	0.10004	0.10000	0.10000	0.10000	0.09969	0.09516	0.09999	0.10000
0.2	0.2	0.19934	0.20134	0.20033	0.19997	0.20006	0.20002	0.19770	0.18127	0.19989	0.19993
0.3	0.3	0.29777	0.30452	0.30113	0.29984	0.30032	0.30008	0.29277	0.25918	0.29947	0.29966
0.4	0.4	0.39475	0.41075	0.40267	0.39951	0.40099	0.40025	0.38402	0.32968	0.39844	0.39898
0.5	0.5	0.48984	0.52110	0.50522	0.49884	0.50238	0.50059	0.47088	0.39347	0.49647	0.49767
0.6	0.6	0.58263	0.63665	0.60904	0.59765	0.60487	0.60121	0.55297	0.45119	0.59321	0.59546
0.7	0.7	0.67275	0.75858	0.71438	0.69577	0.70890	0.70220	0.63013	0.50341	0.68831	0.69211
0.8	0.8	0.75990	0.88811	0.82150	0.79298	0.81499	0.80369	0.70229	0.55067	0.78147	0.78737
0.9	0.9	0.84380	1.02652	0.93068	0.88908	0.92373	0.90581	0.76951	0.59343	0.87239	0.88101
1.0	1.0	0.92423	1.17520	1.04219	0.98385	1.03578	1.00871	0.83191	0.63212	0.96081	0.97280
1.1	1.1	1.00104	1.33565	1.15630	1.07709	1.15187	1.11256	0.88966	0.66713	1.04654	1.06256
1.2	1.2	1.07410	1.50946	1.27331	1.16858	1.27282	1.21752	0.94297	0.69881	1.12939	1.15011
1.3	1.3	1.14334	1.69838	1.39350	1.25813	1.39951	1.32378	0.99207	0.72747	1.20922	1.23531
1.4	1.4	1.20874	1.90430	1.51717	1.34556	1.53294	1.43153	1.03721	0.75340	1.28593	1.31804
1.5	1.5	1.27030	2.12928	1.64463	1.43071	1.67416	1.54098	1.07863	0.77687	1.35946	1.39821

Table 3: Gamma Distribution ($\alpha = \mu = \sigma^2 = 3$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
0.2	0.001149	0.001149	0.001149	0.001149	0.001149	0.001149	0.001149	0.001149	0.001148	0.001149	0.001149
0.4	0.007930	0.007958	0.007958	0.007958	0.007930	0.007930	0.007930	0.007958	0.007926	0.007930	0.007930
0.6	0.023154	0.023386	0.023389	0.023387	0.023154	0.023154	0.023154	0.023382	0.023115	0.023154	0.023154
0.8	0.047607	0.048574	0.048603	0.048589	0.047608	0.047608	0.047608	0.048547	0.047423	0.047608	0.047607
1.0	0.080897	0.083660	0.083807	0.083734	0.080903	0.080903	0.080903	0.083526	0.080301	0.080903	0.080898
1.2	0.122018	0.128240	0.128770	0.128505	0.122043	0.122044	0.122043	0.127775	0.120513	0.122042	0.122023
1.4	0.169720	0.181623	0.183133	0.182376	0.169797	0.169801	0.169799	0.180364	0.166502	0.169793	0.169735
1.6	0.222727	0.242959	0.246598	0.244772	0.222926	0.222940	0.222933	0.240108	0.216642	0.222913	0.222768
1.8	0.279867	0.311309	0.319039	0.315150	0.280311	0.280355	0.280333	0.305661	0.269379	0.280274	0.279958
2.0	0.340126	0.385672	0.400567	0.393049	0.341017	0.341134	0.341075	0.375593	0.323324	0.340923	0.340309
2.2	0.402669	0.465007	0.491580	0.478109	0.404307	0.404583	0.404444	0.448459	0.377286	0.404097	0.403005
2.4	0.466834	0.548243	0.592787	0.570080	0.469636	0.470229	0.469930	0.522867	0.430291	0.469205	0.467406
2.6	0.532113	0.634301	0.705236	0.668829	0.536624	0.537807	0.537209	0.597525	0.481570	0.535810	0.533025
2.8	0.598127	0.722100	0.830341	0.774332	0.605023	0.607241	0.606117	0.671286	0.530546	0.603588	0.599503
3.0	0.664603	0.810587	0.969907	0.886676	0.674688	0.678626	0.676623	0.743165	0.576810	0.672297	0.666578
3.2	0.731349	0.898753	1.126171	1.006056	0.745539	0.752212	0.748801	0.812356	0.620096	0.741748	0.734064
3.4	0.798237	0.985654	1.301844	1.132770	0.817533	0.828395	0.822811	0.878232	0.660260	0.811779	0.801824
3.6	0.865183	1.070435	1.500171	1.267216	0.890639	0.907705	0.898874	0.940334	0.697253	0.882234	0.869753

Table 4: Gamma Distribution ($\alpha = \mu = \sigma^2 = 5$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
0.50	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172	0.000172
0.75	0.001065	0.001065	0.001065	0.001065	0.001065	0.001065	0.001065	0.001065	0.001065	0.001065	0.001065
1.00	0.003660	0.003667	0.003667	0.003667	0.003660	0.003660	0.003660	0.003667	0.003660	0.003660	0.003660
1.25	0.009125	0.009166	0.009166	0.009166	0.009125	0.009125	0.009125	0.009166	0.009124	0.009125	0.009125
1.50	0.018580	0.018750	0.018752	0.018751	0.018580	0.018580	0.018580	0.018748	0.018576	0.018580	0.018580
1.75	0.032917	0.033452	0.033461	0.033457	0.032917	0.032917	0.032917	0.033443	0.032902	0.032917	0.032917
2.00	0.052700	0.054077	0.054116	0.054096	0.052700	0.052700	0.052700	0.054039	0.052653	0.052700	0.052700
2.25	0.078135	0.081181	0.081315	0.081248	0.078136	0.078136	0.078136	0.081057	0.078014	0.078136	0.078135
2.50	0.109099	0.115084	0.115466	0.115275	0.109102	0.109102	0.109102	0.114743	0.108822	0.109102	0.109100
2.75	0.145198	0.155894	0.156847	0.156370	0.145205	0.145206	0.145205	0.155079	0.144621	0.145205	0.145199
3.00	0.185840	0.203537	0.205667	0.204599	0.185859	0.185860	0.185860	0.201801	0.184737	0.185859	0.185842
3.25	0.230317	0.257778	0.262133	0.259946	0.230363	0.230365	0.230364	0.254418	0.228347	0.230361	0.230322
3.50	0.277874	0.318243	0.326510	0.322350	0.277974	0.277980	0.277977	0.312245	0.274555	0.277970	0.277886
3.75	0.327769	0.384433	0.399182	0.391738	0.327971	0.327985	0.327978	0.374440	0.322452	0.327959	0.327793
4.00	0.379315	0.455740	0.480700	0.468054	0.379697	0.379731	0.379714	0.440044	0.371163	0.379669	0.379362
4.25	0.431913	0.531456	0.571834	0.551275	0.432596	0.432670	0.432632	0.508032	0.419882	0.432537	0.431999
4.50	0.485063	0.610790	0.673615	0.641433	0.486225	0.486378	0.486301	0.577360	0.467896	0.486108	0.485214
4.75	0.538375	0.692882	0.787385	0.738623	0.540268	0.540568	0.540417	0.647010	0.514602	0.540049	0.538627
5.00	0.591561	0.776827	0.914844	0.843016	0.594526	0.595086	0.594803	0.716031	0.559507	0.594134	0.591965
5.25	0.644428	0.861694	1.058109	0.954864	0.648909	0.649911	0.649404	0.783565	0.602226	0.648241	0.645050
5.50	0.696860	0.946554	1.219772	1.074514	0.703420	0.705148	0.704271	0.848874	0.642482	0.702328	0.697787

5.2 Linear Failure Rate Distributions

Table 5: Linear Failure Rate Distribution ($\theta_1 = 1$, $\theta_2 = 0.5$, $\mu = .65568$, $\sigma^2 = .25873$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
0.04	0.040789	0.040794	0.040811	0.040803	0.040789	0.040789	0.040789	0.040778	0.039979	0.040789	0.040789
0.08	0.083113	0.083152	0.083296	0.083224	0.083114	0.083117	0.083115	0.083020	0.079833	0.083111	0.083111
0.12	0.126904	0.127029	0.127543	0.127286	0.126908	0.126923	0.126916	0.126576	0.119442	0.126895	0.126895
0.16	0.172094	0.172371	0.173661	0.173015	0.172106	0.172155	0.172131	0.171286	0.158694	0.172063	0.172066
0.20	0.218613	0.219117	0.221779	0.220444	0.218640	0.218766	0.218703	0.216981	0.197481	0.218534	0.218542
0.24	0.266390	0.267193	0.272049	0.269610	0.266443	0.266715	0.266579	0.263482	0.235704	0.266223	0.266241
0.28	0.315355	0.316517	0.324648	0.320557	0.315447	0.315973	0.315709	0.310608	0.273270	0.315038	0.315077
0.32	0.365438	0.366996	0.379783	0.373335	0.365583	0.366520	0.366049	0.358173	0.310094	0.364885	0.364959
0.36	0.416569	0.418525	0.437692	0.428001	0.416783	0.418347	0.417559	0.405992	0.346099	0.415665	0.415796
0.40	0.468678	0.470991	0.498646	0.484621	0.468977	0.471458	0.470206	0.453880	0.381217	0.467277	0.467494
0.44	0.521700	0.524271	0.562954	0.543268	0.522094	0.525875	0.523963	0.501656	0.415384	0.519616	0.519957
0.48	0.575569	0.578230	0.630971	0.604025	0.576064	0.581634	0.578809	0.549145	0.448548	0.572575	0.573091
0.52	0.630220	0.632725	0.703095	0.666983	0.630815	0.638788	0.634732	0.596180	0.480662	0.626047	0.626796
0.56	0.685592	0.687608	0.779779	0.732244	0.686273	0.697412	0.691727	0.642600	0.511688	0.679920	0.680978
0.60	0.741627	0.742720	0.861533	0.799924	0.742363	0.757604	0.749796	0.688256	0.541594	0.734086	0.735539
0.64	0.798268	0.797901	0.948934	0.870147	0.799009	0.819484	0.808953	0.733010	0.570357	0.788434	0.790383
0.68	0.855460	0.852983	1.042632	0.943052	0.856135	0.883199	0.869216	0.776736	0.597959	0.842854	0.845416
0.72	0.913153	0.907798	1.143357	1.018792	0.913660	0.948925	0.930617	0.819319	0.624389	0.897238	0.900546
0.76	0.971298	0.962178	1.251933	1.097534	0.971504	1.016871	0.993194	0.860659	0.649642	0.951479	0.955680
0.80	1.029850	1.015955	1.369287	1.179463	1.029585	1.087281	1.056998	0.900670	0.673720	1.005474	1.010731

Table 6: Linear Failure Rate Distribution ($\theta_1 = 0.5$, $\theta_2 = 4$, $\mu = .38694$, $\sigma^2 = .05191$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
0.025	0.01499	0.01500	0.01500	0.01500	0.01499	0.01499	0.01499	0.01500	0.01489	0.01499	0.01499
0.050	0.03488	0.03500	0.03501	0.03500	0.03489	0.03489	0.03489	0.03499	0.03439	0.03489	0.03488
0.075	0.05956	0.05998	0.06004	0.06001	0.05956	0.05956	0.05956	0.05993	0.05824	0.05956	0.05956
0.100	0.08883	0.08994	0.09012	0.09003	0.08885	0.08885	0.08885	0.08977	0.08607	0.08885	0.08883
0.125	0.12249	0.12484	0.12533	0.12508	0.12253	0.12254	0.12254	0.12441	0.11750	0.12253	0.12250
0.150	0.16029	0.16463	0.16575	0.16519	0.16039	0.16041	0.16040	0.16367	0.15211	0.16037	0.16030
0.175	0.20194	0.20923	0.21155	0.21039	0.20216	0.20221	0.20218	0.20735	0.18942	0.20212	0.20198
0.200	0.24717	0.25855	0.26294	0.26073	0.24757	0.24768	0.24762	0.25516	0.22895	0.24748	0.24723
0.225	0.29564	0.31242	0.32024	0.31630	0.29634	0.29657	0.29645	0.30672	0.27021	0.29616	0.29574
0.250	0.34706	0.37067	0.38385	0.37720	0.34820	0.34863	0.34841	0.36160	0.31271	0.34787	0.34721
0.275	0.40108	0.43304	0.45434	0.44356	0.40286	0.40364	0.40325	0.41932	0.35596	0.40230	0.40130
0.300	0.45741	0.49923	0.53240	0.51555	0.46007	0.46140	0.46073	0.47931	0.39950	0.45914	0.45771
0.325	0.51572	0.56887	0.61894	0.59338	0.51956	0.52178	0.52066	0.54097	0.44289	0.51809	0.51611
0.350	0.57573	0.64153	0.71511	0.67732	0.58110	0.58467	0.58286	0.60369	0.48573	0.57887	0.57621
0.375	0.63716	0.71671	0.82232	0.76770	0.64447	0.65004	0.64720	0.66683	0.52763	0.64118	0.63772
0.400	0.69975	0.79386	0.94233	0.86491	0.70946	0.71793	0.71359	0.72977	0.56829	0.70477	0.70037
0.425	0.76327	0.87235	1.07731	0.96943	0.77589	0.78849	0.78200	0.79189	0.60741	0.76935	0.76390
0.450	0.82752	0.95154	1.22994	1.08182	0.84359	0.86194	0.85244	0.85264	0.64477	0.83469	0.82809
0.475	0.89232	1.03072	1.40347	1.20274	0.91240	0.93865	0.92496	0.91150	0.68018	0.90054	0.89271
0.500	0.95750	1.10920	1.60192	1.33298	0.98218	1.01909	0.99968	0.96803	0.71350	0.96667	0.95757

Table 7: Linear Failure Rate Distribution ($\theta_1 = 3$, $\theta_2 = 1$, $\mu = .28500$, $\sigma^2 = .06378$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
0.02	0.06039	0.06038	0.06044	0.06041	0.06039	0.06039	0.06039	0.06033	0.05861	0.06039	0.06039
0.04	0.12154	0.12145	0.12190	0.12167	0.12153	0.12155	0.12154	0.12105	0.11450	0.12152	0.12153
0.06	0.18339	0.18309	0.18463	0.18386	0.18338	0.18344	0.18341	0.18180	0.16773	0.18332	0.18334
0.08	0.24591	0.24516	0.24890	0.24702	0.24586	0.24608	0.24597	0.24224	0.21839	0.24569	0.24576
0.10	0.30904	0.30754	0.31499	0.31124	0.30894	0.30946	0.30920	0.30208	0.26655	0.30853	0.30869
0.12	0.37276	0.37009	0.38321	0.37659	0.37254	0.37363	0.37308	0.36106	0.31230	0.37173	0.37205
0.14	0.43702	0.43265	0.45390	0.44315	0.43661	0.43865	0.43762	0.41897	0.35571	0.43518	0.43575
0.16	0.50179	0.49510	0.52742	0.51100	0.50107	0.50457	0.50280	0.47560	0.39686	0.49874	0.49968
0.18	0.56703	0.55727	0.60417	0.58025	0.56587	0.57150	0.56865	0.53080	0.43583	0.56230	0.56374
0.20	0.63270	0.61901	0.68459	0.65098	0.63091	0.63956	0.63516	0.58443	0.47271	0.62573	0.62784
0.22	0.69878	0.68019	0.76915	0.72331	0.69613	0.70888	0.70237	0.63638	0.50757	0.68891	0.69186
0.24	0.76523	0.74065	0.85837	0.79734	0.76145	0.77962	0.77031	0.68656	0.54049	0.75171	0.75572
0.26	0.83203	0.80025	0.95280	0.87320	0.82679	0.85198	0.83901	0.73490	0.57156	0.81402	0.81931
0.28	0.89915	0.85886	1.05306	0.95102	0.89205	0.92617	0.90851	0.78135	0.60084	0.87571	0.88254
0.30	0.96657	0.91635	1.15983	1.03093	0.95717	1.00243	0.97887	0.82588	0.62842	0.93670	0.94531
0.32	1.03426	0.97260	1.27384	1.11307	1.02203	1.08104	1.05015	0.86848	0.65437	0.99686	1.00754
0.34	1.10219	1.02748	1.39591	1.19761	1.08657	1.16230	1.12241	0.90914	0.67877	1.05611	1.06914
0.36	1.17035	1.08091	1.52691	1.28470	1.15069	1.24656	1.19573	0.94786	0.70168	1.11436	1.13003
0.38	1.23873	1.13279	1.66784	1.37452	1.21430	1.33420	1.27018	0.98468	0.72318	1.17153	1.19014
0.40	1.30729	1.18304	1.81977	1.46726	1.27732	1.42565	1.34586	1.01962	0.74334	1.22753	1.24940

Table 8: Linear Failure Rate Distribution ($\theta_1 = 1$, $\theta_2 = 5$, $\mu = .31325$, $\sigma^2 = .03922$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
0.025	0.02810	0.02812	0.02813	0.02813	0.02810	0.02810	0.02810	0.02812	0.02773	0.02810	0.02810
0.050	0.06224	0.06248	0.06254	0.06251	0.06225	0.06225	0.06225	0.06242	0.06059	0.06225	0.06224
0.075	0.10219	0.10303	0.10331	0.10317	0.10220	0.10221	0.10220	0.10279	0.09799	0.10220	0.10219
0.100	0.14760	0.14972	0.15056	0.15014	0.14766	0.14768	0.14767	0.14899	0.13929	0.14764	0.14761
0.125	0.19813	0.20243	0.20452	0.20347	0.19828	0.19834	0.19831	0.20072	0.18382	0.19823	0.19814
0.150	0.25336	0.26100	0.26553	0.26325	0.25370	0.25386	0.25378	0.25752	0.23087	0.25357	0.25338
0.175	0.31287	0.32521	0.33404	0.32960	0.31353	0.31389	0.31371	0.31885	0.27973	0.31325	0.31289
0.200	0.37620	0.39475	0.41075	0.40267	0.37739	0.37814	0.37776	0.38402	0.32968	0.37683	0.37622
0.225	0.44292	0.46922	0.49655	0.48269	0.44488	0.44633	0.44560	0.45228	0.38006	0.44387	0.44291
0.250	0.51258	0.54812	0.59264	0.56995	0.51564	0.51826	0.51694	0.52276	0.43022	0.51391	0.51249
0.275	0.58474	0.63086	0.70056	0.66480	0.58930	0.59384	0.59154	0.59458	0.47958	0.58650	0.58451
0.300	0.65901	0.71671	0.82232	0.76770	0.66553	0.67305	0.66922	0.66683	0.52763	0.66119	0.65854
0.325	0.73501	0.80489	0.96044	0.87923	0.74400	0.75604	0.74986	0.73861	0.57392	0.73755	0.73413
0.350	0.81239	0.89449	1.11815	1.00009	0.82441	0.84311	0.83345	0.80906	0.61806	0.81513	0.81090
0.375	0.89086	0.98456	1.29947	1.13110	0.90646	0.93475	0.92002	0.87741	0.65977	0.89352	0.88846
0.400	0.97015	1.07410	1.50946	1.27331	0.98988	1.03170	1.00973	0.94297	0.69881	0.97232	0.96647
0.425	1.05003	1.16212	1.75449	1.42791	1.07439	1.13494	1.10280	1.00516	0.73503	1.05113	1.04460
0.450	1.13031	1.24766	2.04254	1.59637	1.15971	1.24580	1.19957	1.06352	0.76834	1.12959	1.12256
0.475	1.21083	1.32982	2.38363	1.78039	1.24558	1.36601	1.30050	1.11772	0.79873	1.20736	1.20007
0.500	1.29149	1.40781	2.79041	1.98201	1.33169	1.49775	1.40613	1.16755	0.82623	1.28408	1.27689

Table 9: Linear Failure Rate Distribution ($\theta_1 = 1$, $\theta_2 = 10$, $\mu = .23650$, $\sigma^2 = 0.02042$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
0.02	0.02397	0.02400	0.02400	0.02400	0.02397	0.02397	0.02397	0.02400	0.02371	0.02397	0.02397
0.04	0.05571	0.05599	0.05603	0.05601	0.05571	0.05571	0.05571	0.05594	0.05446	0.05571	0.05571
0.06	0.09488	0.09593	0.09615	0.09604	0.09490	0.09490	0.09490	0.09573	0.09154	0.09489	0.09488
0.08	0.14105	0.14375	0.14450	0.14412	0.14112	0.14113	0.14112	0.14311	0.13411	0.14110	0.14106
0.10	0.19371	0.19934	0.20134	0.20033	0.19388	0.19393	0.19391	0.19770	0.18127	0.19384	0.19373
0.12	0.25227	0.26248	0.26708	0.26477	0.25267	0.25280	0.25274	0.25894	0.23203	0.25256	0.25232
0.14	0.31611	0.33287	0.34236	0.33758	0.31691	0.31724	0.31708	0.32610	0.28538	0.31666	0.31619
0.16	0.38458	0.41010	0.42810	0.41901	0.38605	0.38677	0.38641	0.39823	0.34032	0.38552	0.38471
0.18	0.45704	0.49360	0.52561	0.50935	0.45954	0.46099	0.46026	0.47426	0.39589	0.45853	0.45723
0.20	0.53285	0.58263	0.63665	0.60904	0.53684	0.53960	0.53820	0.55297	0.45119	0.53506	0.53309
0.22	0.61142	0.67630	0.76361	0.71863	0.61748	0.62244	0.61992	0.63311	0.50540	0.61449	0.61167
0.24	0.69220	0.77355	0.90962	0.83883	0.70100	0.70954	0.70517	0.71338	0.55780	0.69624	0.69240
0.26	0.77469	0.87316	1.07878	0.97054	0.78699	0.80113	0.79385	0.79252	0.60781	0.77973	0.77471
0.28	0.85844	0.97382	1.27643	1.11490	0.87507	0.89773	0.88596	0.86939	0.65493	0.86442	0.85809
0.30	0.94310	1.07410	1.50946	1.27331	0.96490	1.00017	0.98167	0.94297	0.69881	0.94979	0.94211
0.32	1.02834	1.17259	1.78677	1.44746	1.05615	1.10967	1.08128	1.01241	0.73920	1.03534	1.02633
0.34	1.11393	1.26791	2.11989	1.63946	1.14848	1.22792	1.18525	1.07704	0.77598	1.12062	1.11038
0.36	1.19967	1.35880	2.52371	1.85181	1.24158	1.35719	1.29419	1.13642	0.80910	1.20518	1.19395
0.38	1.28543	1.44418	3.01761	2.08758	1.33512	1.50046	1.40890	1.19026	0.83862	1.28861	1.27674
0.40	1.37110	1.52319	3.62686	2.35040	1.42873	1.66160	1.53034	1.23849	0.86466	1.37053	1.35848

5.3 Pareto Distributions

Table 10: Pareto Distribution ($\theta_1 = 1$, $\theta_2 = 5$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
1	0.031301	0.031746	0.031754	0.031750	0.031301	0.031301	0.031301	0.031738	0.031250	0.031301	0.031301
2	0.134156	0.140969	0.141673	0.141321	0.134207	0.134209	0.134208	0.140358	0.131687	0.134205	0.134166
3	0.251099	0.269252	0.274222	0.271726	0.251687	0.251750	0.251719	0.265461	0.237305	0.251633	0.251238
4	0.364835	0.391887	0.407533	0.399634	0.367028	0.367505	0.367265	0.381367	0.327680	0.366658	0.365380
5	0.473332	0.502937	0.536888	0.519635	0.478109	0.479952	0.479023	0.482630	0.401878	0.476811	0.474466
6	0.577015	0.601904	0.661849	0.631165	0.584628	0.589536	0.587050	0.569694	0.462664	0.581474	0.578511
7	0.676563	0.689815	0.782957	0.734912	0.686347	0.696718	0.691438	0.644447	0.512909	0.680238	0.677642
8	0.772578	0.768030	0.900881	0.831807	0.783057	0.801844	0.792223	0.708902	0.554929	0.772867	0.771918
9	0.865548	0.837866	1.016216	0.922742	0.874650	0.905186	0.889452	0.764829	0.590490	0.859328	0.861385
10	0.955865	0.900487	1.129448	1.008491	0.961137	1.006975	0.983205	0.813693	0.620921	0.939766	0.946116
11	1.043846	0.956891	1.240959	1.089708	1.042628	1.107419	1.073602	0.856680	0.647228	1.014450	1.026226
12	1.129753	1.007919	1.351053	1.166941	1.119306	1.206701	1.160785	0.894745	0.670177	1.083722	1.101871
13	1.213800	1.054278	1.459967	1.240650	1.191402	1.304984	1.244916	0.928661	0.690362	1.147957	1.173240
14	1.296168	1.096564	1.567893	1.311219	1.259170	1.402410	1.326161	0.959052	0.708246	1.207542	1.240543
15	1.377009	1.135279	1.674983	1.378975	1.322877	1.499106	1.404688	0.986427	0.724196	1.262854	1.304002
16	1.456456	1.170849	1.781360	1.444196	1.382789	1.595181	1.480658	1.011205	0.738508	1.314252	1.363841
17	1.534620	1.203636	1.887125	1.507121	1.439164	1.690728	1.554229	1.033734	0.751419	1.362072	1.420285
18	1.611600	1.233951	1.992361	1.567953	1.492252	1.785830	1.625547	1.054301	0.763123	1.406623	1.473551
19	1.687481	1.262060	2.097137	1.626872	1.542285	1.880556	1.694752	1.073149	0.773781	1.448187	1.523850
20	1.762341	1.288192	2.201511	1.684034	1.589483	1.974967	1.761972	1.090483	0.783526	1.487023	1.571378

Table 11: Pareto Distribution ($\theta_1 = 3$, $\theta_2 = 4$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
1	0.003907	0.003914	0.003914	0.003914	0.003907	0.003907	0.003907	0.003914	0.003906	0.003907	0.003907
2	0.025645	0.025932	0.025936	0.025934	0.025645	0.025645	0.025645	0.025928	0.025600	0.025645	0.025645
3	0.062914	0.064516	0.064583	0.064550	0.062917	0.062917	0.062917	0.064453	0.062500	0.062917	0.062914
4	0.108295	0.112626	0.112985	0.112805	0.108323	0.108323	0.108323	0.112306	0.106622	0.108322	0.108301
5	0.157012	0.165191	0.166326	0.165757	0.157124	0.157130	0.157127	0.164229	0.152588	0.157118	0.157038
6	0.206635	0.219178	0.221842	0.220506	0.206943	0.206970	0.206956	0.217040	0.197531	0.206919	0.206709
7	0.256033	0.272856	0.278031	0.275432	0.256696	0.256780	0.256738	0.268924	0.240100	0.256626	0.256198
8	0.304714	0.325260	0.334096	0.329649	0.305918	0.306126	0.306022	0.318896	0.279762	0.305749	0.305017
9	0.352487	0.375870	0.389632	0.382689	0.354418	0.354861	0.354638	0.366463	0.316406	0.354073	0.352971
10	0.399303	0.424430	0.444446	0.434323	0.402120	0.402957	0.402536	0.411423	0.350128	0.401493	0.399994
11	0.445176	0.470840	0.498466	0.484456	0.448991	0.450438	0.449709	0.453742	0.381117	0.447952	0.446073
12	0.490145	0.515091	0.551683	0.533073	0.495014	0.497341	0.496167	0.493486	0.409600	0.493407	0.491213
13	0.534262	0.557228	0.604124	0.580202	0.540174	0.543706	0.541920	0.530770	0.435806	0.537826	0.535427
14	0.577581	0.597329	0.655829	0.625896	0.584456	0.589572	0.586980	0.565736	0.459956	0.581181	0.578725
15	0.620152	0.635486	0.706849	0.670218	0.627846	0.634972	0.631355	0.598537	0.482253	0.623450	0.621117
16	0.662024	0.671799	0.757237	0.713240	0.670334	0.679939	0.675052	0.629326	0.502881	0.664618	0.662612
17	0.703243	0.706371	0.807042	0.755031	0.711908	0.724500	0.718078	0.658252	0.522006	0.704674	0.703215
18	0.743848	0.739304	0.856313	0.795660	0.752562	0.768680	0.760440	0.685454	0.539775	0.743615	0.742932
19	0.783878	0.770695	0.905094	0.835195	0.792292	0.812504	0.802145	0.711064	0.556319	0.781444	0.781771
20	0.823367	0.800637	0.953428	0.873699	0.831098	0.855993	0.843201	0.735204	0.571753	0.818168	0.819738

Table 12: Pareto Distribution ($\theta_1 = 2$, $\theta_2 = 2$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
1	0.114808	0.117647	0.118056	0.117851	0.114861	0.114865	0.114863	0.117284	0.111111	0.114858	0.114822
2	0.276606	0.285714	0.291667	0.288675	0.277341	0.277561	0.277451	0.281250	0.250000	0.277162	0.276772
3	0.430591	0.439024	0.461250	0.450000	0.432639	0.434271	0.433449	0.424800	0.360000	0.431471	0.430748
4	0.574390	0.571429	0.622222	0.596285	0.577027	0.582796	0.579873	0.543210	0.444444	0.573364	0.573351
5	0.709723	0.684932	0.775935	0.729015	0.710641	0.724667	0.717509	0.640358	0.510204	0.702670	0.705132
6	0.838237	0.782609	0.924107	0.850420	0.833967	0.861329	0.847257	0.720703	0.562500	0.819925	0.826832
7	0.961184	0.867257	1.068094	0.962451	0.947595	0.993897	0.969892	0.787913	0.604938	0.925977	0.939151
8	1.079496	0.941176	1.208889	1.066667	1.052196	1.123227	1.086096	0.844800	0.640000	1.021818	1.042772
9	1.193876	1.006211	1.347211	1.164293	1.148478	1.249980	1.196477	0.893484	0.669421	1.108475	1.138372
10	1.304869	1.063830	1.483586	1.256297	1.237151	1.374673	1.301581	0.935571	0.694444	1.186939	1.226616
11	1.412904	1.115207	1.618405	1.343450	1.318899	1.497717	1.401896	0.972287	0.715976	1.258127	1.308136
12	1.518325	1.161290	1.751962	1.426372	1.394364	1.619434	1.497856	1.004581	0.734694	1.322865	1.383532
13	1.621416	1.202847	1.884484	1.505572	1.464136	1.740086	1.589850	1.033195	0.751111	1.381888	1.453356
14	1.722411	1.240506	2.016146	1.581468	1.528756	1.859881	1.678219	1.058716	0.765625	1.435844	1.518117
15	1.821505	1.274788	2.147086	1.654412	1.588709	1.978986	1.763269	1.081614	0.778547	1.485299	1.578276
16	1.918868	1.306122	2.277415	1.724698	1.644432	2.097539	1.845273	1.102271	0.790123	1.530747	1.634253
17	2.014641	1.334873	2.407221	1.792578	1.696318	2.215652	1.924470	1.120997	0.800554	1.572622	1.686423
18	2.108950	1.361345	2.536579	1.858267	1.744718	2.333415	2.001078	1.138050	0.810000	1.611300	1.735127
19	2.201904	1.385797	2.665547	1.921954	1.789944	2.450903	2.075289	1.153642	0.818594	1.647111	1.780668
20	2.293597	1.408451	2.794176	1.983799	1.832278	2.568177	2.147276	1.167953	0.826446	1.680345	1.823322

5.4 Weibull Distributions

Table 13: Weibull Distribution ($\theta = 0.5$, $\mu = 2$, $\sigma^2 = 20$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
0.2	0.50636	0.43991	0.46227	0.45095	0.48975	0.49484	0.49227	0.42561	0.36059	0.48637	0.49489
0.4	0.75242	0.61218	0.67547	0.64305	0.70606	0.72527	0.71546	0.57856	0.46871	0.69529	0.71591
0.6	0.95644	0.73806	0.85441	0.79411	0.87225	0.91367	0.89230	0.68443	0.53911	0.85192	0.88553
0.8	1.13898	0.83921	1.01855	0.92454	1.01079	1.08188	1.04484	0.76589	0.59116	0.97957	1.02637
1.0	1.30798	0.92423	1.17520	1.04219	1.13070	1.23851	1.18178	0.83191	0.63212	1.08782	1.14770
1.2	1.46748	0.99762	1.32806	1.15104	1.23680	1.38803	1.30768	0.88713	0.66561	1.18182	1.25451
1.4	1.61986	1.06210	1.47928	1.25345	1.33205	1.53311	1.42526	0.93432	0.69371	1.26475	1.34992
1.6	1.76668	1.11948	1.63026	1.35094	1.41845	1.67555	1.53635	0.97531	0.71774	1.33879	1.43606
1.8	1.90900	1.17104	1.78195	1.44455	1.49744	1.81658	1.64220	1.01134	0.73858	1.40548	1.51444
2.0	2.04762	1.21772	1.93507	1.53505	1.57008	1.95713	1.74375	1.04332	0.75688	1.46598	1.58623
2.2	2.18310	1.26025	2.09015	1.62299	1.63722	2.09793	1.84171	1.07194	0.77310	1.52119	1.65233
2.4	2.31589	1.29919	2.24763	1.70883	1.69953	2.23953	1.93662	1.09772	0.78758	1.57182	1.71346
2.6	2.44636	1.33501	2.40785	1.79290	1.75755	2.38241	2.02893	1.12108	0.80060	1.61843	1.77019

Table 14: Weibull Distribution ($\theta = 0.75$, $\mu = 1.19064$, $\sigma^2 = 2.59458$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
0.1	0.18206	0.17736	0.17877	0.17806	0.18169	0.18178	0.18174	0.17618	0.16291	0.18161	0.18186
0.2	0.31097	0.29686	0.30355	0.30019	0.30916	0.30989	0.30952	0.29190	0.25849	0.30860	0.30975
0.3	0.42708	0.39990	0.41655	0.40814	0.42254	0.42492	0.42372	0.38880	0.33326	0.42086	0.42361
0.4	0.53622	0.49263	0.52445	0.50829	0.52748	0.53299	0.53020	0.47339	0.39527	0.52391	0.52896
0.5	0.64082	0.57768	0.63027	0.60340	0.62629	0.63680	0.63146	0.54867	0.44822	0.61999	0.62802
0.6	0.74218	0.65650	0.73578	0.69501	0.72016	0.73796	0.72886	0.61641	0.49426	0.71027	0.72193
0.7	0.84109	0.73000	0.84220	0.78410	0.80982	0.83755	0.82330	0.67780	0.53480	0.79551	0.81137
0.8	0.93808	0.79882	0.95045	0.87134	0.89574	0.93638	0.91537	0.73375	0.57083	0.87620	0.89681
0.9	1.03352	0.86344	1.06124	0.95725	0.97825	1.03510	1.00552	0.78493	0.60308	0.95273	0.97855
1.0	1.12769	0.92423	1.17520	1.04219	1.05758	1.13428	1.09411	0.83191	0.63212	1.02538	1.05687
1.1	1.22078	0.98150	1.29288	1.12648	1.13393	1.23440	1.18142	0.87514	0.65839	1.09442	1.13194
1.2	1.31297	1.03551	1.41476	1.21037	1.20743	1.33591	1.26768	0.91500	0.68226	1.16005	1.20396
1.3	1.40438	1.08647	1.54132	1.29406	1.27822	1.43921	1.35310	0.95184	0.70402	1.22248	1.27306

Table 15: Weibull Distribution ($\theta = 2$, $\mu = .88623$, $\sigma^2 = .21460$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
0.05	0.002498	0.002500	0.002500	0.002500	0.002498	0.002498	0.002498	0.002500	0.002497	0.002498	0.002498
0.10	0.009967	0.010000	0.010000	0.010000	0.009967	0.009967	0.009967	0.010000	0.009950	0.009967	0.009967
0.15	0.022333	0.022499	0.022502	0.022500	0.022333	0.022333	0.022333	0.022496	0.022249	0.022333	0.022333
0.20	0.039474	0.039995	0.040011	0.040003	0.039475	0.039475	0.039475	0.039979	0.039211	0.039475	0.039474
0.25	0.061225	0.062480	0.062541	0.062510	0.061230	0.061230	0.061230	0.062422	0.060587	0.061230	0.061226
0.30	0.087379	0.089939	0.090122	0.090030	0.087395	0.087395	0.087395	0.089773	0.086069	0.087394	0.087383
0.35	0.117696	0.122347	0.122807	0.122577	0.117735	0.117737	0.117736	0.121940	0.115294	0.117733	0.117705
0.40	0.151903	0.159660	0.160684	0.160171	0.151988	0.151994	0.151991	0.158787	0.147856	0.151983	0.151923
0.45	0.189707	0.201811	0.203887	0.202846	0.189876	0.189891	0.189883	0.200115	0.183314	0.189863	0.189746
0.50	0.230794	0.248706	0.252612	0.250652	0.231106	0.231140	0.231123	0.245664	0.221199	0.231077	0.230866
0.55	0.274843	0.300214	0.307135	0.303655	0.275383	0.275456	0.275420	0.295100	0.261032	0.275322	0.274965
0.60	0.321526	0.356162	0.367827	0.361947	0.322415	0.322561	0.322488	0.348023	0.302324	0.322297	0.321723
0.65	0.370520	0.416325	0.435182	0.425649	0.371917	0.372194	0.372055	0.403966	0.344594	0.371704	0.370822
0.70	0.421508	0.480426	0.509845	0.494917	0.423623	0.424121	0.423870	0.462403	0.387374	0.423255	0.421954
0.75	0.474186	0.548123	0.592636	0.569945	0.477283	0.478145	0.477710	0.522761	0.430217	0.476675	0.474823
0.80	0.528267	0.619014	0.684594	0.650979	0.532673	0.534110	0.533383	0.584434	0.472708	0.531705	0.529148
0.85	0.583488	0.692629	0.787019	0.738317	0.589592	0.591914	0.590737	0.646799	0.514463	0.588106	0.584671
0.90	0.639605	0.768438	0.901525	0.832326	0.647867	0.651516	0.649659	0.709233	0.555142	0.645654	0.641154
0.95	0.696403	0.845852	1.030103	0.933442	0.707350	0.712943	0.710085	0.771128	0.594445	0.704146	0.698382
1.00	0.753691	0.924234	1.175201	1.042191	0.767917	0.776300	0.771996	0.831909	0.632121	0.763394	0.756164

Table 16: Weibull Distribution ($\theta = 5$, $\mu = .91817$, $\sigma^2 = 0.04423$)

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
0.05	0.002498	0.002500	0.002500	0.002500	0.002498	0.002498	0.002498	0.002500	0.002497	0.002498	0.002498
0.10	0.009967	0.010000	0.010000	0.010000	0.009967	0.009967	0.009967	0.010000	0.009950	0.009967	0.009967
0.15	0.022333	0.022499	0.022502	0.022500	0.022333	0.022333	0.022333	0.022496	0.022249	0.022333	0.022333
0.20	0.039474	0.039995	0.040011	0.040003	0.039475	0.039475	0.039475	0.039979	0.039211	0.039475	0.039474
0.25	0.061225	0.062480	0.062541	0.062510	0.061230	0.061230	0.061230	0.062422	0.060587	0.061230	0.061226
0.30	0.087379	0.089939	0.090122	0.090030	0.087395	0.087395	0.087395	0.089773	0.086069	0.087394	0.087383
0.35	0.117696	0.122347	0.122807	0.122577	0.117735	0.117737	0.117736	0.121940	0.115294	0.117733	0.117705
0.40	0.151903	0.159660	0.160684	0.160171	0.151988	0.151994	0.151991	0.158787	0.147856	0.151983	0.151923
0.45	0.189707	0.201811	0.203887	0.202846	0.189876	0.189891	0.189883	0.200115	0.183314	0.189863	0.189746
0.50	0.230794	0.248706	0.252612	0.250652	0.231106	0.231140	0.231123	0.245664	0.221199	0.231077	0.230866
0.55	0.274843	0.300214	0.307135	0.303655	0.275383	0.275456	0.275420	0.295100	0.261032	0.275322	0.274965
0.60	0.321526	0.356162	0.367827	0.361947	0.322415	0.322561	0.322488	0.348023	0.302324	0.322297	0.321723
0.65	0.370520	0.416325	0.435182	0.425649	0.371917	0.372194	0.372055	0.403966	0.344594	0.371704	0.370822
0.70	0.421508	0.480426	0.509845	0.494917	0.423623	0.424121	0.423870	0.462403	0.387374	0.423255	0.421954
0.75	0.474186	0.548123	0.592636	0.569945	0.477283	0.478145	0.477710	0.522761	0.430217	0.476675	0.474823
0.80	0.528267	0.619014	0.684594	0.650979	0.532673	0.534110	0.533383	0.584434	0.472708	0.531705	0.529148
0.85	0.583488	0.692629	0.787019	0.738317	0.589592	0.591914	0.590737	0.646799	0.514463	0.588106	0.584671
0.90	0.639605	0.768438	0.901525	0.832326	0.647867	0.651516	0.649659	0.709233	0.555142	0.645654	0.641154
0.95	0.696403	0.845852	1.030103	0.933442	0.707350	0.712943	0.710085	0.771128	0.594445	0.704146	0.698382
1.00	0.753691	0.924234	1.175201	1.042191	0.767917	0.776300	0.771996	0.831909	0.632121	0.763394	0.756164

6 Data Application

While numerical results from distributions are important for observing the performance of the approximations, it is desirable to apply the approximations to something more realistic. Given a random sample $X_1, X_2, \ldots X_n$ from an unknown PDF, which in this case could be the lifetime of n objects, we can approximate $M(t)$ as well. It is of interest to observe $M(t)$ at some time instance t_0 where $0 \leq t_0 < \mu$. Since, $f(t)$ is undefined we can estimate μ through the estimator $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$. It is expected that the approximations will work best for $t_0 < \mu$, and this should hold if $t_0 < \bar{X}$ because $\bar{X}$ is a consistent estimator of μ.

To abridge the approximations developed, we consider the random variable Y which will represent the number of elements in the sample where the value of $X_i \leq t_0$. Here, Y is binomial with n trials, and $P = \int_0^{t_0} f(t)dt = F(t_0)$. An estimator for P that we will use is $\hat{P} = \frac{Y}{n}$, which can be substituted into the existing approximations as an estimate of the CDF. For the approximations that use integration, it is necessary to not only to discretize, but also, we require an estimator for $f(t_0)$. We use $\frac{1}{n}$ which is adequate for order statistics. After substitutions, the approximations become:

$$
\begin{aligned}
M_A(t_0) &= \frac{\hat{P}(t_0)}{1 - \frac{1}{2}\hat{P}(t_0)} \\
M_B(t_0) &= \frac{1}{2}\left(\hat{P}(t_0) + \frac{\hat{P}(t_0)}{1 - \hat{P}(t_0)}\right) \\
M_C(t_0) &= \frac{\hat{P}(t_0)}{\sqrt{1 - \hat{P}(t_0)}} \\
M_D(t_0) &= \hat{P}(t_0) + \sum_{i=1}^{Y} \frac{\hat{P}(t_0 - X_i)}{1 - \frac{1}{2}\hat{P}(t_0 - X_i)} \cdot \frac{1}{n} \\
M_E(t_0) &= \hat{P}(t_0) + \sum_{i=1}^{Y} \frac{1}{2}\left(\hat{P}(t_0 - X_i) + \frac{\hat{P}(t_0 - X_i)}{1 - \hat{P}(t_0 - X_i)}\right) \cdot \frac{1}{n} \\
M_F(t_0) &= \hat{P}(t_0) + \sum_{i=1}^{Y} \frac{\hat{P}(t_0 - X_i)}{\sqrt{1 - \hat{P}(t_0 - X_i)}} \cdot \frac{1}{n}
\end{aligned}
$$

$$
\begin{aligned}
M_G(t_0) &= \hat{P}(t_0) + \frac{1}{2}\hat{P}(t_0)^2 \\
M_H(t_0) &= \hat{P}(t_0)\left(1 + \hat{P}\left(\frac{t_0}{2}\right)\right) \\
M_I(t_0) &= \hat{P}(t_0) + \sum_{i=1}^{Y} \hat{P}(t_0 - X_i) + \frac{1}{2}\hat{P}(t_0 - X_i)^2 \cdot \frac{1}{n} \\
M_J(t_0) &= \hat{P}(t_0) + \sum_{i=1}^{Y} \hat{P}(t_0 - X_i)\left(1 + \hat{P}\left(\frac{(t_0 - X_i)}{2}\right)\right) \cdot \frac{1}{n}
\end{aligned}
$$

The approximation techniques were applied to a well known ball bearing failure time data set as described in *Reliability: Probabilistic Models and Statistical Models* (Leemis, 1995). The sample has size $n = 23$ and sample mean $\bar{X} = 72.22$. The sample data is:

17.88	28.92	33.00	41.52	42.14
45.60	48.48	51.84	51.96	54.12
55.56	67.80	68.64	68.64	68.88
84.12	93.12	98.64	105.12	105.84
127.92	128.04	173.40		

Results are provided with comparisons to Xie's approximation:

Table 17: Ball Bearing Failures

Time	Xie Approx.	A	B	C	D	E	F	G	H	I	J
15	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
20	0.04545	0.04444	0.04447	0.04446	0.04348	0.04348	0.04348	0.04442	0.04348	0.04348	0.04348
25	0.04545	0.04444	0.04447	0.04446	0.04348	0.04348	0.04348	0.04442	0.04348	0.04348	0.04348
30	0.09524	0.09091	0.09110	0.09100	0.08696	0.08696	0.08696	0.09074	0.08696	0.08696	0.08696
35	0.15000	0.13953	0.14022	0.13988	0.13043	0.13043	0.13043	0.13894	0.13043	0.13043	0.13043
40	0.15000	0.13953	0.14022	0.13988	0.13237	0.13237	0.13237	0.13894	0.13043	0.13237	0.13241
45	0.27778	0.24390	0.24758	0.24574	0.21932	0.21932	0.21932	0.24102	0.21739	0.21932	0.21936
50	0.43750	0.35897	0.37092	0.36490	0.31023	0.31024	0.31024	0.35066	0.30435	0.31022	0.31043
55	0.76923	0.55556	0.60201	0.57831	0.44471	0.44475	0.44473	0.52930	0.43478	0.44469	0.44514
60	0.91667	0.62857	0.69746	0.66212	0.49436	0.49445	0.49440	0.59263	0.47826	0.49429	0.49519
65	0.91667	0.62857	0.69746	0.66212	0.50712	0.50746	0.50729	0.59263	0.47826	0.50686	0.50900
70	1.87500	0.96774	1.26359	1.10581	0.69705	0.69845	0.69774	0.86484	0.65217	0.69619	0.70050
75	1.87500	0.96774	1.26359	1.10581	0.72446	0.72788	0.72613	0.86484	0.65217	0.72257	0.73058
80	1.87500	0.96774	1.26359	1.10581	0.73804	0.74195	0.73995	0.86484	0.65217	0.73580	0.74571
85	2.28571	1.06667	1.49068	1.26098	0.82484	0.83262	0.82864	0.93762	0.69565	0.82054	0.83768
90	2.28571	1.06667	1.49068	1.26098	0.87192	0.89195	0.88143	0.93762	0.69565	0.86344	0.88946
95	2.83333	1.17241	1.78623	1.44714	0.94789	0.97061	0.95872	1.01229	0.73913	0.93778	0.97008
100	3.60000	1.28571	2.19130	1.67851	1.05406	1.09240	1.07223	1.08885	0.78261	1.03809	1.08375

7 Conclusion

In terms of the approximations applied to defined distributions, the results of the numerical examples suggest that there isn't a clear best approximation method. The results suggest that the best approximation methods are dependent on the distribution used as well as what parameters are chosen; however, we can make some overarching comments about the results that hold consistently. Approximations D, E, F, I, and J will outperform the other approximations in most cases for $t < \mu$, and this was expected because Approximations A, B, C, G, and H are preliminary results necessary to make the renewal equation non-recursive.

We can make some conclusions about specific results. For the Gamma distributions, it appears that Approximations I and J (especially J) provide the least error for $\alpha > 1$, but the same cannot be said about $\alpha \leq 1$. The best approximations in this case range from Approximations D, E, or F. The same rule appears to apply to Weibull distributions based on the parameter θ. Definitively, Approximation D has the best results for any Linear Failure Rate distribution, and Approximations I and J follow. The best approximation approach for the Pareto distribution is not clear; it is dependent on what parameters are used.

While the approximations (especially the ones involving integration) provided generally acceptable results for the distributions for time values less than the mean, the same cannot be said when they are applied to data. Error from these results were higher, but some of the error was to be expected from the sample chosen. The discretization of the integrals in Approximations D, E, F, I, and J especially hindered their accuracy. Choosing a significantly larger sample would possibly allow these results to resemble the accepted value. For the chosen sample, time values $t < 45$ had somewhat acceptable results.

Further avenues will be pursued to improve upon what has already been derived here. One avenue, which hasn't been quite too successful yet, is to use a regression approach to determine the value of a constant to be used in the derived approximations such as in Approximation G where $\frac{1}{2}$ is replaced by a constant c: $M(t) = F(t) + c(F(t))^2$.

The goal is to make the value of c be a function dependent on time and the distribution rather than $\frac{1}{2}$. Future direction includes improving the approximations as well as looking toward finding simple approximations that will perform favorably for values in the close neighborhood of the mean and above the mean.

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