

3-2010

## When is the numerical range of a nilpotent matrix circular?

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### Recommended Citation

Matache, Valentin and Matache, Mihaela Teodora, "When is the numerical range of a nilpotent matrix circular?" (2010). *Mathematics Faculty Publications*. 24.  
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# WHEN IS THE NUMERICAL RANGE OF A NILPOTENT MATRIX CIRCULAR?

MIHAELA T. MATACHE AND VALENTIN MATACHE

ABSTRACT. The problem formulated in the title is investigated. The case of nilpotent matrices of size at most 4 allows a unitary treatment. The numerical range of a nilpotent matrix  $M$  of size at most 4 is circular if and only if the traces  $\operatorname{tr}M^*M^2$  and  $\operatorname{tr}M^*M^3$  are null. The situation becomes more complicated as soon as the size is 5. The conditions under which a  $5 \times 5$  nilpotent matrix has circular numerical range are thoroughly discussed.

2000 *Mathematics Subject Classification.* Primary 15A60, Secondary 47A12

*Keywords.* nilpotent matrix, numerical range.

## 1. INTRODUCTION

A nilpotent operator is a linear operator  $T$  with the property  $T^n = 0$  for some positive integer  $n$ . We say  $n$  is the *order* of  $T$  if  $n$  is the least positive integer with the property  $T^n = 0$ .

Recall that the numerical range  $W(T)$  of a bounded operator  $T$  acting on some complex Hilbert space is the image of the unit sphere of that space under the quadratic form associated to  $T$ , that is  $W(T) = \{\langle Tx, x \rangle : \|x\| = 1\}$ . One of the best known properties of numerical ranges is the so called Toeplitz–Hausdorff theorem (see [4, Ch. 22]), saying that numerical ranges are convex subsets of the complex plane. The supremum  $w(T)$  of the absolute values of the complex numbers in  $W(T)$  is called the numerical radius of  $T$ . The basic facts of the theory of numerical ranges are contained in [3] and [4].

There are several extensions of the notion of numerical range. Some are interesting both in their own right and for the sake of their applications (for instance to quantum information theory, [9]). The numerical ranges of nilpotent operators are insufficiently known as one can deduce from the recent paper [5] where the author uses dilation theory arguments to state and prove that Hilbert space nilpotent operators always have circular numerical ranges. That proof is faulty since even  $3 \times 3$  nilpotent matrices can have non-circular numerical ranges (see [6] for examples of such matrices). Thus it seems both interesting and useful investigating when a  $n \times n$  nilpotent matrix with complex entries has circular numerical range. Here are the first remarks on that issue.

**Remark 1.** *A  $2 \times 2$  nilpotent matrix  $M$  always has circular numerical range. The disk  $W(M)$  is centered at the origin and its radius can be calculated with the formula  $w(M) = \sqrt{\operatorname{tr}(M^*M)}/2$ .*

Indeed, one of the most popular theorems on numerical ranges, the *elliptic range theorem*, says that the numerical range of a  $2 \times 2$  matrix with complex entries is an elliptical disk (possibly degenerate, that is possibly reduced to its focal axis), whose foci are the eigenvalues of that matrix. So, if the matrix is nilpotent, those

eigenvalues equal 0, hence the numerical range is a circular disk centered at the origin (possibly reduced to its center, if we consider the null matrix). The formulas for the semi-axes of the elliptical disk (see [3] or [4]), lead to the equality  $w(M) = \sqrt{\operatorname{tr}(M^*M)}/2$ .

The situation when  $3 \times 3$  nilpotent matrices have circular numerical ranges is completely described in [6, Theorem 4.1], according to which:

**Remark 2.** *A  $3 \times 3$  nilpotent matrix  $M$  has circular numerical range if and only if*

$$(1) \quad \operatorname{tr}(M^*M^2) = 0,$$

*in which case, the numerical radius is computable with the formula*

$$w(M) = \frac{\sqrt{\operatorname{tr}(M^*M)}}{2}.$$

The case of  $4 \times 4$  matrices is covered by [2]. Indeed, a consequence of [2, Corollaries 5 and 6] is the following:

**Remark 3.** *A  $4 \times 4$  nilpotent matrix  $M$  has circular numerical range if and only if*

$$(2) \quad \operatorname{tr}(M^*M^2) = 0 \quad \text{and} \quad \operatorname{tr}(M^*M^3) = 0,$$

*in which case, the numerical radius is computable with the formula*

$$w(M) = \sqrt{\frac{\operatorname{tr}(M^*M) + \sqrt{(\operatorname{tr}(M^*M))^2 - 64 \det(\Re(M))}}{8}}$$

where  $\Re(M) = (1/2)(M + M^*)$  denotes the real part of  $M$ . In all the cases above, the circular disks are centered at the origin. Hence:

**Remark 4.** *A nilpotent  $k \times k$  matrix of size  $2 \leq k \leq 4$  has circular numerical range if and only if*

$$(3) \quad \operatorname{tr}(M^*M^n) = 0, \quad n = 2, 3, 4, 5, \dots$$

If the size of a nilpotent matrix  $M$  is  $k$ , then clearly  $M^n = 0$  if  $n \geq k$ , so the only interesting values of  $n$  in condition (3) are those between 2 and  $k$ . The first reaction is to call (3) the *trace condition* and try to prove that it is the if and only if characterization of the situation when a nilpotent matrix with complex entries has circular numerical range. As we show in the next section, this is not true, and counterexamples can be produced using  $5 \times 5$  matrices. After considering such counterexamples, we completely characterize the situation when a nilpotent  $5 \times 5$  matrix has circular numerical range. This (Theorem 2) is the main result of the current paper. After proving it, we discuss thoroughly when a  $5 \times 5$  nilpotent matrix satisfying, respectively non-satisfying, the trace condition (3) has circular numerical range.

## 2. THE MAIN RESULTS

We begin by noting that the trace condition is not necessary for the circularity of the numerical range of a  $5 \times 5$  matrix.

**Example 1.** *The block-diagonal matrix*

$$M = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

where

$$B_1 = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

has circular numerical range if  $|a|$  is large enough, but  $\text{tr}(M^*M^2) = 1 \neq 0$ .

If  $|a|$  is large enough, then  $W(B_2) \subseteq W(B_1)$  since  $W(B_2)$  is bounded and  $W(B_1)$  is the circular disk centered at the origin and having radius  $|a|/2$  (by Remark 1). In that case, one has that  $W(M) = W(B_1)$  because  $W(M)$  is the convex hull of the union  $W(B_1) \cup W(B_2)$  (a fact valid for any block-diagonal matrix).

In the case of  $5 \times 5$  nilpotent matrices, the trace condition is not sufficient for the circularity of the numerical range. Indeed:

**Example 2.** *The matrix*

$$(4) \quad M = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

satisfies the trace condition, but its numerical range exhibits a flatness on the boundary.

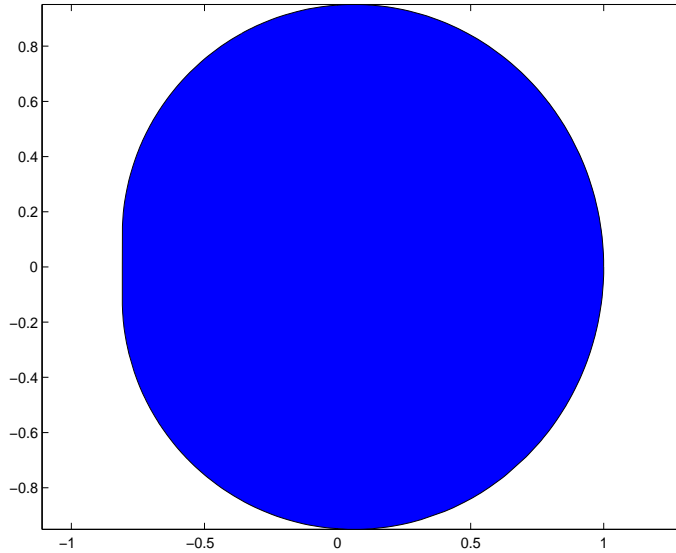


Figure 1. Numerical Range of M.

It is easy to check that the least eigenvalue of  $\Re M$ , the real part of  $M$ , is  $-(\sqrt{5} + 1)/4$ . The eigenspace corresponding to it is 2-dimensional and the compression of the imaginary part  $\Im M = (1/2i)(M - M^*)$  of  $M$  to that subspace is easily seen to be a matrix other than a scalar multiple of the identity, thus  $W(\Im M)$  is an interval of the real axis, non-reduced to a point. These facts combine into showing that the boundary of  $W(M)$  has a flatness along the line  $\Re z = -(\sqrt{5} + 1)/4$ , which, of course, establishes the non-circularity of  $W(M)$ . Recently, the existence of a

flatness on the boundary of numerical ranges started to be seriously investigated. Reference [1] is an interesting source for such results.

Our main goal in this section is to characterize the situation when a  $5 \times 5$  nilpotent matrix has circular numerical range. To reach it, we need to review first a theorem of R. Kippenhahn, [7]. The reader is also referred to [8] for a recent English translation of reference [7].

**Theorem 1** ([8, Theorem 10]). *Let  $M$  be any square matrix with complex entries. Its numerical range  $W(M)$  is the convex hull of the linear envelope of the curve having equation  $p_M(x, y) = 0$ , where*

$$(5) \quad p_M(x, y) = \det(x\Re M + y\Im M + I).$$

**Lemma 1.** *Let  $M$  be an  $n \times n$  nilpotent matrix with complex entries. If we denote  $z = x + iy$ , then equation  $p_M(x, y) = 0$  has the following form*

$$(6) \quad 1 + \sum_{l=1}^k A_l |z|^{2l} + \sum_{l=1}^k \left( \sum_{j=1}^{n-2l} (B_{lj} z^j + \overline{B_{lj} z^j}) \right) |z|^{2l} = 0 \quad \text{if } n = 2k + 1$$

$k = 1, 2, 3, \dots$  respectively

$$(7) \quad 1 + \sum_{l=1}^k A_l |z|^{2l} + \sum_{l=1}^{k-1} \left( \sum_{j=1}^{n-2l} (B_{lj} z^j + \overline{B_{lj} z^j}) \right) |z|^{2l} = 0 \quad \text{if } n = 2k,$$

$k = 2, 3, \dots$  where the coefficients  $A_l$  are real and  $B_{lj}$  complex.

*Proof.* The matrix  $M$  is unitarily equivalent to a nilpotent upper triangular matrix that is to a matrix of the form

$$(8) \quad N = \begin{bmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & a_{(n-1)n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Given that  $M$  and  $N$  are unitarily equivalent, it follows that  $p_M(x, y) = p_N(x, y)$ . On the other hand, with the notation  $z = x + iy$ , one gets that

$$p_N(x, y) = \begin{vmatrix} 1 & \frac{a_{12}\bar{z}}{2} & \frac{a_{13}\bar{z}}{2} & \dots & \frac{a_{1n}\bar{z}}{2} \\ \frac{a_{12}z}{2} & 1 & \frac{a_{23}\bar{z}}{2} & \dots & \frac{a_{2n}\bar{z}}{2} \\ \frac{a_{13}z}{2} & \frac{a_{23}z}{2} & 1 & \dots & \frac{a_{3n}\bar{z}}{2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \frac{a_{1n}z}{2} & \frac{a_{2n}z}{2} & \frac{a_{3n}z}{2} & \dots & 1 \end{vmatrix}.$$

Based on the equality above, one deduces immediately that  $p_M(x, y) = 0$  has the form described in equations (6) and (7).  $\square$

**Proposition 1.** *If the numerical range of a nilpotent matrix  $M$  is circular, then the center of that disk is necessarily the origin.*

This is an immediate consequence of the fact that the real foci of the characteristic curve of the matrix  $M$  are the eigenvalues of  $M$ . See [8, Theorem 11].

Let  $r$  be a positive number and  $k$  an integer larger than or equal to 2. Consider a nilpotent  $n \times n$  matrix  $M$  and the coefficients  $A_l$  and  $B_{lj}$  involved in equation (6) or (7). Let  $K = k$  if  $n = 2k + 1$  and  $K = k - 1$  if  $n = 2k$ . Construct the matrices:

$$A = [A_1 \quad A_2 \quad \dots \quad A_k],$$

$$B = (b_{lj})_{l=1, \dots, K, j=1, \dots, (n-2)} \quad b_{lj} = B_{lj} \quad \text{if } j \leq n-2l, \quad b_{lj} = 0 \quad \text{otherwise,}$$

$$R_1 = \begin{bmatrix} r^{2(k-1)} \\ r^{2(k-2)} \\ \dots \\ r^2 \\ 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} r^{2(k-2)} \\ r^{2(k-3)} \\ \dots \\ r^2 \\ 1 \end{bmatrix}.$$

**Proposition 2.** *Under the assumptions above, if  $W(M) = \{z \in \mathbb{C} : |z| \leq r\}$ ,  $r > 0$ , then necessarily*

$$(9) \quad r^{2k} + AR_1 = 0$$

and

$$(10) \quad B^T R_1 = 0 \quad \text{when } n = 2k+1, \quad \text{respectively } B^T R_2 = 0 \quad \text{when } n = 2k,$$

where  $B^T$  is the transpose of  $B$ .

*Proof.* Let  $n = 2k + 1$ . Since  $W(T)$  is the circle of radius  $r$  centered at the origin and that circle is the linear envelope of the circle of radius  $1/r$  centered at the origin, it follows that substitution of  $z$  by  $(1/r)e^{i\theta}$  in (6) should produce the null trigonometric polynomial. The constant term of that trigonometric polynomial is  $r^{2k} + AR_1$ , so (9) follows. Beside the constant term, the trigonometric polynomial under discussion contains a linear combination of the functions  $e^{\pm i\theta}$ ,  $e^{\pm i2\theta}$ ,  $\dots$ ,  $e^{\pm i(n-2)\theta}$ , which are orthogonal in  $L^2_{[-\pi, \pi]}$ , for which reason their coefficients must be null. The coefficients of  $e^{i\theta}$ ,  $e^{i2\theta}$ ,  $\dots$ ,  $e^{i(n-2)\theta}$  are

$$\frac{1}{r} \left( \frac{1}{r^2} B_{11} + \frac{1}{r^4} B_{21} + \dots + \frac{1}{r^{2k}} B_{k1} \right)$$

$$\frac{1}{r^2} \left( \frac{1}{r^2} B_{12} + \frac{1}{r^4} B_{22} + \dots + \frac{1}{r^{2(k-1)}} B_{(k-1)2} \right)$$

...

$$\frac{1}{r^{n-2}} \left( \frac{1}{r^2} B_{1(n-2)} \right),$$

respectively. The fact that the above coefficients are null is equivalent to the first matrix equation in (10). The case  $n = 2k$  can be treated in an identical way.  $\square$

We are now ready to describe exactly when the numerical range of a  $5 \times 5$  nilpotent matrix is circular. To that aim let  $z = x + iy$  and  $M$  be any  $5 \times 5$  nilpotent matrix. Then, according to Lemma 1,  $p_M(x, y)$  has the following form

$$(11) \quad p_M(x, y) = 1 + A|z|^2 + B|z|^4 + C|z|^2 z + \bar{C}|z|^2 \bar{z} +$$

$$D|z|^2 z^2 + \bar{D}|z|^2 \bar{z}^2 + E|z|^2 z^3 + \bar{E}|z|^2 \bar{z}^3 + F|z|^4 z + \bar{F}|z|^4 \bar{z}.$$

With this notation we prove:

**Theorem 2.** *Let  $M$  be an arbitrary  $5 \times 5$  nilpotent matrix. In order that  $W(M)$  be circular, it is necessary that we have  $D = E = 0$ . If in addition  $C = 0$ , then  $W(M)$  is circular if and only if  $F = 0$  and the numerical radius is the largest positive root of the equation*

$$(12) \quad x^4 + Ax^2 + B = 0 \quad \text{that is} \quad w(M) = \sqrt{\frac{-A + \sqrt{A^2 - 4B}}{2}}.$$

*If  $D = E = 0$ , but  $C \neq 0$ , then the numerical range is circular if and only if the following conditions hold*

$$(13) \quad F/C < 0, \quad 2|C| \leq (2 - AC/F)/(2(-C/F)^{3/2}), \quad \text{and} \quad F^2 - AFC + BC^2 = 0.$$

*In that case, the numerical radius can be calculated with the formula  $w(M) = \sqrt{-F/C}$ .*

*Proof.* If  $W(M)$  is the circular disk of radius  $r > 0$  centered at the origin, then conditions (9) and (10) lead, after a straightforward computation, to  $D = E = 0$ ,

$$(14) \quad r^4 + Ar^2 + B = 0,$$

and

$$(15) \quad Cr^2 + F = 0.$$

If  $C = 0$ , then condition (15) implies that  $F$  must be null if  $W(M)$  is circular. Conversely, if  $D = E = C = F = 0$ , then  $W(M)$  is circular because  $p_M(x, y)$  has the form  $p_M(x, y) = 1 + A|z|^2 + B|z|^4$ , hence the envelope of  $p_A(x, y) = 0$  consists of two concentric circles whose radii satisfy equation (12). The numerical radius is therefore the largest root of that equation.

Let us address the case when  $C \neq 0$ . In order that  $W(M)$  be circular, it is necessary that  $D = E = 0$ .

In that case, if  $W(M)$  is circular then, by (15), the numerical radius is necessarily  $w(M) = \sqrt{-F/C}$  and  $r = \sqrt{-F/C}$  must satisfy equation (12), hence  $F^2 - AFC + BC^2 = 0$ . Conversely, if  $F^2 - AFC + BC^2 = 0$ , then let  $z = x + iy$ ,  $x = \rho \cos \theta$ , and  $y = \rho \sin \theta$ . Equation (11) factors as follows

$$\left(\frac{F}{C}\rho^2 + 1\right) \left(1 + \rho^2 \left(A - \frac{F}{C}\right) + 2\rho^3 (\Re C \cos \theta - \Im C \sin \theta)\right) = 0.$$

Clearly, the envelope of the curve  $\frac{F}{C}\rho^2 + 1$  is the circle  $\rho = \sqrt{-F/C}$  (provided that  $-F/C > 0$ , of course). In order that  $W(M)$  be circular it is necessary and sufficient that the envelope of the curve

$$(16) \quad \left(1 + \rho^2 \left(A - \frac{F}{C}\right) + 2\rho^3 (\Re C \cos \theta - \Im C \sin \theta)\right) = 0$$

be contained in the disk of radius  $\sqrt{-F/C}$ , centered at the origin. This happens if and only if both the circle  $\frac{F}{C}\rho^2 + 1 = 0$  and the origin are contained in the closure of the same connected component of the complement of curve (16). Since, substitution of  $\rho$  by 0 in equation (16) returns the value 1, the if and only if condition for the situation above to occur is

$$(17) \quad 2|C| \leq (2 - AC/F)/(2(-C/F)^{3/2}).$$

Indeed, we are asking that

$$\left(1 + \rho^2 \left(A - \frac{F}{C}\right) + 2\rho^3 (\Re C \cos \theta - \Im C \sin \theta)\right) \geq 0$$

if  $\rho = \sqrt{-C/F}$  for all choices of  $\theta$ . Since the minimum value of  $\Re C \cos \theta - \Im C \sin \theta$  is  $-|C|$ , the previous inequality is equivalent to

$$2|C|\rho^3 \leq 1 + \rho^2(A - F/C)$$

that is to (17).  $\square$

In the following we examine the connection of the circularity of the numerical range of a nilpotent matrix with the trace condition.

**Proposition 3.** *Let  $z = x + iy$  and  $M$  be any  $5 \times 5$  nilpotent matrix. Then, according to (11)*

$$\begin{aligned} p_M(x, y) &= 1 + A|z|^2 + B|z|^4 + C|z|^2z + \bar{C}|z|^2\bar{z} + \\ &D|z|^2z^2 + \bar{D}|z|^2\bar{z}^2 + E|z|^2z^3 + \bar{E}|z|^2\bar{z}^3 + F|z|^4z + \bar{F}|z|^4\bar{z}. \end{aligned}$$

If  $M$  is unitarily equivalent to the triangular matrix

$$(18) \quad \begin{bmatrix} 0 & a & e & h & k \\ 0 & 0 & b & f & j \\ 0 & 0 & 0 & c & g \\ 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

then

$$A = -\frac{1}{4} [ |a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2 + |f|^2 + |g|^2 + |h|^2 + |j|^2 + |k|^2 ]$$

$$\begin{aligned} B &= \frac{1}{16} [ |b|^2|d|^2 + |c|^2|j|^2 + |a|^2|d|^2 + |a|^2|c|^2 + |a|^2|g|^2 + |d|^2|e|^2 + \\ &|e|^2|f|^2 + |e|^2|j|^2 + |g|^2|h|^2 + |b|^2|h|^2 + |h|^2|j|^2 + |f|^2|g|^2 + |c|^2|k|^2 + \\ &|b|^2|k|^2 + |f|^2|k|^2 - 2\Re(\bar{b}d\bar{f}\bar{g} + c\bar{f}\bar{g}j + a\bar{c}\bar{e}f + a\bar{e}\bar{g}j + a\bar{d}\bar{h}j + \bar{d}e\bar{g}h + \\ &\bar{b}e\bar{f}h + \bar{b}e\bar{j}k + \bar{c}gh\bar{k} + \bar{f}h\bar{j}k) ] \end{aligned}$$

$$C = \frac{1}{8} [ \bar{c}d\bar{g} + \bar{b}c\bar{f} + \bar{b}g\bar{j} + \bar{d}\bar{f}j + \bar{a}\bar{b}e + \bar{a}\bar{f}h + \bar{c}\bar{e}h + \bar{a}\bar{j}k + \bar{e}\bar{g}k + \bar{d}hk ]$$

$$D = -\frac{1}{16} [ \bar{b}c\bar{d}j + \bar{a}\bar{b}c\bar{h} + \bar{a}\bar{b}g\bar{k} + \bar{a}\bar{d}\bar{f}k + \bar{c}\bar{d}\bar{e}k ]$$

$$E = \frac{1}{32} [ \bar{a}\bar{b}\bar{c}\bar{d}k ]$$

$$\begin{aligned} F &= -\frac{1}{32} [ |a|^2\bar{c}\bar{d}g + |b|^2\bar{d}\bar{h}k + \bar{a}|c|^2\bar{j}k + \bar{a}\bar{b}|d|^2e + \bar{d}|e|^2\bar{f}j + \bar{e}|f|^2\bar{g}k \\ &+ \bar{a}\bar{f}|g|^2h + \bar{b}\bar{g}|h|^2j + \bar{c}\bar{e}h|j|^2 + \bar{b}\bar{c}\bar{f}|k|^2 - \bar{a}\bar{c}\bar{d}\bar{e}j - \bar{a}\bar{d}\bar{e}\bar{f}g - \bar{b}\bar{d}\bar{e}\bar{h}j - \\ &-\bar{a}\bar{b}\bar{d}\bar{g}h - \bar{a}\bar{c}\bar{j}g\bar{h} - \bar{e}\bar{f}\bar{g}h\bar{j} - \bar{b}\bar{c}\bar{h}j\bar{k} - \bar{a}\bar{c}\bar{f}\bar{g}k - \bar{b}\bar{d}\bar{e}\bar{f}k - \bar{c}\bar{e}\bar{f}\bar{j}k - \bar{b}\bar{f}\bar{g}\bar{h}k ] \end{aligned}$$

Therefore, in terms of the traces:

$$A = -\frac{1}{4}\text{tr}M^*M, \quad C = \frac{1}{8}\overline{\text{tr}M^*M^2} \quad D = -\frac{1}{16}\overline{\text{tr}M^*M^3}, \quad \text{and}$$

$$E = \frac{1}{32}\overline{\text{tr}M^*M^4}.$$



*Proof.* With the notation  $z = x + iy$ , one gets that

$$p_M(x, y) = \begin{vmatrix} 1 & \frac{a\bar{z}}{2} & \frac{e\bar{z}}{2} & \frac{h\bar{z}}{2} & \frac{k\bar{z}}{2} \\ \frac{\bar{a}z}{2} & 1 & \frac{\bar{b}z}{2} & \frac{\bar{f}z}{2} & \frac{\bar{j}z}{2} \\ \frac{\bar{e}z}{2} & \frac{\bar{b}z}{2} & 1 & \frac{\bar{c}z}{2} & \frac{\bar{g}z}{2} \\ \frac{\bar{h}z}{2} & \frac{\bar{f}z}{2} & \frac{\bar{c}z}{2} & 1 & \frac{\bar{d}z}{2} \\ \frac{\bar{k}z}{2} & \frac{\bar{j}z}{2} & \frac{\bar{g}z}{2} & \frac{\bar{d}z}{2} & 1 \end{vmatrix}.$$

Then, the expansion of the determinant above, using its first row and the corresponding  $4 \times 4$  minors  $M_1$ – $M_5$  is

$$(19) \quad p_M(x, y) = M_1 - \frac{a\bar{z}}{2}M_2 + \frac{e\bar{z}}{2}M_3 - \frac{h\bar{z}}{2}M_4 + \frac{k\bar{z}}{2}M_5.$$

One gets the following expressions for the quantities involved in (19):

$$(20) \quad \begin{aligned} M_1 &= 1 - \frac{1}{4} [|b|^2 + |c|^2 + |d|^2 + |f|^2 + |g|^2 + |j|^2] |z|^2 + \\ &+ \frac{1}{16} [|b|^2|d|^2 - \bar{b}\bar{g}fd - \bar{f}\bar{d}bg + |f|^2|g|^2 - \bar{f}\bar{g}jc - \bar{j}\bar{c}fg + |j|^2|c|^2] |z|^4 + \\ &+ \frac{1}{8} [g\bar{c}\bar{d} + f\bar{b}\bar{c} + \bar{b}\bar{g}j + \bar{f}\bar{d}j] |z|^2 z + \\ &+ \frac{1}{8} [\bar{g}cd + \bar{f}bc + \bar{j}bg + \bar{j}fd] |z|^2 \bar{z} - \\ &- \frac{1}{16} [\bar{b}\bar{c}\bar{d}j] |z|^2 z^2 - \\ &- \frac{1}{16} [bcd\bar{j}] |z|^2 \bar{z}^2 \end{aligned}$$

$$(21) \quad \begin{aligned} -\frac{a\bar{z}}{2}M_2 &= -\frac{1}{4}|a|^2|z|^2 + \\ &+ \frac{1}{16} [|a|^2|d|^2 + |a|^2|c|^2 + |a|^2|g|^2 - a\bar{c}\bar{e}f - a\bar{e}\bar{g}j - a\bar{d}\bar{h}j] |z|^4 - \\ &- \frac{1}{32} [|a|^2\bar{c}\bar{d}g - a\bar{c}\bar{d}\bar{e}j] |z|^4 z - \\ &- \frac{1}{32} [|a|^2cd\bar{g} + ab|d|^2\bar{e} - ad\bar{e}f\bar{g} - ab\bar{d}g\bar{h} + af|g|^2\bar{h} - ac\bar{g}\bar{h}j - a\bar{c}f\bar{g}\bar{k} \\ &+ a|c|^2j\bar{k}] |z|^4 \bar{z} + \frac{1}{8} [ab\bar{e} + af\bar{h} + a\bar{k}j] |z|^2 \bar{z} - \\ &- \frac{1}{16} [abc\bar{h} + abg\bar{k} + adf\bar{k}] |z|^2 \bar{z}^2 + \\ &+ \frac{1}{32} [abcd\bar{k}] |z|^2 \bar{z}^3 \end{aligned}$$

$$(22) \quad \begin{aligned} \frac{e\bar{z}}{2}M_3 &= \frac{1}{8} [\bar{a}\bar{b}\bar{e}] |z|^2 z - \\ &- \frac{1}{32} [\bar{a}\bar{b}|d|^2e - \bar{a}\bar{d}\bar{e}f\bar{g} + \bar{d}|e|^2\bar{f}j - \bar{b}\bar{d}\bar{e}\bar{h}j] |z|^4 z - \\ &- \frac{1}{16} [\bar{a}\bar{c}\bar{e}\bar{f} + \bar{a}\bar{e}\bar{g}\bar{j} - |d|^2|e|^2 - |e|^2|f|^2 - |e|^2|j|^2 + \bar{d}\bar{e}\bar{g}\bar{h} + \bar{b}\bar{e}\bar{f}\bar{h} + \bar{b}\bar{e}\bar{k}\bar{j}] |z|^4 + \\ &+ \frac{1}{32} [\bar{a}\bar{c}\bar{d}\bar{e}\bar{j} - d|e|^2\bar{f}\bar{j} + e\bar{f}\bar{g}\bar{h}\bar{j} - c\bar{e}\bar{h}|j|^2 + \bar{b}\bar{d}\bar{e}\bar{f}\bar{k} - e|f|^2\bar{g}\bar{k} + c\bar{e}\bar{f}\bar{k}\bar{j}] |z|^4 \bar{z} - \\ &- \frac{1}{4}|e|^2|z|^2 + \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{8} [ce\bar{h} + eg\bar{k}] |z|^2 \bar{z} - \\
 & - \frac{1}{16} [cde\bar{k}] |z|^2 \bar{z}^2 \\
 (23) \quad & - \frac{h\bar{z}}{2} M_4 = - \frac{1}{16} [\bar{a}\bar{b}\bar{c}h] |z|^2 z^2 + \\
 & + \frac{1}{32} [\bar{a}\bar{b}\bar{d}\bar{g}h - \bar{a}\bar{f}|g|^2 h + \bar{a}\bar{c}\bar{g}h\bar{j} + \bar{e}\bar{f}\bar{g}h\bar{j} - \bar{c}\bar{e}h|j|^2 - \bar{b}\bar{g}|h|^2 j + \bar{b}\bar{c}h\bar{j}\bar{k}] |z|^4 z + \\
 & + \frac{1}{8} [\bar{a}\bar{f}h + \bar{c}\bar{e}h] |z|^2 z - \\
 & - \frac{1}{16} [\bar{a}d\bar{h}\bar{j} + d\bar{e}\bar{g}h + b\bar{e}\bar{f}h - |g|^2|h|^2 - |b|^2|h|^2 - |h|^2|j|^2 + \bar{c}\bar{g}h\bar{k} + \bar{f}h\bar{j}\bar{k}] |z|^4 + \\
 & + \frac{1}{32} [b\bar{d}\bar{e}h\bar{j} - b\bar{g}|h|^2 \bar{j} - |b|^2 d\bar{h}\bar{k} + b\bar{f}\bar{g}h\bar{k}] |z|^4 \bar{z} - \\
 & - \frac{1}{4} |h|^2 |z|^2 + \\
 & + \frac{1}{8} [d\bar{h}\bar{k}] |z|^2 \bar{z} \\
 (24) \quad & \frac{k\bar{z}}{2} M_5 = \frac{1}{32} k\bar{a}\bar{b}\bar{c}\bar{d}|z|^2 z^3 - \frac{1}{16} (k\bar{a}\bar{b}\bar{g} + k\bar{a}\bar{f}\bar{d} + k\bar{e}\bar{c}\bar{d}) |z|^2 z^2 + \\
 & + \frac{1}{32} (k\bar{a}\bar{f}\bar{c}\bar{g} - k\bar{a}\bar{j}|c|^2 + k\bar{b}\bar{e}\bar{f}\bar{d} - k|f|^2 \bar{e}\bar{g} + k\bar{e}\bar{j}\bar{f}\bar{c} - k|b|^2 \bar{h}\bar{d} + k\bar{f}\bar{h}\bar{b}\bar{g} - f|k|^2 \bar{b}\bar{c}) |z|^4 z \\
 & + \frac{1}{16} (-k\bar{b}\bar{e}\bar{j} - k\bar{c}\bar{h}\bar{g} - k\bar{f}\bar{h}\bar{j} + |c|^2 |k|^2 + |b|^2 |k|^2 + |k|^2 |f|^2) |z|^4 \\
 & + \frac{1}{32} (k\bar{b}\bar{c}\bar{h}\bar{j} - b\bar{c}|k|^2 \bar{f}) |z|^4 \bar{z} \\
 & + \frac{1}{8} (k\bar{a}\bar{j} + k\bar{e}\bar{g} + k\bar{h}\bar{d}) |z|^2 z - \frac{1}{4} |k|^2 |z|^2.
 \end{aligned}$$

Adding the quantities in (20) through (24) and taking care of like terms, one gets that  $p_M$  has the form (11) with coefficients  $A$ – $F$  described as in the text of this proposition.  $\square$

Thus the trace condition holds if and only if  $D = E = C = 0$ , in which case  $W(M)$  is circular if and only if  $F = 0$  (by Theorem 2). Examining the matrix  $M$  in Example 2 we note that

$$p_M(x, y) = 1 - \frac{5}{4} |z|^2 + \frac{5}{16} |z|^4 + \frac{1}{32} |z|^4 z + \frac{1}{32} |z|^4 \bar{z}.$$

Thus  $A = -5/4$ ,  $B = 5/16$ ,  $C = D = E = 0$ ,  $F = 1/32$ , which confirms via Theorem 2 the non-circularity of  $W(M)$ .

As for the matrix  $M$  in Example 1, note that, in the case of that matrix, one has

$$p_M(x, y) = 1 - \frac{|a|^2 + 3}{4} |z|^2 + \frac{3|a|^2}{16} |z|^4 + \frac{1}{8} |z|^2 z + \frac{1}{8} |z|^2 \bar{z} - \frac{|a|^2}{32} |z|^4 z - \frac{|a|^2}{32} |z|^4 \bar{z}.$$

So  $D = E = 0$ , but  $C = 1/8 \neq 0$  which means that (as noted before), the trace condition fails. The first and last conditions in (13) hold independent of the value of the complex number  $a$ . The remaining inequality holds if and only if  $|a| \geq 2$ , as one establishes by a straightforward computation. Thus,  $W(M)$  is circular if and only if  $|a| \geq 2$ .

The approach used for  $5 \times 5$  nilpotent matrices can be used for arbitrary nilpotent matrices. It produces progressively more complicated characterizations. More exactly, our method in this paper is considering the system of conditions (9) and (10), the number  $r > 0$  should satisfy so that  $W(M)$  be the disk of radius  $r$ . Then, that system should yield the necessary and sufficient conditions for the circularity of  $W(M)$  and the formula for  $w(M)$ , by considerations similar to those in Theorem 2. Unfortunately, those considerations vary with the size of the given matrix. A unified treatment seems impossible.

To give an example, a  $6 \times 6$  nilpotent matrix has equation (7) of form:

$$\begin{aligned} &1 + A|z|^2 + B|z|^4 + C|z|^6 + D|z|^2z + \overline{D}|z|^2\bar{z} + E|z|^2z^2 + \overline{E}|z|^2\bar{z}^2 \\ &+ F|z|^2z^3 + \overline{F}|z|^2\bar{z}^3 + G|z|^2z^4 + \overline{G}|z|^2\bar{z}^4 + H|z|^4z + \overline{H}|z|^4\bar{z} \\ &+ K|z|^4z^2 + \overline{K}|z|^4\bar{z}^2 = 0. \end{aligned}$$

By Proposition 2, if  $W(M) = \{z : |z| \leq r\}$ ,  $r > 0$ , then necessarily the following hold:

$$r^6 + Ar^4 + Br^2 + C = 0, \quad r^2D + H = 0, \quad r^2E + K = 0, \quad \text{and } F = G = 0.$$

The matrix has a reducing eigenvector, (that is a nonzero vector which is an eigenvector of both  $M$  and  $M^*$ ), if and only if it is unitarily equivalent to a block-diagonal matrix having a  $1 \times 1$  block. This happens if and only if  $C = K = 0$  (which implies  $E = 0$ ), since the complimentary diagonal block is a  $5 \times 5$  nilpotent matrix. In such a case, Theorem 2 fully describes under what circumstances  $W(M)$  is circular. If there are no reducing eigenvectors, a separate discussion (which we leave to the reader), is necessary in order to completely characterize the situation when  $6 \times 6$  nilpotent matrices have circular numerical ranges.

As a final comment, we want to remark that, although Hilbert space nilpotent operators do not always have circular numerical ranges, “about half of their nonzero powers do”. Before explaining this sentence we note that nilpotent operators of order 2 always have circular numerical ranges [10]. The consequence of this fact is that certain powers of nilpotent operators (which are nilpotent operators too), have circular numerical ranges. More formally, what we mean is:

**Remark 5.** *Let  $T$  be a nilpotent operator of order  $n > 1$ . If*

$$n = 2k \quad \text{then} \quad T^k, T^{k+1}, \dots, T^{n-1}$$

*are nonzero nilpotent operators with circular numerical ranges.*

*If  $n = 2k + 1$ , the same holds for the operators*

$$T^{k+1}, \dots, T^{n-1}.$$

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