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A SHORT PROOF OF A CHARACTERIZATION
OF INNER FUNCTIONS IN TERMS OF THE
COMPOSITION OPERATORS THEY INDUCE

VALENTIN MATACHE

ABSTRACT. The paper contains a new proof for the sufficiency in Joel H. Shapiro’s recent characterization of inner functions saying that an analytic self-map \( \varphi \) of the open unit disk is an inner function if and only if the essential norm of the composition operator of symbol \( \varphi \) is equal to \( \sqrt{(1 + |\varphi(0)|)/(1 - |\varphi(0)|)} \). The main ingredient in the proof is a formula for the essential norm of a composition operator in terms of Aleksandrov measures obtained by Cima and Matheson. The necessity was originally proved by Joel Shapiro in 1987. A short proof of the necessity, by Aleksandrov measure techniques, was obtained by Jonathan E. Shapiro in 1998.

For each holomorphic self-map \( \varphi : U \to U \) of the open unit disk \( U \), the composition operator \( C_{\varphi} \) of symbol \( \varphi \) is defined as follows \( C_{\varphi}f = f \circ \varphi, f \in H^2 \). In this definition \( H^2 \) is the Hilbert Hardy space on \( U \), i.e., the set of all analytic functions on \( U \) with square summable Taylor coefficients. It is well known that each such \( \varphi \) induces a bounded composition operator \( C_{\varphi} \) on \( H^2 \). Recently Joel Shapiro obtained the following characterization of inner functions \( \varphi \), (that is of self-maps \( \varphi \) whose radial limit function is unimodular almost everywhere on the unit circle \( T \)).

Theorem 1. The function \( \varphi : U \to U \) is inner if and only if

\[
||C_{\varphi}||_e = \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}},
\]

where \( ||C_{\varphi}||_e \) denotes the essential norm of \( C_{\varphi} \).

The necessity in the previous equivalence was originally proved in [6]. Jonathan Shapiro [5] observed that Aleksandrov measures can be used
to give a very short proof of the aforementioned implication. In this note we provide a new, short proof of the sufficiency in Theorem 1. Our approach is also based on Aleksandrov measures and a formula obtained by Cima and Matheson [1]. For each $\alpha \in \mathbb{T}$ the Aleksandrov measure of index $\alpha$ of $\varphi$ is the positive, finite measure $\tau_\alpha$ on $\mathbb{T}$ whose Poisson integral equals the positive harmonic function

$$
F_\alpha(z) = \Re \frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} = \frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2}, \quad z \in \mathbb{U}.
$$

The fact that $F_\alpha$ is positive and harmonic is a direct consequence of its definition, while the existence and uniqueness of $\tau_\alpha$ is an immediate consequence of the well-known Herglotz theorem [2, Theorem 11.19]. Another well-known fact about the Poisson integral of a measure on $\mathbb{T}$ is that its radial limit function is the absolutely continuous part in its Lebesgue decomposition with respect to the normalized arc-length measure $m$ on $\mathbb{T}$. Therefore, if we denote by $\sigma_\alpha$ the singular part of $\tau_\alpha$, we obtain the equality

$$
d\tau_\alpha(\xi) = \frac{1 - |\varphi(\xi)|^2}{|\alpha - \varphi(\xi)|^2} dm(\xi) + d\sigma_\alpha(\xi).
$$

The main result in [1] is the following formula

$$
\|C_\varphi\|_e = \sup_{\alpha \in \mathbb{T}} \sqrt{\|\sigma_\alpha\|}.
$$

Based on it and the rather evident equality

$$
\sup_{\alpha \in \mathbb{T}} \Re \frac{\alpha + z}{\alpha - z} = \frac{1 + |z|}{1 - |z|},
$$

valid for each $z \in \mathbb{U}$, we are able to give the following short proof to Theorem 1.

**Proof.** If $\varphi$ is inner, then by (3) $\tau_\alpha = \sigma_\alpha$ for any $\alpha \in \mathbb{T}$. Therefore,

$$
\|\sigma_\alpha\| = \|\tau_\alpha\| = F_\alpha(0) = \Re \frac{\alpha + \varphi(0)}{\alpha - \varphi(0)}.
$$
By formula (4) and equality (5) one deduces that (1) holds. This implication is proved by essentially this method in [5]. We included it for the sake of rendering this note self contained. Our original contribution is giving a short proof to the reverse implication, (which was only recently proved, [7]).

Assume (1) holds. For the beginning, we will work under the particular assumption \( \varphi(0) = 0 \). By (4) there exists a sequence \((\alpha_n)\) in \( T \) such that
\[
\|\sigma_{\alpha_n}\|^2 \longrightarrow \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} = 1.
\]
Since \( T \) is compact we will not reduce the generality by assuming that \((\alpha_n)\) is convergent to some \( \alpha_0 \in T \). Since
\[
\|\tau_{\alpha_n}\| = \text{Re} \frac{\alpha_n + \varphi(0)}{\alpha_n - \varphi(0)} = 1, \quad n = 1, 2, \ldots ,
\]
one deduces by (3) that
\[
\lim_{n \to \infty} \int_T \frac{1 - |\varphi(\xi)|^2}{|\alpha_n - \varphi(\xi)|^2} \, dm(\xi) = 0.
\]
Now, by applying the classical Fatou lemma in integration theory, [2, Lemma 1.28], to the sequence
\[
\left( \frac{1 - |\varphi(\xi)|^2}{|\alpha_n - \varphi(\xi)|^2} \right)
\]
one obtains
\[
0 \leq \int_T \frac{1 - |\varphi(\xi)|^2}{|\alpha_0 - \varphi(\xi)|^2} \, dm(\xi) \leq \liminf_{n \to \infty} \int_T \frac{1 - |\varphi(\xi)|^2}{|\alpha_n - \varphi(\xi)|^2} \, dm(\xi) = 0.
\]
Hence \( |\varphi(\xi)| = 1 \) \( m \)-almost everywhere on \( T \), i.e., \( \varphi \) is an inner function. To finish the proof, assume that
\[
\varphi(0) = p \neq 0 \quad \text{and} \quad \|C_\varphi\|_e = \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}.
\]
Denote by \( \alpha_p \) the self-inverse, disk automorphism
\[
\alpha_p(z) = \frac{p - z}{1 - \overline{p} z},
\]
and set \( \psi = \alpha_p \circ \varphi \). Since \( \alpha_p \) is inner we have that \( \| C_{\alpha_p} \|_e = \sqrt{(1 + |p|)/(1 - |p|)} = \| C_{\varphi} \|_e \). Observe that \( \psi(0) = 0 \), \( C_{\psi} = C_{\varphi} C_{\alpha_p} \) and \( C_{\psi} C_{\alpha_p} = C_{\varphi} \), since \( \alpha_p \) is equal to its inverse. Therefore one obtains

\[
\| C_{\varphi} \|_e \leq \| C_{\psi} \|_e \| C_{\alpha_p} \|_e.
\]

We deduce \( \| C_{\psi} \|_e \geq 1 \); hence, \( \| C_{\psi} \|_e = 1 \) because composition operators whose symbol fixes 0 are contractions. By what we proved for symbols fixing 0, it follows that \( \psi \) is an inner function and hence \( \varphi \) is also inner because \( \alpha_p \) is a disk automorphism. \( \square \)

As a final comment we would like to note that, once the essential norm characterization of inner functions in Theorem 1 is proved, one can use it to obtain an operator norm characterization, see [7].

REFERENCES


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