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# A Short Proof of a Characterization of Inner Functions in Terms of the Composition Operators they Induce

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**A SHORT PROOF OF A CHARACTERIZATION  
OF INNER FUNCTIONS IN TERMS OF THE  
COMPOSITION OPERATORS THEY INDUCE**

VALENTIN MATACHE

**ABSTRACT.** The paper contains a new proof for the sufficiency in Joel H. Shapiro's recent characterization of inner functions saying that an analytic self-map  $\varphi$  of the open unit disk is an inner function if and only if the essential norm of the composition operator of symbol  $\varphi$  is equal to  $\sqrt{(1 + |\varphi(0)|)/(1 - |\varphi(0)|)}$ . The main ingredient in the proof is a formula for the essential norm of a composition operator in terms of Aleksandrov measures obtained by Cima and Matheson. The necessity was originally proved by Joel Shapiro in 1987. A short proof of the necessity, by Aleksandrov measure techniques, was obtained by Jonathan E. Shapiro in 1998.

For each holomorphic self-map  $\varphi : \mathbf{U} \rightarrow \mathbf{U}$  of the open unit disk  $\mathbf{U}$ , the composition operator  $C_\varphi$  of symbol  $\varphi$  is defined as follows  $C_\varphi f = f \circ \varphi$ ,  $f \in H^2$ . In this definition  $H^2$  is the Hilbert Hardy space on  $\mathbf{U}$ , i.e., the set of all analytic functions on  $\mathbf{U}$  with square summable Taylor coefficients. It is well known that each such  $\varphi$  induces a bounded composition operator  $C_\varphi$  on  $H^2$ . Recently Joel Shapiro obtained the following characterization of inner functions [7], (that is of self-maps  $\varphi$  whose radial limit function is unimodular almost everywhere on the unit circle  $\mathbf{T}$ ).

**Theorem 1.** *The function  $\varphi : \mathbf{U} \rightarrow \mathbf{U}$  is inner if and only if*

$$(1) \quad \|C_\varphi\|_e = \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}},$$

where  $\|C_\varphi\|_e$  denotes the essential norm of  $C_\varphi$ .

The necessity in the previous equivalence was originally proved in [6]. Jonathan Shapiro [5] observed that Aleksandrov measures can be used

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to give a very short proof of the aforementioned implication. In this note we provide a new, short proof of the sufficiency in Theorem 1. Our approach is also based on Aleksandrov measures and a formula obtained by Cima and Matheson [1]. For each  $\alpha \in \mathbf{T}$  the Aleksandrov measure of index  $\alpha$  of  $\varphi$  is the positive, finite measure  $\tau_\alpha$  on  $\mathbf{T}$  whose Poisson integral equals the positive harmonic function

$$(2) \quad F_\alpha(z) = \operatorname{Re} \frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} = \frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2}, \quad z \in \mathbf{U}.$$

The fact that  $F_\alpha$  is positive and harmonic is a direct consequence of its definition, while the existence and uniqueness of  $\tau_\alpha$  is an immediate consequence of the well-known Herglotz theorem [2, Theorem 11.19]. Another well-known fact about the Poisson integral of a measure on  $\mathbf{T}$  is that its radial limit function is the absolutely continuous part in its Lebesgue decomposition with respect to the normalized arc-length measure  $m$  on  $\mathbf{T}$ . Therefore, if we denote by  $\sigma_\alpha$  the singular part of  $\tau_\alpha$ , we obtain the equality

$$(3) \quad d\tau_\alpha(\xi) = \frac{1 - |\varphi(\xi)|^2}{|\alpha - \varphi(\xi)|^2} dm(\xi) + d\sigma_\alpha(\xi).$$

The main result in [1] is the following formula

$$(4) \quad \|C_\varphi\|_e = \sup_{\alpha \in \mathbf{T}} \sqrt{\|\sigma_\alpha\|}.$$

Based on it and the rather evident equality

$$(5) \quad \sup_{\alpha \in \mathbf{T}} \operatorname{Re} \frac{\alpha + z}{\alpha - z} = \frac{1 + |z|}{1 - |z|},$$

valid for each  $z \in \mathbf{U}$ , we are able to give the following short proof to Theorem 1.

*Proof.* If  $\varphi$  is inner, then by (3)  $\tau_\alpha = \sigma_\alpha$  for any  $\alpha \in \mathbf{T}$ . Therefore,

$$\|\sigma_\alpha\| = \|\tau_\alpha\| = F_\alpha(0) = \operatorname{Re} \frac{\alpha + \varphi(0)}{\alpha - \varphi(0)}.$$

By formula (4) and equality (5) one deduces that (1) holds. This implication is proved by essentially this method in [5]. We included it for the sake of rendering this note self contained. Our original contribution is giving a short proof to the reverse implication, (which was only recently proved, [7]).

Assume (1) holds. For the beginning, we will work under the particular assumption  $\varphi(0) = 0$ . By (4) there exists a sequence  $(\alpha_n)$  in  $\mathbf{T}$  such that

$$\|\sigma_{\alpha_n}\|^2 \longrightarrow \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} = 1.$$

Since  $\mathbf{T}$  is compact we will not reduce the generality by assuming that  $(\alpha_n)$  is convergent to some  $\alpha_0 \in \mathbf{T}$ . Since

$$\|\tau_{\alpha_n}\| = \operatorname{Re} \frac{\alpha_n + \varphi(0)}{\alpha_n - \varphi(0)} = 1, \quad n = 1, 2, \dots,$$

one deduces by (3) that

$$\lim_{n \rightarrow \infty} \int_{\mathbf{T}} \frac{1 - |\varphi(\xi)|^2}{|\alpha_n - \varphi(\xi)|^2} dm(\xi) = 0.$$

Now, by applying the classical Fatou lemma in integration theory, [2, Lemma 1.28], to the sequence

$$\left( \frac{1 - |\varphi(\xi)|^2}{|\alpha_n - \varphi(\xi)|^2} \right)$$

one obtains

$$0 \leq \int_{\mathbf{T}} \frac{1 - |\varphi(\xi)|^2}{|\alpha_0 - \varphi(\xi)|^2} dm(\xi) \leq \liminf_{n \rightarrow \infty} \int_{\mathbf{T}} \frac{1 - |\varphi(\xi)|^2}{|\alpha_n - \varphi(\xi)|^2} dm(\xi) = 0.$$

Hence  $|\varphi(\xi)| = 1$   $m$ -almost everywhere on  $\mathbf{T}$ , i.e.,  $\varphi$  is an inner function. To finish the proof, assume that

$$\varphi(0) = p \neq 0 \quad \text{and} \quad \|C_\varphi\|_e = \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}.$$

Denote by  $\alpha_p$  the self-inverse, disk automorphism

$$\alpha_p(z) = \frac{p - z}{1 - \bar{p}z},$$

and set  $\psi = \alpha_p \circ \varphi$ . Since  $\alpha_p$  is inner we have that  $\|C_{\alpha_p}\|_e = \sqrt{(1+|p|)/(1-|p|)} = \|C_\varphi\|_e$ . Observe that  $\psi(0) = 0$ ,  $C_\psi = C_\varphi C_{\alpha_p}$  and  $C_\psi C_{\alpha_p} = C_\varphi$ , since  $\alpha_p$  is equal to its inverse. Therefore one obtains

$$\|C_\varphi\|_e \leq \|C_\psi\|_e \|C_{\alpha_p}\|_e.$$

We deduce  $\|C_\psi\|_e \geq 1$ ; hence,  $\|C_\psi\|_e = 1$  because composition operators whose symbol fixes 0 are contractions. By what we proved for symbols fixing 0, it follows that  $\psi$  is an inner function and hence  $\varphi$  is also inner because  $\alpha_p$  is a disk automorphism.  $\square$

As a final comment we would like to note that, once the essential norm characterization of inner functions in Theorem 1 is proved, one can use it to obtain an operator norm characterization, see [7].

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