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A SHORT PROOF OF A CHARACTERIZATION OF INNER FUNCTIONS IN TERMS OF THE COMPOSITION OPERATORS THEY INDUCE

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ABSTRACT. The paper contains a new proof for the sufficiency in Joel H. Shapiro's recent characterization of inner functions saying that an analytic self-map φ of the open unit disk is an inner function if and only if the essential norm of the composition operator of symbol φ is equal to $\sqrt{(1+|\varphi(0)|)/(1-|\varphi(0)|)}$. The main ingredient in the proof is a formula for the essential norm of a composition operator in terms of Aleksandrov measures obtained by Cima and Matheson. The necessity was originally proved by Joel Shapiro in 1987. A short proof of the necessity, by Aleksandrov measure techniques, was obtained by Jonathan E. Shapiro in 1998.

For each holomorphic self-map $\varphi : \mathbf{U} \to \mathbf{U}$ of the open unit disk $\mathbf{U}$, the composition operator C_φ of symbol φ is defined as follows $C_\varphi f = f \circ \varphi$, $f \in H^2$. In this definition H^2 is the Hilbert Hardy space on $\mathbf{U}$, i.e., the set of all analytic functions on $\mathbf{U}$ with square summable Taylor coefficients. It is well known that each such φ induces a bounded composition operator C_φ on H^2. Recently Joel Shapiro obtained the following characterization of inner functions [**7**], (that is of self-maps φ whose radial limit function is unimodular almost everywhere on the unit circle $\mathbf{T}$).

Theorem 1. *The function $\varphi : \mathbf{U} \to \mathbf{U}$ is inner if and only if*

$$\|C_\varphi\|_e = \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}, \tag{1}$$

where $\|C_\varphi\|_e$ denotes the essential norm of C_φ.

The necessity in the previous equivalence was originally proved in [**6**]. Jonathan Shapiro [**5**] observed that Aleksandrov measures can be used

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to give a very short proof of the aforementioned implication. In this note we provide a new, short proof of the sufficiency in Theorem 1. Our approach is also based on Aleksandrov measures and a formula obtained by Cima and Matheson [**1**]. For each $\alpha \in \mathbf{T}$ the Aleksandrov measure of index α of φ is the positive, finite measure τ_α on $\mathbf{T}$ whose Poisson integral equals the positive harmonic function

$$F_\alpha(z) = \operatorname{Re}\frac{\alpha+\varphi(z)}{\alpha-\varphi(z)} = \frac{1-|\varphi(z)|^2}{|\alpha-\varphi(z)|^2}, \quad z \in \mathbf{U}. \tag{2}$$

The fact that F_α is positive and harmonic is a direct consequence of its definition, while the existence and uniqueness of τ_α is an immediate consequence of the well-known Herglotz theorem [**2**, Theorem 11.19]. Another well-known fact about the Poisson integral of a measure on $\mathbf{T}$ is that its radial limit function is the absolutely continuous part in its Lebesgue decomposition with respect to the normalized arc-length measure m on $\mathbf{T}$. Therefore, if we denote by σ_α the singular part of τ_α, we obtain the equality

$$d\tau_\alpha(\xi) = \frac{1-|\varphi(\xi)|^2}{|\alpha-\varphi(\xi)|^2}\, dm(\xi) + d\sigma_\alpha(\xi). \tag{3}$$

The main result in [**1**] is the following formula

$$\|C_\varphi\|_e = \sup_{\alpha\in\mathbf{T}} \sqrt{\|\sigma_\alpha\|}\,. \tag{4}$$

Based on it and the rather evident equality

$$\sup_{\alpha\in\mathbf{T}} \operatorname{Re}\frac{\alpha+z}{\alpha-z} = \frac{1+|z|}{1-|z|}, \tag{5}$$

valid for each $z \in \mathbf{U}$, we are able to give the following short proof to Theorem 1.

Proof. If φ is inner, then by (3) $\tau_\alpha = \sigma_\alpha$ for any $\alpha \in \mathbf{T}$. Therefore,

$$\|\sigma_\alpha\| = \|\tau_\alpha\| = F_\alpha(0) = \operatorname{Re}\frac{\alpha+\varphi(0)}{\alpha-\varphi(0)}.$$

By formula (4) and equality (5) one deduces that (1) holds. This implication is proved by essentially this method in [**5**]. We included it for the sake of rendering this note self contained. Our original contribution is giving a short proof to the reverse implication, (which was only recently proved, [**7**]).

Assume (1) holds. For the beginning, we will work under the particular assumption $\varphi(0) = 0$. By (4) there exists a sequence (α_n) in $\mathbf{T}$ such that

$$\|\sigma_{\alpha_n}\|^2 \longrightarrow \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} = 1.$$

Since $\mathbf{T}$ is compact we will not reduce the generality by assuming that (α_n) is convergent to some $\alpha_0 \in \mathbf{T}$. Since

$$\|\tau_{\alpha_n}\| = \operatorname{Re} \frac{\alpha_n + \varphi(0)}{\alpha_n - \varphi(0)} = 1, \quad n = 1, 2, \dots ,$$

one deduces by (3) that

$$\lim_{n\to\infty} \int_{\mathbf{T}} \frac{1 - |\varphi(\xi)|^2}{|\alpha_n - \varphi(\xi)|^2} \, dm(\xi) = 0.$$

Now, by applying the classical Fatou lemma in integration theory, [**2**, Lemma 1.28], to the sequence

$$\left(\frac{1 - |\varphi(\xi)|^2}{|\alpha_n - \varphi(\xi)|^2} \right)$$

one obtains

$$0 \le \int_{\mathbf{T}} \frac{1 - |\varphi(\xi)|^2}{|\alpha_0 - \varphi(\xi)|^2} \, dm(\xi) \le \liminf_{n\to\infty} \int_{\mathbf{T}} \frac{1 - |\varphi(\xi)|^2}{|\alpha_n - \varphi(\xi)|^2} \, dm(\xi) = 0.$$

Hence $|\varphi(\xi)| = 1$ m-almost everywhere on $\mathbf{T}$, i.e., φ is an inner function. To finish the proof, assume that

$$\varphi(0) = p \neq 0 \quad \text{and} \quad \|C_\varphi\|_e = \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}} \, .$$

Denote by α_p the self-inverse, disk automorphism

$$\alpha_p(z) = \frac{p - z}{1 - \bar{p}z},$$

and set $\psi = \alpha_p \circ \varphi$. Since α_p is inner we have that $\|C_{\alpha_p}\|_e = \sqrt{(1+|p|)/(1-|p|)} = \|C_\varphi\|_e$. Observe that $\psi(0) = 0$, $C_\psi = C_\varphi C_{\alpha_p}$ and $C_\psi C_{\alpha_p} = C_\varphi$, since α_p is equal to its inverse. Therefore one obtains

$$\|C_\varphi\|_e \leq \|C_\psi\|_e \|C_{\alpha_p}\|_e.$$

We deduce $\|C_\psi\|_e \geq 1$; hence, $\|C_\psi\|_e = 1$ because composition operators whose symbol fixes 0 are contractions. By what we proved for symbols fixing 0, it follows that ψ is an inner function and hence φ is also inner because α_p is a disk automorphism. □

As a final comment we would like to note that, once the essential norm characterization of inner functions in Theorem 1 is proved, one can use it to obtain an operator norm characterization, see [**7**].

REFERENCES

1. J.A. Cima and A. Matheson, *Essential norms of composition operators and Aleksandrov measures*, Pacific J. Math. **179** (1997), 59–64.

2. W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1966.

3. D. Sarason, *Composition operators as integral operators*, in *Analysis and partial differential equations*, Marcel Dekker, New York, 1990.

4. ———, *Sub-Hardy Hilbert spaces in the unit disk*, Univ. of Arkansas Lecture Notes in Math. Sci., vol. 10, John Wiley & Sons, Inc., New York, 1994.

5. J.E. Shapiro, *Aleksandrov measures used in essential norm inequalities for composition operators*, J. Operator Theory **40** (1998), 133–146.

6. J.H. Shapiro, *The essential norm of a composition operator*, Ann. of Math. **125** (1987), 375–404.

7. ———, *What do composition operators know about inner functions?*, Monatsh. Math. **130** (2000), 57–70.

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