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Research Article

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Abstract: Operators on function spaces of form $C_\phi f = f \circ \phi$, where ϕ is a fixed map are called composition operators with symbol ϕ. We study such operators acting on the Hilbert Hardy space over the right half-plane and characterize the situations when they are invertible, Fredholm, unitary, and Hermitian. We determine the normal composition operators with inner, respectively with Möbius symbol. In select cases, we calculate their spectra, essential spectra, and numerical ranges.

Keywords: Composition operator, Hardy space, Half–plane

MSC: 47B33, 30H10

1 Introduction

Given a linear space $\mathcal{L}$ consisting of complex valued functions with common domain of definition D and a selfmap ϕ of D, we call the composition operator with symbol ϕ and denote C_ϕ the linear map

$$C_\phi f = f \circ \phi \qquad f \in \mathcal{L}.$$

If ψ is an analytic map on D, then the linear function

$$T_{\psi,\phi} f = \psi f \circ \phi \qquad f \in \mathcal{L}$$

is called the weighted composition operator with symbols ψ and φ.

Let Π^+ be the right open half–plane and $H^2(\Pi^+)$ the Hilbert Hardy space over Π^+. This means that $H^2(\Pi^+)$ consists of all analytic functions on Π^+ with the property

$$\sup \left\{ \int_{-\infty}^{+\infty} |f(x+iy)|^2 \, dy : x > 0 \right\} < \infty.$$

The unitary invariant properties of the composition operators on $H^2(\Pi^+)$ can be reduced to the study of the corresponding properties for weighted composition operators on $H^2(\mathbf{U})$ (the Hilbert Hardy space on $\mathbf{U}$, the open unit disc), as follows. Let ϕ be an analytic selfmap of Π^+, $\gamma(z) = (1+z)/(1-z)$, the Cayley transform of $\mathbf{U}$ onto Π^+, and $\varphi = \gamma^{-1} \circ \phi \circ \gamma$ the conformal conjugate of ϕ via Cayley's transform. One of the tools in the study of composition operators on $H^2(\Pi^+)$ is the fact that C_ϕ is unitarily equivalent to $T_{\psi,\varphi}$, $\psi(z) = (1-\varphi(z))/(1-z)$, $z \in \mathbf{U}$, a weighted composition operator acting on $H^2(\mathbf{U})$ [11].

If the angular limit $\varphi(1)$ of φ at 1 equals 1, then the angular derivative $\varphi'(1)$ is known to exist if and only if

$$M := \sup \left\{ \frac{\Re w}{\Re \phi(w)} : w \in \Pi^+ \right\} < \infty,$$

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in which case, the equality $\varphi'(1) = M$ holds and the angular limit $\phi(\infty)$ of ϕ at infinity exists and satisfies the condition $\phi(\infty) = \infty$. For that reason, we denote $M = \phi'(\infty)$ and call it the angular derivative of ϕ at infinity whether finite or not.

In Section 2, we are able to characterize which composition operators on $H^2(\Pi^+)$ are: invertible, Fredholm, and unitary (Theorem 2.4). The spectra, essential spectra, and numerical ranges of the operators in Theorem 2.4 are calculated (Theorems 2.7 and 2.12).

Section 3 is dedicated to normal composition operators. We are able to characterize the Hermitian composition operators on $H^2(\Pi^+)$ and find their spectra, essential spectra, and numerical ranges (Theorem 3.1). We determine which Möbius, respectively which inner maps induce normal composition operators (Theorem 3.4 and Proposition 3.5). In the sequel we use the notation $\sigma(C_\phi)$, $\sigma_p(C_\phi)$, $\sigma_e(C_\phi)$, and $W(C_\phi)$ for: the spectrum, point spectrum, essential spectrum, and numerical range of composition operators C_ϕ, respectively.

2 Invertible composition operators

Unlike the case of Hardy spaces of a disc (where all analytic selfmaps of that disc induce bounded composition operators), $H^2(\Pi^+)$ supports fewer bounded composition operators. A first hint in that direction is the fact that only few Möbius selfmaps of Π^+ induce bounded composition operators [9] (see Corollary 2.3 in this paper). Furthermore, the main result in [11] is:

Theorem 2.1 ([11, Theorem 3.1]). *The Hardy spaces over a half–plane cannot support compact composition operators.*

While boundedness of composition operators on $H^2(\Pi^+)$ was characterized in terms of Carleson measures [9], that criterion is considered hard to use in practice and so, a practical boundedness criterion, the norm, essential norm, and spectral radius formulas can be found in [5] (see also [12]):

Theorem 2.2 ([5, Theorems 3.1 and 3.4]). *The operator C_ϕ is bounded if and only if $\phi(\infty) = \infty$ and $\phi'(\infty) < \infty$, in which case the following equalities hold.*

$$||C_\phi|| = r(C_\phi) = ||C_\phi||_e = \sqrt{\phi'(\infty)}. \tag{1}$$

An immediate consequence is the following result, originally proved (in the framework of the upper half–plane), in [9] using Carleson measure techniques:

Corollary 2.3 ([9, pp. 66]). *The only bounded composition operators on $H^2(\Pi^+)$ induced by Möbius selfmaps of Π^+ are operators C_ϕ with symbol of form*

$$\phi(w) = aw + b \qquad w \in \Pi^+, \tag{2}$$

where $a > 0$ and $\Re b \geq 0$.

This corollary leads to the characterization of invertible composition operators on $H^2(\Pi^+)$. Indeed, according to Corollary 2.3, besides the identity, there are only two kinds of half–plane conformal automorphisms that induce bounded composition operators on $H^2(\Pi^+)$, namely

$$\phi(w) = w + ir \qquad r \in \mathbf{R} \setminus \{0\} \qquad \text{parabolic} \tag{3}$$

and

$$\phi(w) = Kw + ir \qquad K > 0,\ K \neq 1,\ r \in \mathbf{R} \qquad \text{hyperbolic.} \tag{4}$$

The composition operators induced by the half-plane automorphisms will be designated by the term **automorphic** composition operators. Clearly, such conformal automorphisms ϕ induce invertible composition operators C_ϕ with inverse $C_{\phi^{-1}}$, since the inverse of a parabolic, respectively hyperbolic half–plane automorphism is an automorphism of the same kind. Furthermore:

Theorem 2.4. *A bounded composition operator C_ϕ on $H^2(\Pi^+)$ is invertible if and only if ϕ is the identity, a parabolic, or a hyperbolic half–plane automorphism. That operator is unitary if and only if ϕ is the identity or a parabolic half–plane automorphism. Finally, C_ϕ is a Fredholm operator if and only if C_ϕ is invertible.*

Proof. The if part of the characterization of invertibility is obvious. For the only if part, note that, if the symbol conformally conjugated to ϕ is φ, it is known (as noted in the introduction), that the operators C_ϕ and $T_{\psi,\varphi}$ where $\psi(z) = (1-\varphi(z))/(1-z)$ are unitarily equivalent. Thus, if C_ϕ is a bounded, invertible operator, so is the weighted composition operator $T_{\psi,\varphi}$. According to [6], φ must be a disc automorphism and hence, ϕ must be one of the conformal automorphisms of Π^+ inducing bounded composition operators. Also, according to [12], only the identity and parabolic half–plane automorphisms induce isometric composition operators. Finally, C_ϕ is a Fredholm operator if and only if $T_{\psi,\varphi}$ is such an operator, in which case, φ needs to be a disc automorphism according to [8, Theorem 3.6]. Hence C_ϕ must be invertible. □

Given an analytic selfmap of ϕ of Π^+ inducing a bounded composition operator, we call it a **map of parabolic type** if $\phi'(\infty) = 1$, respectively a **map of hyperbolic type** if $\phi'(\infty) \neq 1$. If the radial limits of ϕ which must exist a.e. on the imaginary axis are a.e. of the form ix, with $x \in \mathbf{R}$ we call ϕ an **inner selfmap** of Π^+. A map ϕ of parabolic type is called a map of **parabolic automorphic** type if its conformal conjugate φ has pseudo–hyperbolically separated orbits. The reader is sent to paper [1] for more details on that notion. With this terminology, the isometric composition operators have the following characterization:

Theorem 2.5 ([12, Proposition 4])**.** *If ϕ is an analytic selfmap of Π^+, then C_ϕ is an isometry if and only if ϕ is an inner map of parabolic type.*

The numerical range of isometric composition operators on $H^2(\Pi^+)$ is easy to describe in certain cases [14, Proposition 1]:

Remark 2.6. *If ϕ, an analytic selfmap of Π^+ other than the identity, is an inner map of parabolic automorphic type, then $W(C_\phi)$, the numerical range of C_ϕ equals $\mathbf{U}$. In particular $W(C_\phi) = \mathbf{U}$ if C_ϕ is unitary, and $C_\phi \neq I$.*

The spectrum and essential spectrum of unitary composition operators on $H^2(\Pi^+)$ are easy to determine:

Theorem 2.7. *If $C_\phi \neq I$ is a unitary composition operator on $H^2(\Pi^+)$, then $\sigma(C_\phi) = \sigma_e(C_\phi) = \mathbf{T}$ (the unit circle).*

Proof. If φ is the selfmap of $\mathbf{U}$ conformally conjugated to ϕ and $\psi(z) = (1-\varphi(z))/(1-z)$, $z \in \mathbf{U}$, then $\sigma(C_\phi) = \sigma(T_{\psi,\varphi}) = \mathbf{T}$ by [2, Theorem 7]. The essential spectrum must be equal to the spectrum since the point spectrum of this kind of operator is empty. Indeed, by Remark 2.6, C_ϕ cannot have eigenvalues on the unit circle, since eigenvalues belong to the numerical range. □

We consider now the other type of invertible composition operators: those induced by hyperbolic half–plane automorphisms. First, let us recall the description of the spectra of hyperbolic automorphic composition operators on $H^2(\mathbf{U})$:

Theorem 2.8 ([15, Theorem 6])**.** *If φ is a hyperbolic disc automorphism with attractive fixed point $a \in \mathbf{T}$, then the spectrum $\sigma(C_\varphi)$ is given by*

$$\sigma(C_\varphi) = \{z \in \mathbf{C} : 1/r \leq |z| \leq r\} \tag{5}$$

where $r = 1/\sqrt{\varphi'(a)}$.

Thus, the spectrum of a hyperbolic composition operator acting on $H^2(\mathbf{U})$ is a circular annulus centered at the origin with nonempty interior.

Remark 2.9. *The point spectrum of a hyperbolic automorphic composition operator on $H^2(\mathbf{U})$ is the interior of its spectrum.*

Indeed, this fact was proved in the particular case of hyperbolic automorphic composition operators with antipodal fixed points in [10], but holds in general because any hyperbolic automorphic composition operator on $H^2(\mathbf{U})$ is similar to a hyperbolic automorphic composition operator with antipodal fixed points. In the case of hyperbolic automorphic composition operators on $H^2(\Pi^+)$, the situation is different, the spectrum being a circle, as we prove in the following. First recall a well known fact, namely:

Remark 2.10. *If $b \in \mathbf{T}$ is the repulsive fixed point of φ, a hyperbolic disc automorphism having attracting fixed point $a \in \mathbf{T}$, then $\varphi'(a)\varphi'(b) = 1$.*

The fact contained by Remark 2.10 has elementary proofs. It can also be obtained as a consequence of Theorem 2.8. Indeed, φ^{-1} is a hyperbolic disc automorphism with attractive fixed point b. By the spectral mapping theorem and (5), one gets $\sigma(C_\varphi) = \sigma(C_\varphi)^{-1} = \sigma(C_{\varphi^{-1}})$, which implies $\sqrt{\varphi'(a)} = \sqrt{(\varphi^{-1})'(b)}$, and clearly, $(\varphi^{-1})'(b) = 1/\varphi'(b)$. Next we determine the spectrum of hyperbolic automorphic composition operators on $H^2(\Pi^+)$. We start with a technical lemma:

Lemma 2.11. *If $T_{\psi,\varphi}$ is a bounded, nonzero weighted composition operator on $H^2(\mathbf{U})$ so that the composition operator C_φ has a nonconstant, inner, invariant function (i.e. $C_\varphi u = u$ for some nonconstant, inner u), then either $\sigma_p(T_{\psi,\varphi}) = \emptyset$ or all the eigenspaces of $T_{\psi,\varphi}$ are infinite–dimensional.*

Proof. Let u be a nonconstant, inner invariant function of C_φ and f an eigenfunction of $T_{\psi,\varphi}$ associated to the eigenvalue λ. It is elementary to prove that the infinite set $\{u^n f : n = 1, 2, 3, \dots\}$ is a linearly independent set consisting of eigenfunctions of $T_{\psi,\varphi}$ corresponding to eigenvalue λ. □

Theorem 2.12. *Let $\phi(z) = Kz + ir$ where K and $r \neq 0$ are real numbers and $K > 0$, $K \neq 1$ be a hyperbolic half–plane automorphism. Then $\sigma(C_\phi) = \sigma_e(C_\phi) = \{z \in \mathbf{C} : |z| = 1/\sqrt{K}\}$, $\sigma_p(C_\phi) = \emptyset$, and $W(C_\phi) = (1/\sqrt{K})\mathbf{U}$.*

Proof. Let $T_{\psi,\varphi}$ be the weighted composition operator on $H^2(\mathbf{U})$ unitarily equivalent to C_ϕ described in the proof of Theorem 2.4. In our case φ is a hyperbolic disc automorphism with fixed points 1 and $q \in \mathbf{T}$. The map ψ is both bounded and bounded away from 0 and $\psi(1) = \varphi'(1) = 1/K$, whereas $\psi(q) = 1$. Hence $\varphi'(q) = K$, and so

$$\psi(1)/\sqrt{\varphi'(1)} = \psi(q)/\sqrt{\varphi'(q)} = \frac{1}{\sqrt{K}} = r(C_\phi),$$

hence, by [8, Theorem 4.8], $\sigma(T_{\psi,\varphi}) = \{z \in \mathbf{C} : |z| = 1/\sqrt{K}\}$.

If, arguing by contradiction, one assumes that $\sigma(C_\phi) \neq \sigma_e(C_\phi)$, then $\sigma(T_{\psi,\varphi}) \neq \sigma_e(T_{\psi,\varphi})$, and one gets that there is a complex number λ so that $|\lambda| = 1/\sqrt{K}$ and $T_{\psi,\varphi} - \lambda I$ is a noninvertible Fredholm operator having null Fredholm index. By Lemma 2.11, this is a contradiction. Indeed, it is known that hyperbolic automorphic composition operators on $H^2(\mathbf{U})$ have nonconstant, inner, invariant functions [13].

To prove $\sigma_p(C_\phi) = \emptyset$, argue by contradiction and assume the contrary fact holds. Then one can consider an eigenvalue λ of C_ϕ having absolute value $1/\sqrt{K}$ and an associated eigenfunction f. Let $g = f \circ \gamma$. According to [7, pp. 127], g is in $H^2(\Pi^+)$ and, as one can easily check, $C_\varphi g = \lambda g$, which is a contradiction, given Remark 2.9.

Finally, one has $W(C_\phi) = W(T_{\psi,\varphi})$, for which reason, $W(C_\phi)$ is a disc centered at the origin [14]. Given that $||C_\phi|| = r(C_\phi) = 1/\sqrt{K}$, it follows that the numerical radius of C_ϕ is less than or equal to $1/\sqrt{K}$, since the numerical radius of an operator is less than or equal the norm of that operator and the spectrum of that same operator is contained in the closure of its numerical range. Thus, the only thing to prove is that $W(C_\phi)$ is the open disc of radius $1/\sqrt{K}$, not the closed one. If, arguing by contradiction, one assumes that $|\langle C_\phi f, f\rangle| = 1/\sqrt{K}$, for some f having norm 1, then a standard application of the Cauchy–Schwarz inequality shows that

$$1/\sqrt{K} \leq ||C_\phi f||\,||f|| \leq ||C_\phi|| = 1/\sqrt{K}$$

so, the Cauchy–Schwarz inequality being in our case an equality, it follows that f and $C_\phi f$ are linearly dependent, that is, f is an eigenvector of C_ϕ, which is a contradiction, given that we proved $\sigma_p(C_\phi) = \emptyset$. □

3 Normal composition operators

If one denotes C_{ϕ_r} the unitary composition operator with symbol $\phi_r(z) = z + ir$, $r \in \mathbf{R}$, one should note that $\phi_r \circ \phi_\rho = \phi_{r+\rho}$ $r, \rho \in \mathbf{R}$, and so, the unitary composition operators on $H^2(\Pi^+)$ form a one parameter group of operators isomorphic to the additive group $\mathbf{R}$.

Another interesting composition operator semigroup is that of translations. Let T_w denote the symbol $T_w(z) = z + w$, $z \in \Pi^+$. The obvious relation $T_w \circ T_u = T_{u+w}$, $w, u \in \Pi^+$ shows that $\mathcal{T} = \{C_{T_w} : w \in \Pi^+\}$ is a semigroup of operators isomorphic to the additive semigroup Π^+. As we will prove in this section, $\mathcal{T}$ is a semigroup of normal contractions. Its sub semigroup $\mathcal{H} = \{C_{T_b} : b \geq 0\}$ consists of all Hermitian composition operators on $H^2(\Pi^+)$. Indeed:

Theorem 3.1. *The Hermitian composition operators acting on $H^2(\Pi^+)$ are exactly those belonging to the unital semigroup $\mathcal{H} = \{C_{T_b} : b \geq 0\}$ and*

$$\sigma(C_{T_b}) = \sigma_e(C_{T_b}) = [0, 1] \qquad b > 0. \tag{6}$$

The numerical range of Hermitian composition operators acting on $H^2(\Pi^+)$ is given by the equality

$$W(C_{T_b}) = (0, 1) \qquad b > 0. \tag{7}$$

Proof. Assume C_ϕ is a Hermitian composition operator on $H^2(\Pi^+)$. Then, C_ϕ is unitarily equivalent to the weighted composition operator $T_{\psi,\varphi}$ acting on $H^2(\mathbf{U})$, where as before, $\psi(z) = (1-\varphi(z))/(1-z)$. The composition symbol φ of that operator is the conformal conjugate of ϕ and since, the operator $T_{\psi,\varphi}$ must be Hermitian, its composition symbol needs to be a linear fractional selfmap of $\mathbf{U}$ by [3, Theorem 2.1]. Therefore ϕ must be a linear fractional selfmap of Π^+, that is ϕ must be of the form $\phi(w) = aw + b$, with $a > 0$ and $\Re b \geq 0$ (Corollary 2.3). It is straightforward to find φ the conformal conjugate of ϕ and ψ, getting:

$$\varphi(z) = \frac{(a-b+1)z + a + b - 1}{(a-b-1)z + a + b + 1}, \quad \psi(z) = \frac{2}{(a-b-1)z + a + b + 1}, \tag{8}$$

and

$$\varphi'(z) = \frac{(a-b+1)(a+b+1) - (a-b-1)(a+b-1)}{((a-b-1)z + a + b + 1)^2}.$$

Let $c = \psi(0)$ and $a_0 = \varphi(0)$. According to [3, Theorem 2.1], in order that $T_{\psi,\varphi}$ be Hermitian, it is necessary that c and $a_1 = \varphi'(0)$ be real numbers. Given that $c = 2/(a + b + 1)$ and a is a real number, it follows that b too must be real, more exactly $b \geq 0$. In that case, the condition $\varphi'(0) \in \mathbf{R}$ is clearly satisfied as well. Another condition $T_{\psi,\varphi}$ must satisfy in order to be Hermitian is $\psi(z) = c/(1 - \overline{a_0}z)$, [3, Theorem 2.1]. By a straightforward computation one can see that $\psi(z) = c/(1 - \overline{a_0}z)$ holds if and only if

$$\frac{2}{(a-b-1)z + a + b + 1} = \frac{\frac{2}{1+a+b}}{1 - \frac{a+b-1}{a+b+1}z} \qquad z \in \mathbf{U},$$

that is if and only if $a = 1$. Thus, if C_ϕ is Hermitian, the map ϕ must be of the form $\phi(w) = w + b$ and

$$b \geq 0, \quad a_0 = b/(2+b), \quad c = 2/(2+b), \quad \text{and} \quad a_1 = 4/(2+b)^2. \tag{9}$$

Then

$$\varphi(z) = \frac{(2-b)z + b}{2 + b - bz} \quad \text{and} \quad \psi(z) = \frac{2}{2 + b - bz} \qquad z \in \mathbf{U}. \tag{10}$$

For the converse implication, recall the following. According to [3, Theorem 2.1], any analytic selfmap $\tilde{\varphi}$ of $\mathbf{U}$ and any map $\tilde{\psi}$ analytic on $\mathbf{U}$ induce a Hermitian weighted composition operator $T_{\tilde{\psi},\tilde{\varphi}}$ if $\tilde{\varphi}$ and $\tilde{\psi}$ have the form

$$\tilde{\varphi}(z) = a_0 + \frac{a_1 z}{1-\overline{a_0}z} \quad \tilde{\psi}(z) = \frac{c}{1-\overline{a_0}z}, \tag{11}$$

and the constants a_0, a_1, and c satisfy the conditions

$$a_0 \in \mathbf{U}, a_1 \in \mathbf{R}, \quad \text{and} \quad c \in \mathbf{R}. \tag{12}$$

Considering the values of a_0, a_1, and c given by (9), one can easily check that condition (12) is satisfied. Substitute the chosen values of a_0, a_1, and c in (11). A straightforward computation establishes the equalities

$$\varphi = \tilde{\varphi} \quad \text{and} \quad \psi = \tilde{\psi},$$

thus proving the fact that all operators of form $T_{\psi,\varphi}$ are Hermitian. Therefore, the Hermitian composition operators acting on $H^2(\Pi^+)$ are those belonging to the set $\mathcal{H} = \{C_{T_b} : b \geq 0\}$. For such operators, Corollary 5.13 in [3], combines with the fact that C_{T_b} is unitarily equivalent to $T_{\psi,\varphi}$ with ψ and φ given by equality (10), into proving that $\sigma(C_{T_b}) = [0,1]$, $b > 0$. The fact that the spectrum and essential spectrum of $\sigma(C_{T_b})$ coincide follows like in previous proofs from the fact that $\sigma(C_{T_b}) = [0,1]$ has no interior points or isolated points. Finally, these operators being Hermitian, their numerical range must be an interval of the line with endpoints 0 and 1 (since the closure of the numerical range must be equal to the convex hull of the spectrum, a rather well known property of normal operators). An endpoint of $[0,1]$ must be an eigenvalue of C_{T_b} if it belongs to $W(C_{T_b})$. The point 0 cannot be an eigenvalue because C_{T_b} is obviously injective. The point 1 cannot be an eigenvalue either because, if one assumes $C_{T_b} f = f$ for some $f \in H^2(\Pi^+)$, then one gets $f(z) = f(z+nb)$ for all $z \in \Pi^+$ and $n = 1,2,3\ldots$ Letting $n \to \infty$, one gets $f = 0$, because $H^2(\Pi^+)$–functions have null limit if $z \to \infty$ through a half–plane properly contained in Π^+ (see [4, Ch11, Corollary 2]). This proves equality (7). □

The analytic semigroups studied so far are sub–semigroups of the Möbius semigroup of composition operators, that is of the class of composition operators with symbols given by (2). The next question we address is which composition operators with Möbius symbol are normal. The answer is those with symbols having property $a = 1$ or $\Re b = 0$. To prove the announced result, we introduce more notation, and prove some preliminary lemmas.

Recall that $H^2(\mathbf{U})$ is a reproducing kernel Hilbert space with kernel functions

$$K_\lambda(z) = \frac{1}{1-\overline{\lambda}z} \qquad z \in \mathbf{U}.$$

If ϕ has form (2), then its conformal conjugate φ and the map $\psi(z) = (1-\varphi(z))/(1-z)$ are given by (8) and one can see that the following equality holds:

$$\psi(z) = \frac{1-\varphi(z)}{1-z} = cK_\lambda(z) \qquad z \in \mathbf{U},$$

where

$$c = \frac{2}{a+b+1} \qquad \text{and} \qquad \lambda = \frac{\overline{b}+1-a}{\overline{b}+1+a}. \tag{13}$$

The consequence of these considerations is:

Lemma 3.2. *The operator C_ϕ is normal if and only if the weighted composition operator $T_{K_\lambda,\varphi}$ is normal, where $\phi(z) = az + b$ and λ is given by* (13).

Proof. As noted before, C_ϕ is unitarily equivalent to the weighted composition operator $T_{cK_\lambda,\varphi}$, where c and λ are given by (13). Obviously, one has that $T_{cK_\lambda,\varphi} = cT_{K_\lambda,\varphi}$. □

Next, we observe the simple fact that two nonzero weighted composition operators on $H^2(\mathbf{U})$ coincide if and only if they have identical symbols. More formally:

Lemma 3.3. *If T_{ψ_1,φ_1} and T_{ψ_2,φ_2} are nonzero, bounded weighted composition operators on $H^2(\mathbf{U})$ then $T_{\psi_1,\varphi_1} = T_{\psi_2,\varphi_2}$ if and only if $\psi_1 = \psi_2$ and $\varphi_1 = \varphi_2$.*

Proof. If $T_{\psi_1,\varphi_1} = T_{\psi_2,\varphi_2} \neq 0$, then $\psi_1 = T_{\psi_1,\varphi_1}(1) = T_{\psi_2,\varphi_2}(1) = \psi_2$ and $\psi_1 = \psi_2$ is a nonzero, $H^2(\mathbf{U})$–function. Also, the equality $T_{\psi_1,\varphi_1} = T_{\psi_2,\varphi_2}$ implies that $\psi_1\varphi_1 = T_{\psi_1,\varphi_1}(z) = T_{\psi_2,\varphi_2}(z) = \psi_2\varphi_2$, hence $\varphi_1 = \varphi_2$. Clearly, $T_{\psi_1,\varphi_1} = T_{\psi_2,\varphi_2}$ if $\psi_1 = \psi_2$ and $\varphi_1 = \varphi_2$. □

Based on all that, we prove the announced characterization of normal composition operators with Möbius symbols:

Theorem 3.4. *The operator C_ϕ with symbol $\phi(z) = az + b$, $a > 0$ and $\Re b \geq 0$ is a normal operator on $H^2(\Pi^+)$ if and only if $a = 1$ or $\Re b = 0$.*

Proof. Given a Möbius selfmap $f(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ of $\mathbf{U}$, let $\sigma(z) = \frac{\overline{\alpha} z-\overline{\gamma}}{-\overline{\beta} z+\overline{\delta}}$. According to [2, Proposition 12] the weighted composition operator $T_{K_{\sigma(0)},f}$ is normal if and only if

$$Tg_1, f_1 = Tg_2, f_2 \tag{14}$$

where

$$g_1 = \frac{|\delta|^2}{|\delta|^2 - |\beta|^2 - (\overline{\beta}\alpha - \overline{\delta}\gamma)z} \qquad g_2 = \frac{|\delta|^2}{|\delta|^2 - |\gamma|^2 - (\overline{\beta}\delta - \overline{\alpha}\gamma)z}$$

$$f_1 = \sigma \circ f \quad \text{and} \quad f_2 = f \circ \sigma.$$

Note that, we are interested in the particular case $f = \varphi$, where φ is given by (8) and hence, the map σ is

$$\sigma(z) = \frac{z(a+1-\overline{b}) - (a-1-\overline{b})}{-(a+\overline{b}-1)z + (a+\overline{b}+1)}.$$

For that particular choice of f, $\sigma(0) = \lambda$ (see (13)), and so, C_ϕ is normal if and only if $T_{K_{\sigma(0)},\varphi}$ is normal. This will happen if and only if (14) holds. For the choice $f = \varphi$, one has that $\delta = a+b+1 \neq 0$ and hence, the operators involved in (14) are nonzero weighted composition operators. Therefore, (14) holds if and only if

$$g_1 = g_2 \qquad \text{and} \qquad f_1 = f_2.$$

It is easy to see that $g_1 = g_2$ is equivalent to

$$\overline{\beta}(\alpha - \delta) = \gamma(\overline{\delta} - \overline{\alpha}) \quad \text{and} \quad |\beta| = |\gamma|, \tag{15}$$

whereas condition $f_1 = f_2$ is equivalent to

$$\sigma \circ \varphi = \varphi \circ \sigma. \tag{16}$$

The equality $|\beta| = |\gamma|$ is equivalent to

$$a = 1 \quad \text{or} \quad \Re b = 0.$$

If $a = 1$, then

$$\varphi(z) = \frac{z(2-b)+b}{(2+b)-bz} \qquad \text{and} \qquad \sigma(z) = \frac{z(2-\overline{b})+\overline{b}}{(2+\overline{b})-\overline{b}z},$$

and it is a routine computation to check that relations (15) and (16) hold for all $b \in \mathbf{C}$, $\Re b \geq 0$.

Let us consider the case $\Re b = 0$ now. Let $t \in \mathbf{R}$ and $b = it$. Then

$$\varphi(z) = \frac{(a-it+1)z + a + it - 1}{(a-it-1)z + a + it + 1} \quad \text{and} \quad \sigma(z) = \frac{(a+it+1)z - (a+it-1)}{-(a-it-1)z + a - it + 1},$$

and again, it is a routine computation to check that relations (15) and (16) hold for all $t \in \mathbf{R}$. □

As an application of Theorem 3.4, we prove the following:

Proposition 3.5. *The only normal composition operators on $H^2(\Pi^+)$ whose symbols are inner functions are the automorphic composition operators.*

Proof. Indeed, if C_ϕ is normal and ϕ inner, then φ, the conformal conjugate of ϕ is inner too and, by [2, Proposition 3], φ needs to be univalent. Hence φ must be a conformal automorphism. This means C_ϕ must be an automorphic composition operator and, by Theorems 2.4 and 3.4 in this paper, all automorphic composition operators on $H^2(\Pi^+)$ are normal. □

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