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S-Toeplitz Composition Operators

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Abstract. Operators on function spaces acting by composition to the right with a fixed selfmap φ of some set are called composition operators of symbol φ. Isometric operators S on a Hilbert space with the property that the sequence $\{S^{*n}\}$ tends to 0 pointwise are called forward unilateral shifts. A Hilbert space operator T is called S–Toeplitz if $S^*TS = T$ and S–uniformly asymptotically Toeplitz, (S–UAT), S–strongly asymptotically Toeplitz, (S–SAT), respectively S–weakly asymptotically Toeplitz, (S–WAT), if the sequence $\{S^{*n}TS^n\}$ is convergent uniformly, strongly, respectively weakly. We study when composition operators on the Hilbert Hardy space H^2 are M_ϕ–Toeplitz, M_ϕ–UAT, M_ϕ–SAT, or M_ϕ–WAT, where ϕ is a nonconstant inner function and M_ϕ the multiplication operator induced by that function.

1 INTRODUCTION

Let H^2 denote the Hilbert Hardy space on the open unit disk $\mathbb{U}$, that is the space of all functions f analytic in $\mathbb{U}$ satisfying the condition

$$\|f\|_2 := \sup_{0<r<1} \left(\int_{\partial \mathbb{U}} |f(r\zeta)|^2 \, dm(\zeta) \right)^{1/2} < \infty, \tag{1}$$

where m is the normalized Lebesgue measure.

It is well known that $\| \quad \|_2$ is a Hilbert norm on H^2 with alternative description

$$\|f\|_2 = \sqrt{\sum_{n=0}^{\infty} |c_n|^2}, \tag{2}$$

where $\{c_n\}$ is the sequence of Maclaurin coefficients of f.

Recall that H^2 is a reproducing kernel Hilbert space with kernel–functions $K_p(z) = 1/(1-\overline{p}z)$, that is the functions K_p have the *reproducing property:*

$$f(p) = < f, K_p > \qquad f \in H^2,\ p \in \mathbb{U}.$$

For each analytic selfmap φ of $\mathbb{U}$, *the composition operator of symbol* φ is the following operator

$$C_\varphi f = f \circ \varphi \qquad f \in H^2. \tag{3}$$

Composition operators on H^2 are bounded, as a consequence of Littlewood's Subordination Principle, [7, Theorem 1.7], which says that composition operators whose symbols fix the origin are contractions. The space L^2 is the Lebesgue space of index two on the unit circle $\partial\mathbb{U}$ and L^∞ denotes the space of essentially bounded measurable functions on $\partial\mathbb{U}$. Any function f in H^2 has nontangential limits a.e. on $\partial\mathbb{U}$. The nontangential limit–function is in L^2 and we will denote it by the same symbol as the function itself, relying on context to distinguish between the two notions. The L^2–norm of f coincides to $\|f\|_2$.

As noted in [6, Ch 7.], Toeplitz operators were originally studied as operators on separable, infinite-dimensional Hilbert spaces, (actually spaces of square-summable sequences) having matrices with constant diagonals with respect to some orthonormal basis of choice. We will say that such an operator has a Toeplitz matrix. Denoting by M_z the multiplication operator on H^2 having symbol the coordinate function, that is

$$M_z f(z) = zf(z) \qquad z \in \mathbb{U},\ f \in H^2,$$

one can easily see that an operator A on H^2 has a Toeplitz matrix with respect to the monomial basis of that space if and only if $M_z^* A M_z = A$. On the other hand, Brown and Halmos [4], noted that a bounded operator A has a Toeplitz matrix with respect to the monomial basis of H^2 if and only if A is the compression to H^2 of a multiplication operator L_ϕ on L^2; that is, there is some ϕ in L^∞ called the symbol of the multiplication operator L_ϕ so that

$$Af = PL_\phi f = P(\phi f) \qquad f \in H^2, \tag{4}$$

where P is the orthogonal projection of L^2 onto H^2. We call the operator A in (4) the M_z–Toeplitz operator of symbol ϕ and denote $A = T_\phi$. These operators are considered the classical Toeplitz operators, due to the result above.

Now, the matricial approach to the notion of Toeplitz operator can benefit from the fact that any orthonormal basis of an infinite-dimensional, separable Hilbert space can be understood as the orbit of a unit vector under a unilateral, forward shift of multiplicity 1.

Here are some explanations of this statement. A unilateral forward shift is any isometric operator S on a separable, infinite-dimensional Hilbert space H with the property that $\{S^{*n}\}$ tends strongly to 0 on H. The closed subspace $L = H \ominus SH$ is called the wandering subspace of the shift S because

$$S^n L \perp S^m L \qquad m \neq n \geq 0$$

and

$$H = L \oplus SL \oplus S^2 L \oplus \cdots \oplus S^n L \oplus \ldots \tag{5}$$

The Hilbert dimension $\dim L$ is called the multiplicity of the shift S. When the multiplicity is 1, there is essentially only one unit vector e in L, (modulo multiplication with unimodular scalars). The orbit $\{e, Se, S^2 e, \ldots, S^n e, \ldots\}$ of e under S is then an orthonormal basis. Conversely, given any orthonormal basis $\{e_0, e_1, \ldots, e_n, \ldots\}$, the equation

$$Se_n = e_{n+1} \qquad n \geq 0$$

uniquely determines a unilateral forward shift S on H having multiplicity 1 and such that the given basis is the orbit of e_0 under S. The fact that the matrix, with respect to the basis above, of some operator, is a Toeplitz matrix is equivalent to the equation $S^*TS = T$, [15]. Therefore, given some unilateral forward shift S of any multiplicity, we follow [15] and call a Hilbert space operator S–Toeplitz if the aforementioned equation holds. It should be added that, if the multiplicity of S is more than 1, the S–Toeplitz operators have a block-matrix with constant diagonals with respect to the direct sum decomposition (5).

The classical Toeplitz operators are called M_z–Toeplitz operators in this paper because, as the reader will easily note, M_z is a unilateral forward shift of multiplicity 1 on H^2. Note that we distinguish between multiplication operators on L^2 and those on H^2 by denoting the former by L and the latter by M. The symbol of the multiplication operator is specified as a subscript in both cases. Obviously, the symbols of multiplication operators on H^2 need to belong to H^∞, the space of bounded analytic functions on $\mathbb{U}$, so that those operators will be bounded.

Barria and Halmos [1] introduced a natural asymptotic generalization of the notion of M_z–Toeplitz operator defining strong asymptotically Toeplitz operators. Other authors, [8] extended their definition considering the other usual topologies on the space of linear, bounded operators. More exactly, an operator T is called uniformly asymptotically Toeplitz, strongly asymptotically Toeplitz, respectively weakly asymptotically Toeplitz if the sequence $\{M_z^{*n}TM_z^n\}$ is convergent in the uniform operator topology, the strong operator topology, respectively the weak operator topology.

We introduce the corresponding notions of S–asymptotic Toeplitzness by substituting above the forward shift M_z by any unilateral, forward shift S, with the comment that the limit A is necessarily an S–Toeplitz operator. That operator is called the *asymptotic image* of T. We use the abreviations S–UAT, S–SAT, respectively S–WAT for S–uniformly asymptotically Toeplitz, S–strongly asymptotically Toeplitz, respectively S–weakly asymptotically Toeplitz.

Recently, Nazarov and Shapiro studied M_z–asymptotic Toeplitzness for composition operators on H^2 [12]. The current paper is inspired by that paper and starts with the elementary observation that some of the results proved there are easily extendable to S–Toeplitzness with respect to forward shifts of the form $S = M_\phi$ where ϕ is a nonconstant inner function. This string of immediate generalizations of results from [12] is in the fourth section of this paper. This introductory section is dedicated to setting up the notation and introducing the main concepts. In the second section we study which composition operators C_φ can be M_ϕ–Toeplitz. We are able to give a complete answer in the case when φ has a fixed point in $\mathbb{U}$, (Theorem 1). It turns out that only composition operators of inner symbol can be M_ϕ–Toeplitz. Furthermore, ϕ must be an invariant inner function of C_φ. The third section is dedicated to operators that are M_ϕ–Toeplitz with respect to Guyker shifts, that is with respect to shifts of the form M_{α_p} where $\alpha_p(z) = (p - z)/(1 - \bar{p}z)$, p being any fixed constant in $\mathbb{U}$. It turns out that an operator on H^2 is M_z–Toeplitz, respectively M_z–UAT if and only if it is M_{α_p}–Toeplitz, respectively M_{α_p}–UAT, where $p \in \mathbb{U}$ is arbitrary and fixed. These facts and the characterization of M_z–UAT composition operators [12, Theorem 1.1], combine into showing that the only M_{α_p}–UAT composition operators are the compact composition operators and the identity operator, (Corollary 2). In section 4 the question of when a composition operator is M_ϕ–SAT or M_ϕ–WAT is studied. Section 5 is dedicated to operators of form $C_\varphi^* C_\varphi$. It is shown that they are M_ϕ–Toeplitz if and only if φ is inner, which extends a recent result in [2]. The operators under consideration are always M_ϕ–WAT, no matter the nonconstant inner function ϕ (Theorem 5). The situations when they are M_ϕ–UAT or SAT are also studied.

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Note that what makes M_z a forward shift is the fact that z is a non-constant inner function, that is a bounded analytic function with nontangential limit–function unimodular a.e. on $\partial\mathbb{U}$. Indeed:

Proposition 1. *If ϕ is a non–constant inner function, then the multiplication operator*

$$M_\phi f = \phi f \qquad f \in H^2$$

is a unilateral, forward shift. That shift has finite multiplicity if and only if ϕ is a finite Blaschke product, (i.e. it is a finite product of factors of form $\alpha_p(z) = (p-z)/(1-\overline{p}z)$, with $p \in \mathbb{U}$ times a unimodular constant). In that case, the multiplicity of M_ϕ equals the number of factors of the finite Blaschke product.

Proof. Obviously, M_ϕ is isometric if ϕ is inner. As is well known, $M_\phi^{*n} = PL_{\overline{\phi^n}}\,|_{H^2}$. Let k be a fixed, nonnegative integer.

$$\|PL_{\overline{\phi^n}}e^{ik\theta}\|_2^2 = \|Pe^{ik\theta}\overline{\phi^n}\|_2^2 = \sum_{j=0}^{k}\left(\frac{|\,(\phi^n)^{(j)}(0)\,|}{j!}\right)^2 \to 0,$$

since $\phi^n \to 0$ uniformly on compacts. Thus $M_\phi^{*n} f \to 0$ for each analytic polynomial f. Since the sequence $\{M_\phi^{*n}\}$ is norm-bounded and the analytic polynomials dense in H^2, it follows that $\{M_\phi^{*n}\}$ tends strongly to 0, that is, M_ϕ is a unilateral forward shift. The fact that $\dim(H^2 \ominus \phi H^2)$ is finite if and only if ϕ is a finite Blaschke product, (in which case, it equals the number of factors of that product), is well known [14, Theorem 3.14]. □

There is only one kind of M_ϕ-shifts, (modulo multiplication by a unimodular constant), having multiplicity 1, namely *the Guyker shifts* M_{α_p}, named that way because the normalized kernel $k_p = \sqrt{1-|p|^2}K_p$ is a unit vector in the wandering subspace $L = H^2 \ominus \alpha_p H^2$ of M_{α_p} and hence the orbit $\{\alpha_p^n k_p : n = 0, 1, 2, \dots\}$ of k_p under M_{α_p} is an orthonormal basis of H^2, (known in the literature as *the Guyker basis of index p*, since it was introduced by J. Guyker [9]). Visibly, if one takes $p = 0$, the corresponding Guyker shift is $M_{-z} = -M_z$ and hence, one studies classical Toeplitz–concepts.

In [12], the authors show that the only M_z–Toeplitz composition operator is the identity. This result extends to Guyker shifts, as we will soon prove. First, we note that nontrivial composition operators can be M_ϕ–Toeplitz as early as multiplicity 2, for example example, C_{α_p} is $M_{z\alpha_p}$–Toeplitz because $C_{\alpha_p}M_{z\alpha_p} = M_{z\alpha_p}C_{\alpha_p}$. The example above is typical, as we prove in the following.

Theorem 1. *If φ is an analytic selfmap of $\mathbb{U}$ and ϕ a non–constant inner function then, if C_φ is M_ϕ–Toeplitz, φ needs to be an inner function. If φ is inner, then C_φ is M_ϕ–Toeplitz if and only if ϕ is an invariant function of C_φ, (that is $C_\varphi\phi = \phi$) or, equivalently, if and only if M_ϕ and C_φ commute. If φ is an inner function, other than the identity, that fixes a point $p \in \mathbb{U}$ then the only situations when C_φ can be M_ϕ–Toeplitz with respect to some shift M_ϕ are when φ is an elliptic disk automorphism with property $\varphi(z) = \alpha_p(\lambda\alpha_p(z))$ where λ is some root of unity. Such a composition operator is M_ϕ–Toeplitz if and only if ϕ has the form*

$$\phi(z) = (B \circ \alpha_p)\,(S_\nu \circ \alpha_p)\,. \tag{6}$$

Above B is a, (possibly constant) Blaschke product with properties

$$B(z) = cz^{kN}\prod_{a\in Z}\left(\frac{|a|}{a}\,\frac{a-z}{1-\overline{a}z}\right)^{j(a)} \qquad z \in \mathbb{U},$$

where $k \geq 0$ is a fixed integer, each $j(a) > 0$ is an integer, c is a unimodular constant, N is the order of the root of unity λ, and Z is a countable, (possibly empty) subset of

$\mathbb{U}\setminus\{0\}$ with the properties

$$\lambda Z = Z, \qquad \sum_{a\in Z} 1-|a| < +\infty, \qquad \text{and} \qquad j(a)=j(b),$$

if $a,b\in Z$ and $a=\lambda^n b$ for some integer n. If $Z=\emptyset$, $B(z)=cz^{kN}$.

In relation (6), S_ν must be a, (possibly constant) singular inner function induced by a λz–invariant measure ν, that is

$$S_\nu(z) = c_1 \exp\left(-\int_{\partial\mathbb{U}} \frac{u+z}{u-z}\,d\nu(u)\right) \qquad z\in\mathbb{U}$$

where c_1 is a unimodular constant and ν is a Borel measure, singular with respect to Lebesgue measure, and with the additional property

$$\nu(\lambda E) = \nu(E)$$

valid for each Borel subset E of $\mathbb{U}$.

Proof. If C_φ is M_ϕ–Toeplitz, then $M_\phi^* C_\varphi M_\phi(z) = C_\varphi(z)$ that is $P(\overline{\phi}(\phi\circ\varphi)\varphi)=\varphi$. If arguing by contradiction, one assumes that φ is not inner, one obtains the contradiction

$$\|\varphi\|_2^2 = \|P(\overline{\phi}(\phi\circ\varphi)\varphi)\|_2^2 \le \int_{\partial\mathbb{U}} |\phi|^2|\phi\circ\varphi|^2|\varphi|^2\,dm < \int_{\partial\mathbb{U}} |\varphi|^2\,dm,$$

where the fact that ϕ is a non–constant inner function and the maximum modulus principle were used.

Assume that φ is inner and C_φ is M_ϕ–Toeplitz. In that case

$$1 = < C_\varphi(1), 1 > = < M_\phi^* C_\varphi M_\phi(1), 1 > = < \phi\circ\varphi, \phi > \le \|\phi\circ\varphi\|_2\|\phi\|_2 \le 1$$

by the Cauchy–Schwarz inequality and the fact that ϕ is inner. As is well known, the Cauchy–Schwarz inequality is an equality if and only if the vectors involved in it are colinear. Therefore, there is some complex number λ such that $\phi\circ\varphi=\lambda\phi$ and, since $1=<\phi\circ\varphi,\phi>$, it follows that $\lambda=1$. Thus ϕ must be an invariant function of C_φ, that is $C_\varphi\phi=\phi$. It is elementary to see that ϕ is an invariant function of C_φ if and only if M_ϕ and C_φ commute, and in that case, C_φ is necessarily M_ϕ–Toeplitz.

Assume now that the inner function φ, not the identity, fixes a point $p\in\mathbb{U}$ and C_φ is M_ϕ–Toeplitz. For the beginning, assume $p=0$. Denoting $\varphi^{[n]}=\varphi\circ\cdots\circ\varphi$, n–times, note that, by the Schwarz lemma in classical complex analysis, $\varphi^{[n]}(z)\to 0$ pointwise, unless φ is a rotation or φ is the identity. If φ is neither a rotation nor the identity, then $\phi\circ\varphi^{[n]}=\phi$, for all n, so $\phi\equiv\phi(0)$, that is ϕ is constant, a contradiction. If $\varphi(z)=\lambda z$ for some unimodular λ, not a root of unity, then it is easy to see that C_φ has no nonconstant invariant vectors. If λ is a root of unity having order $N>1$, we begin by noting that a singular inner function is an eigenfunction of C_φ if and only if that function is induced by a singular measure ν invariant under the rotation $\varphi(z)=\lambda z$. This is a direct consequence of [11, Theorem 3.2] where the singular eigenfunctions of automorphic composition operators are determined. More exactly, if φ is a disk automorphism, then S_ν is an eigenfunction of C_φ if and only if, the pull–back measure $\nu\varphi^{-1}$ is absolutely continuous with respect to ν and $d\nu\varphi^{-1}(u)=P(\varphi(0),u)\,d\nu(u)$, where $P(\varphi(0),u)$ is the Poisson kernel evaluated at $\varphi(0)$

and $u \in \partial\mathbb{U}$. In our case, this means $\nu(\overline{\lambda}E) = \nu(E)$ for each Borel set $E \subseteq \partial\mathbb{U}$, that is ν must be invariant under the rotation $\varphi(z) = \lambda z$. It is straightforward to see that any singular inner function induced by a measure that is invariant under the rotation $\varphi(z) = \lambda z$ is actually left invariant by C_φ. This and the fact that the composite of a Blaschke product respectively a singular inner function and φ is also a Blaschke product, respectively a singular inner function, leads to the conclusion that, if ϕ is an inner invariant function of C_φ, then the Blaschke product in its representation as a product of a Blaschke product times a singular inner function must be invariant under C_φ. The obvious relation

$$\frac{|a|}{a}\frac{a-\lambda z}{1-\overline{a}\lambda z} = \frac{|\overline{\lambda}a|}{\overline{\lambda}a}\frac{\overline{\lambda}a - z}{1-\overline{\overline{\lambda}a}z} \qquad a\,,z \in \mathbb{U}\,,\lambda \in \partial\mathbb{U}\,,a \neq 0$$

implies then that the Blaschke product under consideration must have the properties in the text of this theorem. To end the proof, note that, in the case $p \neq 0$, $\alpha_p \circ \varphi \circ \alpha_p$ is a rotation λz. A function $f \in H^2$ is left invariant by C_φ if and only if $f \circ \alpha_p$ is left invariant by $C_{\lambda z}$. □

Corollary 1. *If S is any Guyker shift, the only S–Toeplitz composition operator is the identity.*

Proof. Let $S = M_{\alpha_p}$. If C_φ is S–Toeplitz then $\alpha_p \circ \varphi = \alpha_p$. Since α_p is selfinverse, one gets that $\varphi(z) = z$, $z \in \mathbb{U}$. □

It should be added that, if φ fixes $p \in \mathbb{U}$ and is α_p–conformally conjugated to a rotation by a root λ of unity having order $N > 1$, then nonconstant, C_φ–invariant, singular inner functions always exist. Indeed, consider the singular inner function induced by the measure $\nu = \delta_1 + \delta_\lambda + \delta_{\lambda^2} + \cdots + \delta_{\lambda^{N-1}}$, where δ_u denotes the Dirac unit mass concentrated at u, for each $u \in \partial\mathbb{U}$. Clearly ν is λz–invariant, so $S_\nu \circ \alpha_p$ is a nonconstant singular inner invariant function of C_φ. Nonconstant C_φ–invariant Blaschke products obviously exist.

The next thing is to see when fixed point–free, inner maps φ of $\mathbb{U}$ can be M_ϕ–Toeplitz, for some inner, nonconstant ϕ. We begin by recalling a noted theorem.

Theorem 2 (Denjoy–Wolff). *Let φ be an analytic selfmap of $\mathbb{U}$ other than the identity or an elliptic disk automorphism. Then the sequence of iterates $\{\varphi^{[n]}\}$ converges uniformly on compacts to a constant $\omega \in \overline{\mathbb{U}}$ called the Denjoy–Wolff point of φ.*

Thus, if a function φ, as above, has a fixed point in $\mathbb{U}$, then that point is exactly its Denjoy–Wolff point. If φ is fixed point–free, then its Denjoy–Wolff point is on the unit circle $\partial\mathbb{U}$. We are interested in inner functions with Denjoy–Wolff point on $\partial\mathbb{U}$. For any selfmap φ and any $z \in \mathbb{U}$ we denote $O_\varphi(z) = \{z, \varphi(z), \ldots, \varphi^{[n]}(z), \ldots\}$ the orbit of z under φ. We say that the orbit $O_\varphi(z)$ *is Blaschke* if the condition $\sum_{\lambda \in O_\varphi(z)}(1 - |\lambda|) < +\infty$ is satisfied. If all orbits under φ are Blaschke, we say φ *has Blaschke orbits*. Our interest in these notions is explained in the following.

Proposition 2. *If φ is an analytic selfmap of $\mathbb{U}$ and C_φ has a nonconstant, bounded, analytic, invariant function ϕ then φ must have Blaschke orbits.*

If φ is an inner function with Denjoy–Wolff point $\omega \in \partial\mathbb{U}$, then the following are equivalent.

(i) *The orbit* $O_\varphi(0)$ *is Blaschke.*

(ii) $\varphi^{[n]} \to \omega$ *a.e. on* $\partial\mathbb{U}$.

(iii) *The map* φ *has Blaschke orbits.*

Proof. Let φ be an analytic selfmap of $\mathbb{U}$ such that C_φ has a nonconstant, bounded, analytic, invariant function ϕ. Then ϕ must be constant on each orbit of φ. Let p be an arbitrary point in $\mathbb{U}$. Then the elements of the orbit $O_\varphi(p)$ are zeros of the bounded analytic function $\alpha_{\phi(p)} \circ \phi$. Hence $O_\varphi(p)$ is Blaschke. Sice p was arbitrarily chosen, this takes care of the first statement above.

Let now φ be an inner function with Denjoy–Wolff point $\omega \in \partial\mathbb{U}$. The equivalence (i)$\Longleftrightarrow$(ii) was established in [3, Theorem 4.2]. Since clearly, (iii)$\Longrightarrow$(i), one only needs to show that (iii) holds if (i) holds. Let p be an arbitrary point in $\mathbb{U}$. Note that, if we denote $\psi = \alpha_p \circ \varphi \circ \alpha_p$, then

$$\psi^{[n]} = \alpha_p \circ \varphi^{[n]} \circ \alpha_p \qquad n = 1, 2, 3, \ldots$$

and hence, ψ is an inner function having Denjoy–Wolff point $\eta = \alpha_p(\omega)$ with the property $\psi^{[n]} \to \eta$ a.e. on $\partial\mathbb{U}$. Thus $O_\psi(0)$ is Blaschke. By the identity

$$1 - |\psi^{[n]}(0)|^2 = \frac{(1-|p|^2)(1-|\varphi^{[n]}(p)|^2)}{|1-\bar{p}\varphi^{[n]}(p)|^2} \qquad n = 1, 2, 3, \ldots$$

it follows that $O_\varphi(p)$ is Blaschke. □

If the Denjoy–Wolff point ω of an analytic selfmap φ is on $\partial\mathbb{U}$ then the angular derivative $\varphi'(\omega)$ is known to exist and satisfy the condition $0 < \varphi'(\omega) \le 1$, (which, of course, means that the aforementioned derivative is necessarily a real number). If $\varphi'(\omega) < 1$ then φ is called *of hyperbolic type.* If $\varphi'(\omega) = 1$ then φ is called *of parabolic type.* Analytic selfmaps of parabolic type are classified into two categories. The first is selfmaps of *parabolic automorphic type.* This means that the selfmap φ of parabolic type has *hyperbolically separated orbits,* that is,

$$\lim_{n\to+\infty} \rho(\varphi^{[n+1]}(z), \varphi^{[n]}(z)) > 0 \qquad z \in \mathbb{U}, \tag{7}$$

where ρ is the pseudohyperbolic distance $\rho(z,w) = |\alpha_w(z)|$, $z, w \in \mathbb{U}$. Either all the orbits of an analytic selfmap of parabolic type are hyperbolically separated or all of them are hyperbolically non–separated that is

$$\lim_{n\to+\infty} \rho(\varphi^{[n+1]}(z), \varphi^{[n]}(z)) = 0 \qquad z \in \mathbb{U}, \tag{8}$$

(see [3, Section 2.5] for a more thorough discussion of this phenomenon). In case (8) holds, φ is called *of parabolic non–automorphic type.* The limits in (7) or (8) necessarily exist because the sequence under scrutiny is decreasing, by the Schwarz–Pick lemma [16, Section 4.3], saying that analytic selfmaps of $\mathbb{U}$ are contractive under the pseudohyperbolic distance, that is, if φ is such a map, then

$$\rho(\varphi(z), \varphi(w)) \le \rho(z, w) \qquad z, w \in \mathbb{U}.$$

According to [3, Theorem 4.4], inner functions of hyperbolic type or of parabolic automorphic type have Blaschke orbits. On the other hand, some inner functions of parabolic non–automorphic type may have non–Blaschke orbits, (e.g. $\varphi(z) = \exp\left\{-2\frac{1-z}{1+z}\right\}$, see [3, Example 4.6]). By Proposition 2, if φ is such a map, then C_φ cannot be M_ϕ–Toeplitz, no matter the nonconstant inner map ϕ.

Recall that, if φ is a disk automorphism then φ is a parabolic disk automorphism if and only if it has only one fixed point $\omega \in \partial\mathbb{U}$. The automorphism φ is called hyperbolic if it has exactly 2 fixed points, both on $\partial\mathbb{U}$. Since the composite of a Blaschke product and a parabolic or hyperbolic disk automorphism is again a Blaschke product and a similar property holds for singular inner functions then:

Remark 1. *An automorphic composition operator C_φ with parabolic or hyperbolic symbol is M_ϕ–Toeplitz if and only if the Blascke product B and the singular inner function S in the representation $\phi = cBS$, $|c| = 1$ are eigenfunctions of C_φ corresponding to reciprocal eigenvalues.*

The Blaschke products, respectively the singular inner functions that are eigenfunctions of the parabolic or hyperbolic composition operators are characterized in [11, Section 3]. Here are some simple examples. If $\varphi(z) = (2z+1)/(z+2)$, then φ is hyperbolic of fixed points ± 1 and

$$B(z) = z \prod_{n=-\infty\, n\neq 0}^{+\infty} \operatorname{sign}(n)\varphi^{[n]}$$

is a convergent Blaschke product with the property $C_\varphi B = -B$, so C_φ is M_{B^2}–Toeplitz. The notation $\varphi^{[-n]}$ means $(\varphi^{-1})^{[n]}$, $n = 1, 2, 3, \ldots$

For a second example, consider the parabolic disk automorphism

$$\varphi(z) = \frac{(1-\pi i)z + \pi i}{1 + \pi i - \pi i z}.$$

It is straightforward to check that the atomic singular inner function

$$S(z) = \exp\left(-\frac{1+z}{1-z}\right)$$

is invariant under C_φ and hence C_φ is M_S–Toeplitz.

The determination of all inner invariant functions of nonautomorphic composition operators whose symbols have no fixed points in $\mathbb{U}$ is an open problem, beyond the scope of this paper.

3 GUYKER SHIFTS

The operator L_{α_p} will be referred to as the bilateral Guyker shift of symbol α_p. We begin by showing that, indeed L_{α_p} is a bilateral shift. This means that L_{α_p} is a unitary operator with the property that there is some closed subspace L of L^2 so that $L^n_{\alpha_p} L \perp L^m_{\alpha_p} L$ for any integers $m \neq n$ and L^2 is the direct sum of all subspaces $L^n_{\alpha_p} L$, $n = 0, \pm 1, \pm 2, \ldots$. It is not hard to see that, for any distinct integers $m \neq n$, one has that $\alpha_p^m k_p \perp \alpha_p^n k_p$, so, what we need to prove is that

$$\sum_{n=-\infty}^{\infty} \oplus \alpha_p^n L = L^2$$

where L is the linear subspace spanned by k_p. Let $\mathcal{B}_p := \{\alpha_p^n k_p, \overline{\alpha_p}^n k_p : n = 0,1,2,\dots\} = \{\alpha_p^n k_p : n = 0, \pm 1, \pm 2, \dots\}$, (where the equality $\overline{\alpha_p}^n k_p = \alpha_p^{-n} k_p$, $n \geq 0$ is in the sense of boundary–value functions), and assume that some $f \in L^2$ is orthogonal to $\mathcal{B}_p$. Then $f \in L^2 \ominus H^2$, since f is orthogonal to the Guyker basis of index p. Thus $\overline{f}$ is in H^2 and $\overline{f}(0) = 0$. Also, one has that

$$\overline{f} \perp \alpha_p^n \overline{k_p} \qquad \text{that is} \qquad k_p \overline{f} \perp \alpha_p^n \qquad n = 1,2,3,\dots$$

or, in other words

$$< C_{\alpha_p}^*(k_p\overline{f}), z^n >= 0 \qquad n = 1,2,3,\dots$$

It follows that $C_{\alpha_p}^*(k_p\overline{f}) \equiv c$ for some $c \in \mathbb{U}$. To calculate c note that

$$c =< C_{\alpha_p}^*(k_p\overline{f}), 1 >=< k_p\overline{f}, 1 >= 0.$$

Hence $k_p\overline{f} \equiv 0$, since $C_{\alpha_p}^*$ is injective and so, $f \equiv 0$.

Therefore, for each $p \in \mathbb{U}$, the set $\mathcal{B}_p := \{\alpha_p^n k_p : n = 0, \pm 1, \pm 2, \dots\}$ is an orthonormal basis of L^2 we call *the bilateral Guyker basis of index p*. Since L_{α_p} is a bilateral shift of multiplicity 1, the commutant $\{L_{\alpha_p}\}'$ of L_{α_p} is the algebra $\mathcal{L}$ of all multiplication operators on L^2. Indeed, by [14, Theorem 1.20], $\mathcal{L}$ is maximal abelian, that is $\mathcal{L}$ is abelian and the only operators on L^2 commuting with all the operators in $\mathcal{L}$ are the operators in $\mathcal{L}$ themselves. Since L_z and L_{α_p} are bilateral forward shifts of multiplicity 1, they are unitarily equivalent, that is there is some unitary operator U such that $U^*L_zU = L_{\alpha_p}$. Therefore $\{L_{\alpha_p}\}' = U^*\{L_z\}'U = U^*\mathcal{L}U$, [14, Theorem 3.2]. Hence, $\{L_{\alpha_p}\}'$ is a maximal abelian operator algebra containing $\mathcal{L}$. The consequence is $\{L_{\alpha_p}\}' = \mathcal{L}$. Keeping this in mind, one can borrow from the proof of [4, Theorem 4] to prove

Proposition 3. *Let $p \in \mathbb{U}$ be fixed. A bounded operator on H^2 is M_z–Toeplitz if and only if it is M_{α_p}–Toeplitz.*

Proof. If T is M_z–Toeplitz then there is some $\varphi \in L^\infty$ so that $Tf = P(\varphi f)$, $f \in H^2$. Then for any inner, nonconstant ϕ

$$< M_\phi^* T M_\phi f, g >=< \varphi\phi f, \phi g >=< \varphi f, g > \qquad f, g \in H^2,$$

hence $M_\phi^* T M_\phi = T$. Thus, M_z–Toeplitz operators are M_ϕ–Toeplitz for any ϕ inner and nonconstant.

For $\phi = \alpha_p$, the converse is true. Assume $M_{\alpha_p}^* T M_{\alpha_p} = T$. If we show that T is the compression to H^2 of a multiplication operator on L^2, the proof is over. Note that T has a Toeplitz matrix with respect to the Guyker basis of index p, that is, for all integers $i, j \geq 0$

$$< T\alpha_p^{n+j} k_p, \alpha_p^{n+i} k_p >=< T\alpha_p^j k_p, \alpha_p^i k_p > \qquad n \geq 0.$$

Imitating part of the proof of [4, Theorem 4] with the bilateral Guyker basis $\mathcal{B}_p$ in place of the monomial basis of L^2, we introduce the operator sequence $\{T_n = L_{\alpha_p}^{*n} T P L_{\alpha_p}^n\}$ and claim it converges weakly to some A. To see this, note that, if $i, j \geq 0$ are integers, then

$$< T_n\alpha_p^j k_p, \alpha_p^i k_p >=< T\alpha_p^{n+j} k_p, \alpha_p^{n+i} k_p >=< T\alpha_p^j k_p, \alpha_p^i k_p > \qquad n \geq 0. \qquad (9)$$

If i and j are any integral values, then there is $n_0 \geq 0$ so that $n+i, n+j \geq 0$ if $n \geq n_0$ so, the computation above can be repeated, and thus, for each choice of i and j, the sequence above is eventually constant. Given that $\{T_n\}$ is norm-bounded and $\mathcal{B}_p$ is an orthonormal basis of L^2, it follows that $\{T_n\}$ tends weakly to some A which commutes with L_{α_p} because

$$< L^*_{\alpha_p} A L_{\alpha_p} \alpha_p^j k_p, \alpha_p^i k_p > = \lim_{n\to\infty} < L^{*(n+1)}_{\alpha_p} TPL^{n+1}_{\alpha_p} \alpha_p^j k_p, \alpha_p^i k_p > =$$

$$< A\alpha_p^j k_p, \alpha_p^i k_p > \qquad i, j = 0, \pm 1, \pm 2, \ldots$$

Thus $A \in \mathcal{L}$ and its compression to H^2 is T, by identity (9). □

Let us note that Feintuch's theorem [8, Theorem 4.1], characterizing M_z–UAT operators extends to:

Theorem 3. *If S is a forward unilateral shift of finite multiplicity on a Hilbert space, then an operator on that space is S–UAT if and only if it is a compact perturbation of an S–Toeplitz operator.*

Proof. The proof is practically the same as the original one. We include it for the sake of completeness. If S is a forward shift, then so is S^k for any k. For each such S, $P = SS^*$ is the projection on SH, (where H is the Hilbert space, on which S acts), and so, $Q = I - P$ is the projection on $L = H \ominus SH$, the wandering subspace of S, [15, 1.3]. Thus, the projections $Q_n = I - S^n S^{*n}$ are finite-dimensional projections because S has finite multiplicity and hence, so does any of the shifts S^n. Denote $P_n = S^n S^{*n}$. Assume T is S–UAT with asymptotic image A. That image must be S–Toeplitz, hence $S^{*n}(T-A)S^n = S^{*n}TS^n - A$ and so, $\|S^{*n}(T-A)S^n\| \to 0$. This implies that $\|P_n(T-A)P_n\| \to 0$ due to the estimate

$$\|P_n(T-A)P_n\| = \|S^n S^{*n}(T-A)S^n S^{*n}\| \leq \|S^{*n}(T-A)S^n\| \qquad n = 1, 2, 3, \ldots$$

On the other hand

$$P_n(T-A)P_n = (I-Q_n)(T-A)(I-Q_n) = T - A + F_n \qquad n = 1, 2, 3, \ldots$$

where $F_n = -Q_n(T-A)(I-Q_n) - (T-A)Q_n$ is obviously a sequence of finite rank operators, tending uniformly to $-(T-A)$. This shows $T-A$ is compact if T is S–UAT with asymptotic image A.

The converse is easier. If S is any forward shift, (not necessarily one of finite multiplicity), a compact perturbation $T+K$ of a S–Toeplitz operator T is necessarily S–UAT with asymptotic image T because $\|S^{*n}KS^n\| \to 0$. When K is a finite rank operator, this is a consequence of the fact that $S^{*n} \to 0$ strongly. Since arbitrary compact operators are uniform limits of finite rank operators it follows that $\|S^{*n}KS^n\| \to 0$ for any compact K. □

By [12, Theorem 1.1], the consequence of the two theorems above is the following:

Corollary 2. *Let $p \in \mathbb{U}$ be arbitray and fixed. An operator T on H^2 is M_z–UAT if and only if it is M_{α_p}–UAT, in particular, a composition operator is M_{α_p}–UAT if and only if it is compact or the identity.*

Proof. The fact that T is M_z–UAT if and only if it is M_{α_p}–UAT is a direct consequence of Proposition 3 and Theorem 3. The fact that a composition operator is M_{α_p}–UAT if and only if it is compact or the identity is a consequence of the fact that, by [12, Theorem 1.1], those are the only M_z–UAT composition operators. □

4 M_ϕ–ASYMPTOTICALLY TOEPLITZ COMPOSITION OPERATORS

This section contains extensions of results in [12] originally proved for M_z–Toeplitzness concepts. Those results can be extended with minimal technical efforts to M_ϕ–Toeplitzness concepts. Some of the results included in this section require that the following condition be satisfied.

Basic Assumption: *Let ϕ be an inner function and φ an analytic selfmap of $\mathbb{U}$. We say that φ and ϕ satisfy the basic assumption if there is a non–constant, analytic selfmap ψ of $\mathbb{U}$ such that $C_\varphi\phi = \psi\phi$. In other words, the maps φ and ϕ should satisfy the condition*

$$\left|\frac{\phi\circ\varphi(z)}{\phi(z)}\right| \le 1 \qquad z\in\mathbb{U},\ \phi(z)\neq 0.$$

The basic assumption extends the situation when one studies some concept of M_z–Toeplitzness and requires that the operator C_φ be such that $\varphi(0)=0$. Indeed, in such a case, the basic assumption is satisfied by $\phi(z)=z$ and $\psi(z)=\varphi(z)/z$.

Remark 2. *If some inner ϕ and some analytic selfmap φ of $\mathbb{U}$ satisfy the basic assumption, one has that*

$$M_\phi^{*n}C_\varphi M_\phi^n = T_{\psi^n,\varphi} \qquad n=1,2,3,\dots$$

where $T_{\psi^n,\varphi}$ is the weighted composition operator of symbols ψ^n and φ, that is $T_{\psi^n,\varphi} = M_{\psi^n}C_\varphi$.

Proof. Since ϕ is inner, one has $\overline{\phi^n}\phi^n = 1$ a.e. Hence $M_\phi^{*n}C_\varphi M_\phi^n f = P(\overline{\phi^n}\phi^n\psi^n f\circ\varphi) = T_{\psi^n,\varphi}f$, for any $f\in H^2$. □

With this remark we are ready to begin our string of extensions of results in [12]. The following is an extension of [12, Proposition 3.1].

Proposition 4. *If ϕ is inner and non–constant and φ is an analytic selfmap of $\mathbb{U}$ with the property $|\varphi|<1$ a.e., then C_φ is M_ϕ–SAT with null asymptotic image.*

Proof. Note that

$$\|M_\phi^{*n}C_\varphi M_\phi^n f\|^2 \le \|C_\varphi M_\phi^n f\|^2 = \int_{\partial\mathbb{U}} |(\phi\circ\varphi)^{2n}||f\circ\varphi|^2\,dm \to 0,$$

by Lebesgue's dominated convergence theorem, since $(\phi\circ\varphi)^{2n}\to 0$ a.e. □

Our next result extends [12, Theorem 4.2].

Proposition 5. *If φ is an analytic selfmap of $\mathbb{U}$ and ϕ a non–constant inner function with the property that there is a non–constant, analytic selfmap ψ of $\mathbb{U}$ such that $C_\varphi\phi = \psi\phi$, then C_φ^* is M_ϕ–SAT with null asymptotic image.*

Proof. Since the set of all kernel–functions is a spanning subset of H^2 and $\{M_\phi^{*n}C_\varphi^*M_\phi^n\}$ is norm–bounded, it is enough to prove that $\|M_\phi^{*n}C_\varphi^*M_\phi^n K_p\|\to 0$ for any $p\in\mathbb{U}$. It is well known that, the action of the adjoint of a weighted composition operator on the kernel–functions is described by

$$T_{\psi,\varphi}^*K_p = \overline{\psi(p)}K_{\varphi(p)} \qquad p\in\mathbb{U}.$$

Therefore, by Remark 2, one has

$$\|M_\phi^{*n} C_\varphi^* M_\phi^n K_p\| = |\psi^n(p)|\, \|K_{\varphi(p)}\| \to 0$$

because $\|\psi\|_\infty \leq 1$ and ψ is not constant. □

As an immediate consequence we obtain an extension of [12, Proposition 2.1].

Corollary 3. *If φ and ϕ satisfy the basic assumption, then C_φ is M_ϕ–WAT with null asymptotic image.*

The next result is an extension of [12, Proposition 3.2].

Proposition 6. *If φ is an analytic selfmap of $\mathbb{U}$ and ϕ a non–constant inner function with the property that there is a non–constant, analytic selfmap ψ of $\mathbb{U}$ such that $C_\varphi \phi = \psi\phi$ and C_φ is M_ϕ–SAT, then necessarily, $|\varphi| < 1$ a.e.*

Proof. By Corollary 3, $\|M_\phi^{*n} C_\varphi M_\phi^n 1\| \to 0$. But, by Remark 2,

$$\|M_\phi^{*n} C_\varphi M_\phi^n 1\|^2 = \int_{\partial\mathbb{U}} |\psi|^{2n}\, dm \geq m(E_\psi),$$

where we denote

$$E_\psi = \{\zeta \in \partial\mathbb{U} : |\psi(\zeta)| = 1\} \qquad E_\varphi = \{\zeta \in \partial\mathbb{U} : |\varphi(\zeta)| = 1\}.$$

Letting $n \to \infty$ one gets $m(E_\psi) = 0$. On the other hand, by the basic assumption, $E_\varphi \subseteq E_\psi$ a.e. □

The following is an extension of [12, Proposition 4.1].

Proposition 7. *If $|\varphi| < 1$ a.e. and ϕ is inner and non-constant, then C_φ^* is M_ϕ–SAT.*

Proof. For fixed $f \in H^2$ and n, choose $g \in H^2$, a unit vector with property $\|M_\phi^{*n} C_\varphi^* M_\phi^n f\| =< M_\phi^{*n} C_\varphi^* M_\phi^n f, g >$. One has

$$\|M_\phi^{*n} C_\varphi^* M_\phi^n f\| =< M_\phi^n f, C_\varphi M_\phi^n g >= \int_{\partial\mathbb{U}} \phi^n f \overline{(\phi\circ\varphi)^n g\circ\varphi}\, dm \leq$$

$$\sqrt{\int_{\partial\mathbb{U}} |\phi\circ\varphi|^{2n} |f|^2\, dm}\; \|C_\varphi\| \to 0,$$

by the Cauchy–Schwarz inequality and Lebesgue's dominated convergence theorem. □

5 THE OPERATOR $C_\varphi^* C_\varphi$

Operators of form $C_\varphi^* C_\varphi$ are frequently M_ϕ–asymptotically Toeplitz. We study them in this section. Let

$$P(z,u) = \Re\frac{u+z}{u-z} = \frac{1-|z|^2}{|u-z|^2} \qquad u \in \partial\mathbb{U},\ z \in \mathbb{U}$$

be the usual Poisson kernel. In [2] the operator $C_\varphi^* C_\varphi$ is shown to be the M_z–Toeplitz operator $T_{P(\varphi(0),u)}$, provided that φ be inner, [2, Proposition 3]. The proof is based on a representation, (due to Cowen [5]), of adjoints of composition operators with linear fractional symbols as a product of two Toeplitz operators and a composition operator. In the next theorem, we give a very short proof of that result, based on a pull–back measure formula of Nordgren [13], and prove an extra fact: namely that $C_\varphi^* C_\varphi$ is a M_ϕ–Toeplitz operator only if φ is inner.

Theorem 4. *Let ϕ be a nonconstant inner function. The operator $C_\varphi^* C_\varphi$ is a M_ϕ–Toeplitz operator if and only if φ is inner, in which case $C_\varphi^* C_\varphi = T_{P(\varphi(0),u)}$.*

Proof. If φ is inner then by [13], $dm\varphi^{-1}(u) = P(\varphi(0), u)dm(u)$ and, using this pull–back measure formula, one has

$$< C_\varphi^* C_\varphi f, g >= \int_{\partial \mathbb{U}} f \circ \varphi(u) \bar{g} \circ \varphi(u)\, dm(u) = \int_{\partial \mathbb{U}} f(u) P(\varphi(0), u) \bar{g}(u)\, dm(u),$$

for any $f, g \in H^2$. Hence

$$C_\varphi^* C_\varphi = T_{P(\varphi(0),u)}. \tag{10}$$

If ϕ is a fixed, nonconstant, inner function it is straightforward to establish that M_z–Toeplitz operators are also M_ϕ–Toeplitz. To show only inner functions φ have the property that $C_\varphi^* C_\varphi$ is M_ϕ– Toeplitz, assume that $C_\varphi^* C_\varphi$ is M_ϕ–Toeplitz. Then

$$< M_\phi^* C_\varphi^* C_\varphi M_\phi f, f >=< C_\varphi^* C_\varphi f, f > \qquad f \in H^2$$

that is

$$\|\phi \circ \varphi f \circ \varphi\|_2 = \|f \circ \varphi\|_2 \qquad f \in H^2. \tag{11}$$

Taking $f \equiv 1$ in (11), one obtains $\|\phi \circ \varphi\|_2 = 1$, hence $\phi \circ \varphi$ is inner. Indeed, $1 - |\phi \circ \varphi(u)|^2 \geq 0$ a.e. and $\int_{\partial \mathbb{U}} (1 - |\phi \circ \varphi(u)|^2)\, dm(u) = 0$. This implies that φ is inner, because ϕ is a nonconstant inner function and hence, if one assumes $m(E_\varphi) < 1$, then, by the maximum modulus principle, it follows that $m(E_\phi) < 1$, a contradiction. □

As an application, we obtain the following extension in the H^2–context of part of [2, Proposition 4].

Corollary 4. *If φ is inner then the spectrum of $C_\varphi^* C_\varphi$ is the line-interval*

$$\left[\frac{1 - |\varphi(0)|}{1 + |\varphi(0)|}, \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\right].$$

Proof. This is a direct consequence of formula (10), the Hartman–Wiener theorem, ([6, Theorem 7.20]), and the well known formulas

$$\sup \{P(\varphi(0), u) : u \in \partial \mathbb{U}\} = \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}$$

$$\inf \{P(\varphi(0), u) : u \in \partial \mathbb{U}\} = \frac{1 - |\varphi(0)|}{1 + |\varphi(0)|}.$$

□

The formula $C_\varphi C_\varphi^* = \left(T_{P(\varphi^{-1}(0),u)}\right)^{-1}$ (valid if φ is a disk–automorphism), can be obtained as a particular case of [2, Proposition 2]. The formula can also be obtained as an immediate consequence of (10):

Corollary 5. *If φ is a disk-automorphism, then $C_\varphi C_\varphi^* = \left(T_{P(\varphi^{-1}(0),u)}\right)^{-1}$.*

Proof. $(C_\varphi C_\varphi^*)^{-1} = C_{\varphi^{-1}}^* C_{\varphi^{-1}} = T_{P(\varphi^{-1}(0),u)}$, by formula (10). □

Symbols φ with the property $|\varphi| < 1$ a.e. are somehow at the opposite side of the spectrum compared to inner symbols. Clearly $C_\varphi^* C_\varphi$ cannot be M_ϕ–Toeplitz if φ is such a symbol but it can be M_ϕ–WAT, M_ϕ–SAT, respectively M_ϕ–UAT, as we show in the following.

Theorem 5. *Let ϕ be a nonconstant inner function and φ an analytic selfmap of $\mathbb{U}$. Then:*
(i) $C_\varphi^* C_\varphi$ is always M_ϕ–WAT. Its asymptotic image is the M_z–Toepliz operator T_ψ where ψ has Fourier coefficients $\{c_n\}$ given by

$$c_n = \int_{E_\varphi} \overline{\varphi^n}\, dm \qquad n = 0, \pm 1, \pm 2, \dots \tag{12}$$

(ii) $C_\varphi^* C_\varphi$ is M_ϕ–WAT with asymptotic image $0 \iff C_\varphi^* C_\varphi$ is M_ϕ–SAT with asymptotic image $0 \iff |\varphi| < 1$ a.e.
(iii) If φ is not an inner function and $(\phi \circ \varphi)^n \to 0$ uniformly a.e. on $E_\varphi^c = \partial\mathbb{U} \setminus E_\varphi$ (that is $\operatorname{ess\,sup}\{|\phi \circ \varphi(\zeta)|^n : \zeta \in E_\varphi^c\} \to 0$), then $C_\varphi^* C_\varphi$ is M_ϕ–UAT. In particular, $C_\varphi^* C_\varphi$ is M_ϕ–UAT with asymptotic image 0 if $\|\phi \circ \varphi\|_\infty < 1$.

Proof. (i) Note that $|\phi \circ \varphi| < 1$ a.e. on E_φ^c, since ϕ is inner and nonconstant. Hence, for any $f, g \in H^2$ one has

$$< M_\phi^{*n} C_\varphi^* C_\varphi M_\phi^n f, g > = < C_\varphi M_\phi^n f, C_\varphi M_\phi^n g > =$$

$$\int_{\partial\mathbb{U}} |\phi \circ \varphi|^{2n} f \circ \varphi \overline{g \circ \varphi}\, dm \to \int_{E_\varphi} f \circ \varphi \overline{g \circ \varphi}\, dm,$$

by Lebesgue's dominated convergence theorem. Since the weak limit above is the same no matter ϕ, it must be an M_z–Toeplitz operator T_ψ whose symbol is a.e. real since T_ψ is a non-negative operator. Keeping that in mind, one can calculate the Fourier coefficients of ψ by using the identity

$$< T_\psi f, g > = \int_{E_\varphi} f \circ \varphi \overline{g \circ \varphi}\, dm \qquad f, g \in H^2. \tag{13}$$

Indeed, for an arbitrary integer $n \geq 0$, taking $g = z^n$ and $f = 1$ in (13), one obtains (12). Since, ψ is real a.e. $c_{-n} = \overline{c_n}$.

(ii) If $|\varphi| < 1$ a.e. then note that, for each $f \in H^2$

$$\|M_\phi^{*n} C_\varphi^* C_\varphi M_\phi^n f\|_2 \leq \|C_\varphi\| \|C_\varphi M_\phi^n f\|_2 = \|C_\varphi\| \sqrt{\int_{\partial\mathbb{U}} |\phi \circ \varphi|^{2n} |f \circ \varphi|^2\, dm} \to 0,$$

by Lebesgue's dominated convergence theorem. Thus $C_\varphi^* C_\varphi$ is M_ϕ–SAT with asymptotic image 0 if $|\varphi| < 1$ a.e. Clearly, $C_\varphi^* C_\varphi$ is M_ϕ–WAT with asymptotic image 0 if $C_\varphi^* C_\varphi$ is M_ϕ–SAT with asymptotic image 0, so the only thing left to prove is that, if $C_\varphi^* C_\varphi$ is M_ϕ–WAT with asymptotic image 0, then $|\varphi| < 1$ a.e. This is a consequence of the fact that $T_\psi = 0$. Indeed, the Fourier coefficient $c_0 = m(E_\varphi)$ of ψ must be null, that is $|\varphi| < 1$ a.e.

For the proof of (iii), recall that the norm $\|T\|$ and the numerical radius $w(T) = \sup\{| < Tf, f > | : \|f\| = 1\}$ of a Hilbert space operator T satisfy the inequality $\|T\| \leq 2w(T)$, [10, Ch. 22]. Keeping this in mind, consider any fixed $f \in H^2$, $\|f\|_2 = 1$. One has the following estimates

$$| < (M_\phi^{*n+k} C_\varphi^* C_\varphi M_\phi^{n+k} - M_\phi^{*n} C_\varphi^* C_\varphi M_\phi^n) f, f > | =$$

$$\left| \|C_\varphi M_\phi^{n+k} f\|^2 - \|C_\varphi M_\phi^n f\|^2 \right| = \left| \int_{\partial \mathbb{U}} |\phi \circ \varphi|^{2n} (|\phi \circ \varphi|^{2k} - 1) |f \circ \varphi|^2 \, dm \right|$$

$$= \int_{E_\varphi^c} |\phi \circ \varphi|^{2n} (1 - |\phi \circ \varphi|^{2k}) |f \circ \varphi|^2 \, dm$$

$$\leq \text{ess sup}\{|\phi \circ \varphi(\zeta)|^{2n} : \zeta \in E_\varphi^c\} \|C_\varphi\|^2 \qquad f \in H^2, \ \|f\|_2 = 1.$$

Given our assumption that $\text{ess sup}\{|\phi \circ \varphi(\zeta)|^{2n} : \zeta \in E_\varphi^c\} \to 0$ and the relation between norms of operators and their numerical radii, it follows that the sequence $\{M_\phi^{*n} C_\varphi^* C_\varphi M_\phi^n\}$ is Cauchy and hence, norm convergent.

Clearly, $\text{ess sup}\{|\phi \circ \varphi(\zeta)|^{2n} : \zeta \in E_\varphi^c\} \to 0$ if $\|\phi \circ \varphi\|_\infty < 1$. Actually, in that case $E_\varphi^c = \partial \mathbb{U}$ a.e. that is $|\varphi| < 1$ a.e. so, by (ii), the asymptotic image is 0. □

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