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Unitary Weighted Composition Operators

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UNITARY WEIGHTED COMPOSITION OPERATORS

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ABSTRACT. A necessary and sufficient condition for a weighted composition operator to be unitary is given in terms of the weight-function and of the composition function.

1. INTRODUCTION

The composition operators appeared in some of Koopman's works dealing with classical mecanics, [1]. Starting with 1966 they made the object of an interesting study carried out by several mathematicians among whom we mention Nordgren whose survey-work [2] , contains the basic results obtained as far as 1978. The study of this class of operators is still actual.

Let (X,S,m) be a σ-finite measure space and $T : X \to X$ a nonsingular measure transformation. By the composition operator induced by T we understand the operator

$$C_T : L^2(m) \longrightarrow L^2(m) \qquad (1)$$
$$C_T f = f \circ T \qquad (f \in L^2(m))$$

The problem which measure transformation T induces a bounded composition operator C_T is solved long ago (see [2]). If mT^{-1} is the measure naturally induced by T, we know that $mT^{-1} \ll m$ and the Nicodym derivative

$$f_o = dmT^{-1}/ dm \qquad (2)$$

must be in $L^{\infty}(m)$.

If θ is another mapping from $L^{\infty}(m)$ we denote by M_{θ} the multiplication operator induced by θ on $L^2(m)$, that is the opera-

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tor

(3) $$M_\theta f = f\theta \qquad (f \in L^2(m)).$$

We call weighted composition operator induced by T with weight θ the following operator

(4) $$C_T^\theta = C_T M_\theta .$$

We aim to characterize unitary weighted composition operators.

2. UNITARY C_T^θ

We shall begin by claiming that C_T^θ be isometric. It is an easy calculation to prove

PROPOSITION 1. *The necessary and sufficient condition that C_T^θ be an isometry is*

(5) $$f_o|\theta|^2 = 1, \qquad \text{a.e.}$$

Proof. We claim that $(C_T^\theta)^* C_T^\theta = I$ where I denotes the identity on $L^2(m)$. That is

$$M_{\bar\theta} C_T^* C_T M_\theta = I .$$

Singh shows in [3] that $C_T^* C_T = M_{f_o}$ and the condition above becomes

$$M_{|\theta|^2 f_o} = I.$$

As we require that m be σ-finite, there are functions f in $L^2(m)$ which are a.e. different of zero, hence

$$|\theta|^2 f_o = 1 \qquad \text{a.e.},$$

which concludes the proof.

We shall denote by $T^{-1}(S)$ the σ-subalgebra of S induced by T that is the collection of all sets $T^{-1}(E)$ for any E from S.

PROPOSITION 2. *The necessary and sufficient condition that C_T^θ be unitary is*

(6) $$T^{-1}(S) = S \qquad \textit{and}$$
$$|\theta|^2 f_o = 1 \qquad \textit{a.e.}$$

Proof. We begin by observing that condition $T^{-1}(S) = S$ is necessary and sufficient for C_T to have dense range, as it follows from [4] that is $\overline{R(C_T)} = L^2(m)$. If (6) holds then $(C_T^\theta)^* C_T^\theta = I$ according to proposition 1. Furthermore the fact that $\overline{R(C_T)} =$ $= L^2(m)$ implies the equality

$$C_T C_T^* = M_{f_o} \circ T \qquad \text{(see [5]).}$$

Then we can show the equality $C_T^\theta (C_T^\theta)^* = I$. Indeed

$$C_T^\theta (C_T^\theta)^* = C_T M_\theta M_{\bar\theta} C_T^* = C_T M_{|\theta|^2} C_T^* = M_{(|\theta|^2 \circ T)} C_T C_T^* = M_{(|\theta|^2 \circ T)(f_o \circ T)} =$$

$$= M_{(|\theta|^2 f_o) \circ T} = I.$$

For the converse suppose now that C_T is unitary hence isometric. We obtain by proposition 1, that

$$|\theta|^2 f = 1 \qquad \text{a.e.}$$

Suppose that $\overline{R(C_T)} \neq L^2(m)$, then there is $f \in L^2(m)$ such that $C_T^* f = 0$ a.e. but f is not a.e.-equal to 0. So

$$C_T^\theta (C_T^\theta)^* f = C_T M_\theta M_{\bar\theta} C_T^* f = 0 ,$$

which is absurd aslong as C_T^θ is supposed to be unitary. We conclude that C_T has dense range, hence (6) is fulfilled.

3. WEIGHTED SEQUENCE SPACES AND C_T^θ

The case of weighted sequence spaces is perhaps the easyest to handle. We take $X = N$ and $S = P(N)$. Let $w = (w_n)_{n \in N}$ be a fixed sequence of real positive numbers. We set

$$m(E) = \sum_{n \in E} w_n \qquad (E \subseteq N)$$

The space $L^2(m)$ obtained is also denoted by l_w^2. The composition operators on such spaces made the object of some of the author's latest works [6] and [7].

Let us adapt the results of paragraph 2 to this particular case, which could provide us a lot of easy concrete examples. Thus

C_T^θ will be isometric iff T is surjective and

(7) $$m(T^{-1}(\{n\}))/m(\{n\}) = |\theta(n)|^{-2} .$$

Indeed, in [6] we proved that $f_o(n) = m(T^{-1}(\{n\}))/m(\{n\})$ for $n \in T(N)$ and $f_o(n)$ is zero for $n \in N-T(N)$. Taking into account proposition 1. (7) easyly follows.

Next, C_T will be unitary if T is bijective and (7) holds which is obvious from the considerations above and proposition 2, noticing that $T^{-1}(S) = S$ is in this particular case fulfilled iff T is bijective. See also [6] .

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