

1987

Unitary Weighted Composition Operators

Valentin Matache

University of Nebraska at Omaha, vmatache@unomaha.edu

Follow this and additional works at: <https://digitalcommons.unomaha.edu/mathfacpub>



Part of the [Mathematics Commons](#)

Recommended Citation

Matache, Valentin, "Unitary Weighted Composition Operators" (1987). *Mathematics Faculty Publications*. 36.
<https://digitalcommons.unomaha.edu/mathfacpub/36>

This Article is brought to you for free and open access by the Department of Mathematics at DigitalCommons@UNO. It has been accepted for inclusion in Mathematics Faculty Publications by an authorized administrator of DigitalCommons@UNO. For more information, please contact unodigitalcommons@unomaha.edu.



UNITARY WEIGHTED COMPOSITION OPERATORS

by

VALENTIN MATACHE

ABSTRACT. A necessary and sufficient condition for a weighted composition operator to be unitary is given in terms of the weight-function and of the composition function.

1. INTRODUCTION

The composition operators appeared in some of Koopman's works dealing with classical mechanics, [1]. Starting with 1966 they made the object of an interesting study carried out by several mathematicians among whom we mention Nordgren whose survey-work [2], contains the basic results obtained as far as 1978. The study of this class of operators is still actual.

Let (X, S, m) be a σ -finite measure space and $T : X \rightarrow X$ a nonsingular measure transformation. By the composition operator induced by T we understand the operator

$$(1) \quad \begin{aligned} C_T : L^2(m) &\longrightarrow L^2(m) \\ C_T f &= f \circ T \end{aligned} \quad (f \in L^2(m))$$

The problem which measure transformation T induces a bounded composition operator C_T is solved long ago (see [2]). If mT^{-1} is the measure naturally induced by T , we know that $mT^{-1} \ll m$ and the Nicodym derivative

$$(2) \quad f_0 = dmT^{-1} / dm$$

must be in $L^\infty(m)$.

If θ is another mapping from $L^\infty(m)$ we denote by M_θ the multiplication operator induced by θ on $L^2(m)$, that is the opera-

AMS (MOS) subject classification: Primary: 47B37, Secondary: 47B38.

tor

$$(3) \quad M_{\theta} f = f \theta \quad (f \in L^2(m)).$$

We call weighted composition operator induced by T with weight θ the following operator

$$(4) \quad C_T^{\theta} = C_T M_{\theta}.$$

We aim to characterize unitary weighted composition operators.

2. UNITARY C_T^{θ}

We shall begin by claiming that C_T^{θ} be isometric. It is an easy calculation to prove

PROPOSITION 1. *The necessary and sufficient condition that C_T^{θ} be an isometry is*

$$(5) \quad f_{\theta} |\theta|^2 = 1, \quad \text{a.e.}$$

Proof. We claim that $(C_T^{\theta})^* C_T^{\theta} = I$ where I denotes the identity on $L^2(m)$. That is

$$M_{\theta} C_T^* C_T M_{\theta} = I.$$

Singh shows in [3] that $C_T^* C_T = M_{f_{\theta}}$ and the condition above becomes

$$M_{|\theta|^2} f_{\theta} = I.$$

As we require that m be σ -finite, there are functions f in $L^2(m)$ which are a.e. different of zero, hence

$$|\theta|^2 f_{\theta} = 1 \quad \text{a.e.}$$

which concludes the proof.

We shall denote by $T^{-1}(S)$ the σ -subalgebra of S induced by T that is the collection of all sets $T^{-1}(E)$ for any E from S .

PROPOSITION 2. *The necessary and sufficient condition that C_T^{θ} be unitary is*

$$(6) \quad \begin{aligned} T^{-1}(S) &= S && \text{and} \\ |\theta|^2 f_{\theta} &= 1 && \text{a.e.} \end{aligned}$$

Proof. We begin by observing that condition $T^{-1}(S) = S$ is necessary and sufficient for C_T to have dense range, as it follows from [4] that $\overline{R(C_T)} = L^2(m)$. If (6) holds then $(C_T^\theta)^\times C_T^\theta = I$ according to proposition 1. Furthermore the fact that $\overline{R(C_T)} = L^2(m)$ implies the equality

$$C_T C_T^\times = M_{f_0} \circ T \quad (\text{see [5] }).$$

Then we can show the equality $C_T^\theta (C_T^\theta)^\times = I$. Indeed

$$\begin{aligned} C_T^\theta (C_T^\theta)^\times &= C_T M_\theta M_\theta^\times C_T^\times = C_T M_{|\theta|^2} C_T^\times = M_{(|\theta|^2 \circ T)} C_T C_T^\times = M_{(|\theta|^2 \circ T)} (f_0 \circ T) = \\ &= M_{(|\theta|^2 f_0)} \circ T = I. \end{aligned}$$

For the converse suppose now that C_T is unitary hence isometric. We obtain by proposition 1, that

$$|\theta|^2 f = 1 \quad \text{a.e.}$$

Suppose that $\overline{R(C_T)} \neq L^2(m)$, then there is $f \in L^2(m)$ such that $C_T^\times f = 0$ a.e. but f is not a.e.-equal to 0. So

$$C_T^\theta (C_T^\theta)^\times f = C_T M_\theta M_\theta^\times C_T^\times f = 0,$$

which is absurd as long as C_T^θ is supposed to be unitary. We conclude that C_T has dense range, hence (6) is fulfilled.

3. WEIGHTED SEQUENCE SPACES AND C_T^θ

The case of weighted sequence spaces is perhaps the easiest to handle. We take $X = N$ and $S = P(N)$. Let $w = (w_n)_{n \in N}$ be a fixed sequence of real positive numbers. We set

$$m(E) = \sum_{n \in E} w_n \quad (E \subseteq N)$$

The space $L^2(m)$ obtained is also denoted by l_w^2 . The composition operators on such spaces made the object of some of the author's latest works [6] and [7].

Let us adapt the results of paragraph 2 to this particular case, which could provide us a lot of easy concrete examples. Thus

C_T^θ will be isometric iff T is surjective and

$$(7) \quad m(T^{-1}(\{n\}))/m(\{n\}) = |\theta(n)|^{-2}.$$

Indeed, in [6] we proved that $f_\theta(n) = m(T^{-1}(\{n\}))/m(\{n\})$ for $n \in T(N)$ and $f_\theta(n)$ is zero for $n \in N - T(N)$. Taking into account proposition 1. (7) easily follows.

Next, C_T will be unitary if T is bijective and (7) holds which is obvious from the considerations above and proposition 2, noticing that $T^{-1}(S) = S$ is in this particular case fulfilled iff T is bijective. See also [6].

REFERENCES

- [1] B.O. KOOPMAN, Hamiltonian systems and transformations in Hilbert spaces, Proc. Nat. Acad. Sci. U.S.A. 17 (1931), 315-318.
- [2] E. A. NORDGREN, Composition operators in Hilbert spaces, in Hilbert space operators, Lecture notes in mathematics, 693, Springer Verlag, Berlin, 1978, 37-63.
- [3] R.K. SINGH, Compact and quasinormal composition operators, Proc. Amer. Math. Soc., 49 (1974), 82-85.
- [4] R.K. SINGH and A. KUMAR, Characterization of invertible, unitary and normal composition operators, Bull. Austral. Math. Soc., 19 (1978), 81-95.
- [5] R.K. SINGH, and A. KUMAR, Compact weighted composition operators on $L^2(\lambda)$, Acta, Sci. Math. Szeged, 49 (1985), 339-344.
- [6] V. MATACHE, The spectrum of a composition operator on weighted sequence spaces, S.L.O.H.A. 3 (1985).
- [7] V. MATACHE, Composition operators on weighted sequence spaces, S.L.O.H.A. 2 (1985).

10.05.1987

Dept. of Mathematics
University of Timisoara
Bul. V. Pärvan
1900 TIMISOARA
ROMANIA