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Composition Operators Whose Symbols Have Orthogonal Powers

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ABSTRACT. Composition operators on the Hilbert Hardy space H^2 whose symbols are analytic selfmaps of the open unit disk having orthogonal powers are considered. The spectra and essential spectra of such operators are described. In the general case of an arbitrary analytic selfmap of the open unit disk, it is proved that the composition operator induced by that map has essential spectral radius less than 1 if and only if the map under consideration is a non–inner map with a fixed point in the unit disk. The canonical decomposition of a non–unitary composition contraction is determined.

1. INTRODUCTION

Let H^2 denote the Hilbert Hardy space on the open unit disk $\mathbb{U}$, that is the space of all functions f analytic in $\mathbb{U}$ satisfying the condition

$$(1) \qquad \|f\| := \sup_{0<r<1} \left(\int_{\partial \mathbb{U}} |f(r\zeta)|^2 \, dm(\zeta) \right)^{1/2} < +\infty,$$

where m is the normalized Lebesgue measure.

The norm above is computable in terms of the Maclaurin coefficients $\{c_n\}$ of f by the formula

$$\|f\| = \sqrt{\sum_{n=0}^{+\infty} |c_n|^2}.$$

Visibly $H^\infty \subseteq H^2$, where H^∞ is the space of bounded analytic functions. P. Fatou proved that bounded analytic functions have radial limits a.e. That result was extended to H^2–functions by F. and M. Riesz. Radial limit–functions

will be denoted by the same symbols as the functions themselves. A bounded analytic function whose radial limit–function is unimodular a.e. is called an inner function. We refer to [9] for the essential facts on Hardy–space theory. By the term *symbol* we designate any analytic selfmap φ of $\mathbb{U}$ and call the operator

$$C_\varphi f = f \circ \varphi \qquad f \in H^2$$

the *composition operator* with symbol φ. The fact that any symbol φ induces a bounded composition operator on H^2 is well–known. We say that a symbol φ has *orthogonal powers* if the set $\{1, \varphi, \varphi^2, \ldots, \varphi^n, \ldots\}$ is an orthogonal subset of H^2. In the literature, such symbols are sometimes called simply *orthogonal* or even symbols satisfying *Rudin's orthogonality condition*. A noted problem related to them was raised in 1988 by W. Rudin. He asked if the only analytic selfmaps of $\mathbb{U}$ having orthogonal powers are the constant multiples of inner functions fixing the origin. The negative answer was obtained by Bishop [1], respectively Sundberg [19], who worked independently of each other. Those papers raised the interest in the class of composition operators whose symbols have orthogonal powers.

In the second section of this paper we determine when such composition operators are compact and find multiple formulas for their essential norm. We also find the spectra and essential spectra of composition operators whose symbols have orthogonal powers. We show that the essential norm and essential spectral radius of such operators coincide. The third section contains some spectral properties characteristic to composition operators induced by non–inner symbols fixing a point (not necessarily symbols with orthogonal powers). Among those properties we note the following new result (Theorem 3.3), saying that the inequality $r_e(C_\varphi) < 1$ holds if and only if φ is a non–inner map fixing a point. Of course, r_e denotes the essential spectral radius of an operator. The current section is dedicated to briefly outlining the content of this paper and introducing the main concepts. We conclude it by introducing more notation.

Let $m\varphi^{-1}$ be the pull–back measure of m under φ, that is the Borel measure on $\overline{\mathbb{U}}$ given by $m\varphi^{-1}(E) = m(\varphi^{-1}(E))$. Bishop [1], proved that φ, a symbol fixing the origin, has orthogonal powers if and only if $\int_{\overline{\mathbb{U}}} \log(1/|z|)\, dm\varphi^{-1}(z) < +\infty$ and $m\varphi^{-1}$ is rotation–invariant (he used the term *radial* for such a measure), that is $m\varphi^{-1}(\lambda E) = m\varphi^{-1}(E)$ for each unimodular number λ and each measurable set E. Let us denote by E_φ the subset of the unit circle consisting of all points where the radial limit–function of φ is unimodular. As a final remark in this introductory section, we wish to mention that, in [1], the author constructed (among other things), symbols φ with orthogonal powers and the property that both E_φ and its complement $\partial\mathbb{U} \setminus E_\varphi$ have positive arc–length measure. The

relevance of this remark will become evident in the next section which starts with the characterization of compact composition operators having orthogonal symbols.

2. The main results

The compactness of composition operators can be understood by using the formulas for their essential norm. Two main alternatives exist: an asymptotic formula in terms of the Nevanlinna counting function [17], respectively a formula in terms of the Aleksandrov measures of the symbol [6]. In practical cases, each of those formulas can be hard to use. For that reason, we prove easier ones for the particular case of composition operators whose symbols have orthogonal powers.

Theorem 2.1. *If the symbol φ of the composition operator C_φ has orthogonal powers then C_φ is compact if and only if $|\varphi| < 1$ a.e. The essential norm $\|C_\varphi\|_e$ of C_φ is computable with the formulas*

$$(2) \qquad \|C_\varphi\|_e = \lim_{n \to +\infty} \|\varphi^n\| = \sqrt{m\varphi^{-1}(\partial\mathbb{U})} = \inf\left\{\|C_\varphi f\| : \|f\| = 1\right\}.$$

The operator C_φ is in the Schatten class S_p, $0 < p < +\infty$ if and only if

$$(3) \qquad \sum_{n=1}^{+\infty} \|\varphi^n\|^p < +\infty.$$

Proof. Indeed, by the polar decomposition theorem [10, Ch. 16], C_φ is compact if and only if $\sqrt{C_\varphi^* C_\varphi}$ is compact. On the other hand, obviously, $C_\varphi^* C_\varphi$ is a diagonal operator with respect to the monomial basis of H^2 if and only if φ has orthogonal powers. In that case, $\sqrt{C_\varphi^* C_\varphi}$ is the diagonal operator having diagonal entries $\{\|\varphi^n\|\}$. By Lebesgue's bounded convergence theorem, if $|\varphi| < 1$ a.e. then the diagonal entries above tend to 0 and hence $\sqrt{C_\varphi^* C_\varphi}$ is compact, [16, Corollary 1.5]. It is well–known that, if C_φ is compact, then $|\varphi| < 1$ a.e. In our case, this is particularly evident because, if the condition $|\varphi| < 1$ a.e. is not satisfied, then the set $E_\varphi = \{\zeta \in \partial\mathbb{U} : |\varphi(\zeta)| = 1\}$ has positive Lebesgue measure and its measure is visibly a lower bound of the diagonal entries of $\sqrt{C_\varphi^* C_\varphi}$. Therefore, the operator $\sqrt{C_\varphi^* C_\varphi}$ (and hence C_φ too), fails to be compact, if $m(E_\varphi) > 0$.

We obtain the first equality in the formula (2) by applying to the diagonal operator $C_\varphi^* C_\varphi$ the following property:

If the diagonal operator T acting on an infinite–dimensional, separable Hilbert space has diagonal entries $\{\lambda_n\}$ with property $\lambda_n \to \lambda$, then $\|T\|_e = |\lambda|$.

The property above is an immediate consequence of [16, Corollary 1.5]. Indeed, by that corollary, the operator $K = T - \lambda I$ is compact, hence $\|T\|_e = \|\lambda I\|_e = |\lambda|$.

The second equality in (2) is the consequence of the following formula [1, Lemma 6.1]:

$$(4) \qquad \|C_\varphi f\|^2 = \int_{\mathbb{U}} |f|^2 \, dm\varphi^{-1} + m\varphi^{-1}(\partial\mathbb{U})\|f\|^2 \qquad f \in H^2.$$

Indeed, substitute f by z^n in (4) obtaining

$$\|\varphi^n\|^2 = \int_{\mathbb{U}} |z|^{2n} \, dm\varphi^{-1} + m\varphi^{-1}(\partial\mathbb{U}) \qquad n = 1, 2, 3, \dots$$

Letting $n \to +\infty$ above and applying Lebesgue's bounded convergence theorem leads to the equality $\lim_{n\to+\infty} \|\varphi^n\|^2 = m\varphi^{-1}(\partial\mathbb{U})$. For the third equality in (2), choose any $f \in H^2$, denote by $\{c_n\}$ the sequence of Maclaurin coefficients of f, and note that, due to the orthogonality of the set $\{\varphi^n : n = 0, 1, 2 \dots\}$, one can write

$$\|C_\varphi f\|^2 = \sum_{n=0}^{+\infty} |c_n|^2 \|\varphi^n\|^2 \geq \inf_{n \geq 1} \|\varphi^n\|^2 \sum_{n=0}^{+\infty} |c_n|^2 = \|C_\varphi\|_e^2 \|f\|^2 \qquad f \in H^2,$$

since $\{\|\varphi^n\|\}$ is a non–increasing sequence. Thus, the infimum in (2) is larger than or equal to $\|C_\varphi\|_e$. The converse inequality is obtained by noting that

$$\|\varphi^n\| = \|C_\varphi(z^n)\| \geq \inf \{\|C_\varphi f\| : \|f\| = 1\} \qquad n = 1, 2, 3, \dots$$

and letting $n \to +\infty$.

Relation (3) is a direct consequence of the fact that $\sqrt{C_\varphi^* C_\varphi}$ is the diagonal operator with diagonal entries $\{\|\varphi^n\|\}$. $\qquad \square$

Here are some interesting consequences of the theorem above.

Corollary 2.2. *Assume φ is an analytic selfmap of $\mathbb{U}$ having orthogonal powers, other than a rotation. If $\|C_\varphi\|_e > 0$, then 0 is an interior point of the spectrum $\sigma(C_\varphi)$ of C_φ. Thus, if φ has orthogonal powers, then C_φ is a Riesz operator (that is $r_e(C_\varphi) = 0$), if and only if C_φ is compact. Also, C_φ is a closed range operator if and only if it is non–compact.*

PROOF. Indeed, the properties above are consequences of the fact that C_φ is bounded bellow if $\|C_\varphi\|_e > 0$. Since φ is not a disk automorphism, clearly $0 \in \sigma(C_\varphi)$. As is well–known, for each operator, the approximate point spectrum contains the boundary of the spectrum. Thus 0 must be an interior point of $\sigma(C_\varphi)$. Since Riesz composition operators are spectrally indistinguishable from compact composition operators, it follows that C_φ is not Riesz if $\|C_\varphi\|_e > 0$. Injective operators (and all composition operators with nonconstant symbols are

injective), have closed range if and only if they are bounded below, that is (in our particular case), if and only if $\|C_\varphi\|_e > 0$, by (2). $\qquad\square$

Let us also denote by C_φ the operator

$$C_\varphi f = f \circ \varphi \qquad f \in \mathcal{H}(\mathbb{U})$$

acting on the space $\mathcal{H}(\mathbb{U})$ of all analytic functions on $\mathbb{U}$. With this notation, the object of study of paper [5] is: *Given a specific subspace $\mathcal{L} \subseteq \mathcal{H}(\mathbb{U})$ which is left invariant by C_φ, when is it true that $C_\varphi^{-1}(\mathcal{L}) = \mathcal{L}$?* Relative to this question, we wish to note the following:

Corollary 2.3. *Let φ be an analytic selfmap of $\mathbb{U}$ having orthogonal powers. Then $C_\varphi^{-1}(H^2) = H^2$ if and only if C_φ is non–compact.*

PROOF. Indeed, this is a direct consequence of (2) combined with [1, Corollary 6.3] where it is proved that $C_\varphi^{-1}(H^2) = H^2 \iff m\varphi^{-1}(\partial\mathbb{U}) > 0$. $\qquad\square$

The spectrum of a compact composition operator and that of an operator induced by an inner function fixing a point in $\mathbb{U}$ have well–known descriptions. Thus, from a spectral prospective, we are mainly interested in the case when φ has orthogonal powers, is not inner, and $\|C_\varphi\|_e > 0$. We begin with the essential spectrum $\sigma_e(C_\varphi)$. We will show below that, if φ is a symbol with orthogonal powers, other than a rotation, then $\sigma_e(C_\varphi)$ is the closed disk centered at the origin having radius $\|C_\varphi\|_e$. We will break the proof into some preliminary lemmas to make it easier to follow.

First denote $P = \sqrt{C_\varphi^* C_\varphi}$ and let U be the partial isometry $UPf := C_\varphi f$, $f \in H^2$, in the polar representation $C_\varphi = UP$ of C_φ. Since composition operators with nonconstant symbols are injective and visibly the kernel of C_φ and that of P coincide, it follows that P has dense range and hence U is actually an isometry (not just a partial isometry), whenever φ is not a constant function. After these preliminary comments, we prove:

Lemma 2.4. *If φ is a symbol fixing the origin then, for each $j = 1, 2, 3 \ldots$*

$$(5) \qquad\qquad C_\varphi^*(z^j) = \sum_{k=1}^{j} <z^j, \varphi^k> z^k.$$

If in addition, φ has orthogonal powers and $\|C_\varphi\|_e > 0$, then

$$(6) \qquad\qquad U^*(z^j) = \sum_{k=1}^{j} \frac{<z^j, \varphi^k>}{\|\varphi^k\|} z^k.$$

PROOF. Let c_n be the n-th Maclaurin coefficient of $C_\varphi^*(z^j)$. Clearly $c_n = <z^j, \varphi^n>$ and, since φ has a zero at the origin, c_n can be nonzero only if $n = 1, 2, \ldots, j$, which proves (5). In case φ has orthogonal powers and $\|C_\varphi\|_e > 0$, then P is the invertible diagonal operator with diagonal entries $\{\|\varphi^n\|\}$ and hence, P^{-1} is the diagonal operator with diagonal entries $\{1/\|\varphi^n\|\}$. Since both the afore-mentioned operators are selfadjoint, the equation $C_\varphi = UP$ leads to $U^* = P^{-1}C_\varphi^*$ and hence to (6). $\qquad\square$

Recall that a forward shift is an isometry S acting on a separable, infinite–dimensional Hilbert space, with the property $S^{*n} \to 0$ in the strong operator topology. The Hilbert dimension of the subspace $\ker S^*$ is called the multiplicity of the shift. Each isometry V on an infinite–dimensional, separable Hilbert space H has a unique Wold decomposition. This means that there is a unique pair of orthogonal, complementary, closed subspaces $H_0 \oplus H_1 = H$ of H, which reduce V, and $V|H_0$ is unitary, whereas $V|H_1$ is a forward shift. Clearly, if V is unitary, then $H = H_0$ and $H_1 = 0$. Nordgren [14] found the Wold decomposition of a composition isometry (that is of a composition operator whose symbol is an inner function fixing the origin), in the interesting case when the operator is not unitary (that is when the symbol is not a rotation). For such a composition operator, the Wold decomposition is $H^2 = \mathbb{C} \oplus zH^2$ (where $\mathbb{C}$ denotes the subspace of constant functions). The obvious relation $C_\varphi 1 = 1$, valid for any composition operator, makes it evident that $C_\varphi|zH^2$ is the forward shift in the Wold decomposition of C_φ. Note also that $U = C_\varphi$ if C_φ is isometric so, our next lemma extends Nordgren's result mentioned above. In the text of the next lemma, the term *non–automorphic* means, of course, that φ should not be a conformal disk automorphism.

Lemma 2.5. *Let φ be a non–automorphic symbol with orthogonal powers having the property $\|C_\varphi\|_e > 0$. Then the Wold decomposition of the isometry U is $H^2 = \mathbb{C} \oplus zH^2$ and $U|zH^2$ is a forward shift of infinite multiplicity.*

PROOF. For each symbol φ fixing the origin, it is easy to see that $\mathbb{C}$ reduces C_φ. Hence C_0, the composition operator with null symbol or, in other words, the orthogonal projection on $\mathbb{C}$, commutes with both C_φ and its adjoint. Hence it commutes with P and therefore with both U and U^*, since $U^* = P^{-1}C_\varphi^*$. Thus, $\mathbb{C}$ is a reducing subspace of U.

The next thing to show is that $U|zH^2$ is a forward shift. If φ is inner, the result was proved by Nordgren, so assume φ is non–inner. Since the sequence

U^{*n} is norm–bounded, it will suffice to show that

$$(7) \qquad\qquad\qquad U^{*n}(z^j) \to 0 \qquad j = 1, 2, \ldots$$

Working by induction, we start with $j = 1$. By (6), one has

$$\|U^{*n}z\| = \left| \frac{< z, \varphi >}{\|\varphi\|} \right|^n \to 0$$

because $| < z, \varphi > /\|\varphi\|| < 1$. Indeed, if arguing by contradiction, one assumes $| < z, \varphi > /\|\varphi\|| = 1$, then the Cauchy–Schwartz inequality $| < z, \varphi > | \leq \|\varphi\|$ is an equality and hence, the vectors involved in it must be colinear, that is $\varphi = \lambda z$ with $|\lambda| = 1$ (since $\|C_\varphi\|_e > 0$), a contradiction with our assumption that φ is not inner. A similar argument shows that, under our assumptions, $| < z^j, \varphi^j > /\|\varphi^j\|| < 1$, for each $j = 1, 2, \ldots$, a fact that will be tacitly used in the sequel.

Assuming now that (7) holds for $j = 1, \ldots, k - 1$, let us prove it also holds for k. Note that, by (6), one has that

$$U^{*n}(z^k) = U^{*(n-1)} \left(\sum_{j=1}^{k-1} \frac{< z^k, \varphi^j >}{\|\varphi^j\|} z^j \right) + \frac{< z^k, \varphi^k >}{\|\varphi^k\|} U^{*(n-1)}(z^k),$$

that is $U^{*n}(z^k)$ is representable as the sum of $(< z^k, \varphi^k > /\|\varphi^k\|)U^{*(n-1)}(z^k)$ and a quantity that is norm–convergent to 0 when $n \to +\infty$. Iterating, note that, for each fixed $1 < j < n$, one has a representation of the form

$$(8) \qquad\qquad U^{*n}(z^k) = Q_n(j) + \left(\frac{< z^k, \varphi^k >}{\|\varphi^k\|} \right)^j U^{*(n-j)}(z^k)$$

where $Q_n(j)$ tends to 0 as $n \to +\infty$. For arbitrary fixed $\epsilon > 0$ choose such a j, large enough that $|(< z^k, \varphi^k > /\|\varphi^k\|)|^j < \epsilon/2$. Then choose a positive integer n_0 so that $\|Q_n(j)\| < \epsilon/2$ if $n \geq n_0$. Given the relation (8), it follows that $\|U^{*n}(z^k)\| < \epsilon$ if $n \geq n_0$, which ends the inductive argument proving that (7) holds. Thus, $U|zH^2$ is a forward shift.

To finish the proof, we need to prove that the multiplicity of that shift is infinite. Note that, by the relation $U^* = P^{-1}C_\varphi^*$, the kernels of U^* and C_φ^* coincide. On the other hand, $\|C_\varphi\|_e > 0$, hence C_φ is an operator bounded below (by (2)), which is not a Fredholm operator (that is C_φ is not a closed–range operator with finite–dimensional kernel and cokernel). This is a consequence of the fact that the only Fredholm composition operators on H^2 are the automorphic composition operators [8, Theorem 3.39]. It follows that C_φ^* has infinite–dimensional kernel and so, the forward shift $U|zH^2$ must have infinite multiplicity. $\qquad\square$

Prior to (finally), describing the essential spectrum $\sigma_e(C_\varphi)$ of C_φ, we wish to review a (rather well–known) property of forward shifts of infinite multiplicity.

Remark. If S is such a shift then, for each $|\lambda| < 1$, the operator $S - \lambda I$ is not a Fredholm operator.

Indeed, the adjoint of $S - \lambda I$ has infinite dimensional kernel, which is a consequence of the fact that a shift of infinite multiplicity is unitarily equivalent to a direct sum of infinitely many copies of M_z, the multiplication operator with the coordinate function acting on H^2. As one can readily see, the evaluation kernel at λ, that is the H^2–function $k_\lambda(z) = 1/(1 - \overline{\lambda}z) = \sum_{n=0}^{+\infty} \overline{\lambda}^n z^n$ is an eigenfunction of M_z^* corresponding to the eigenvalue $\overline{\lambda}$. Putting the aforementioned facts together, the reader can see that the conclusion of the remark above holds. Keeping this in mind, we describe in the following the essential spectrum of a composition operator whose symbol has orthogonal powers.

Theorem 2.6. *If φ is a non–automorphic symbol with orthogonal powers then*

$$\sigma_e(C_\varphi) = \|C_\varphi\|_e \overline{\mathbb{U}}. \tag{9}$$

Consequently

$$r_e(C_\varphi) = \|C_\varphi\|_e. \tag{10}$$

PROOF. The relations above are trivially true if C_φ is compact. If $\|C_\varphi\|_e > 0$ then, consider any $|\lambda| < \|C_\varphi\|_e$ and note that $C_\varphi^* - \lambda I = P(U^* - \lambda P^{-1})$. Since P is invertible, $C_\varphi^* - \lambda I$ and $U^* - \lambda P^{-1}$ are simultaneously Fredholm or non–Fredholm. The same holds for $U^* - \lambda P^{-1}$ and $U^* - (\lambda/\|C_\varphi\|_e)I$, given the obvious relation $U^* - \lambda P^{-1} = (U^* - (\lambda/\|C_\varphi\|_e)I) + ((\lambda/\|C_\varphi\|_e)I - \lambda P^{-1})$, because $(\lambda/\|C_\varphi\|_e)I - \lambda P^{-1}$ is a diagonal operator whose diagonal entries tend to 0, hence a compact operator. Finally, note that, by the remark above, $U^* - (\lambda/\|C_\varphi\|_e)I$ is not Fredholm, that is $\|C_\varphi\|_e \mathbb{U} \subseteq \sigma_e(C_\varphi)$. $\qquad \square$

Combining the theorem above, with some standard Fredholm theory facts (for which the reader is referred to [15, Section 1]), and a theorem about the point–spectrum of any composition operator whose non–automorphic symbol fixes a point (due to G. Koenigs), one can find the spectrum $\sigma(C_\varphi)$ of C_φ.

Theorem 2.7. *If φ is a non–automorphic symbol with orthogonal powers then*

$$\sigma(C_\varphi) = r_e(C_\varphi)\overline{\mathbb{U}} \cup \{(\varphi'(0))^n : n = 1, 2, 3, \dots\} \cup \{1\}. \tag{11}$$

PROOF. Description (11) of $\sigma(C_\varphi)$ is known when C_φ is compact. Thus, the interesting case is when one has $0 < \|C_\varphi\|_e$. In that case, note that, if $|\lambda| > \|C_\varphi\|_e$,

then $C_\varphi - \lambda I$ is a Fredholm operator. Thus the Fredholm index, that is the map $i(\lambda) = \dim(\ker(C_\varphi - \lambda I)) - \dim(\ker(C_\varphi - \lambda I)^*)$, is continuous on the arcwise–connected set $\{C_\varphi - \lambda I : |\lambda| > \|C_\varphi\|_e\}$. Since the map i is valued in a discrete set, this means that it is constant on $\{C_\varphi - \lambda I : |\lambda| > \|C_\varphi\|_e\}$, namely null since, for $|\lambda|$ large enough, $C_\varphi - \lambda I$ is invertible. The conclusion is that, if $\lambda \in \sigma(C_\varphi)$ and $|\lambda| > \|C_\varphi\|_e$, then λ is an eigenvalue of C_φ. On the other hand, a noted theorem of G. Koenigs [12] (see also [17, Section 6.1]), says that the point spectrum $\sigma_p(C_\varphi)$ of C_φ is a subset of the set $\{(\varphi'(0))^n : n = 1, 2, 3, \ldots\} \cup \{1\}$. The aforementioned set is known to be a subset of $\sigma(C_\varphi)$ for any symbol φ fixing the origin [8, Proposition 7.32]. These considerations combined with (9) and (10) end the proof. □

To our knowledge, descriptions of spectra for composition operators induced by symbols fixing a point are known in the following cases: (a) the case of compact composition operators [4] (see also [17, Section 6.2]), (b) the case of composition operators whose non–automorphic symbols fix a point and are extendable by analyticity to an open neighborhood of $\overline{\mathbb{U}}$ [11] (see also [8, Theorem 7.36]), (c) the case of composition operators with non–automorphic, univalent symbol fixing a point [7] (see also [8, Theorem 7.30]), and (d) the case of composition operators whose symbols are inner maps fixing a point [8, Section 7.8]. In the interesting case when φ is a non–inner symbol inducing a noncompact composition operator C_φ, Theorem 2.7 is rather complementary to the aforementioned results. Indeed, the only known example of maps which are analytic on a neighborhood of $\overline{\mathbb{U}}$, leave $\mathbb{U}$ invariant, and are symbols with orthogonal powers is that of finite Blaschke products fixing the origin [1]. On the other hand, the only univalent symbols with orthogonal powers are the constant multiples of z [2].

3. Final remarks

In this section we show that some spectral properties true for composition operators whose non–inner symbols have orthogonal powers are more general, being valid for any composition operator whose non–inner symbol fixes a point. First recall some known results:

Theorem 3.1 ([18, Theorem 4.1]). *The analytic selfmap φ of $\mathbb{U}$ is inner if and only if*

$$(12) \qquad \|C_\varphi\|_e = \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}.$$

Given that composition operators whose symbols fix the origin are contractions (by Littlewood's subordination principle, [9, Theorem 1.7]), an immediate consequence is:

Corollary 3.2. *If the analytic selfmap φ of $\mathbb{U}$ is not inner and fixes the origin, then*

$$(13) \qquad\qquad r_e(C_\varphi) < 1.$$

Relative to that, we prove:

Theorem 3.3. *The inequality*

$$(14) \qquad\qquad r_e(C_\varphi) < 1$$

holds if and only if φ is a non–inner map fixing a point in $\mathbb{U}$.

PROOF. Note that, if φ is inner, then

$$(15) \qquad\qquad \|C_\varphi^n\|_e \geq 1 \qquad n = 1, 2, 3, \ldots$$

by formula (12) and the fact that the iterates of an inner function are inner functions. Also, if φ is fixed point free, having Denjoy–Wolff point $\omega \in \partial\mathbb{U}$ then (15) holds as well. Indeed, an immediate consequence of [6, (3.1)] is

$$(16) \qquad\qquad \|C_\varphi\|_e \geq \frac{1}{\sqrt{\varphi'(\omega)}}.$$

The angular derivative at the Denjoy–Wolff point is a positive number, less than or equal to 1. This and the fact that each iterate of φ has the same Denjoy–Wolff point, namely ω, imply that (15) must hold. The consequence is that $r_e(C_\varphi) \geq 1$ (since $r_e(C_\varphi) = \lim_{n\to+\infty} \sqrt[n]{\|C_\varphi^n\|_e}$), for all analytic selfmaps φ of $\mathbb{U}$ except the non-inner ones fixing a point. As we noted in Corollary 3.2, if φ is not inner and fixes the origin, then (14) holds. In case φ is not inner and fixes $p \in \mathbb{U} \setminus \{0\}$, let $\alpha_p(z) = (p - z)/(1 - \bar{p}z)$ and note that this is a selfinverse disk automorphism inducing the operator similarity

$$(17) \qquad\qquad C_{\alpha_p} C_\varphi C_{\alpha_p} = C_\psi$$

where $\psi = \alpha_p \circ \varphi \circ \alpha_p$. Visibly ψ is non–inner (since φ is not inner), and fixes 0. Thus $r_e(C_\psi) = r_e(C_\varphi) < 1$.   □

The existence of a fixed point in the case of symbols inducing compact composition operators, was originally proved by Caughran and Schwartz [4]. That result was eventually extended to Riesz composition operators by Bourdon and Shapiro [3]. Theorem 3.3 is visibly an extension of those results.

Let us recall another known fact:

Theorem 3.4 ([18, Theorem 5.1]). *Let φ be an analytic selfmap of $\mathbb{U}$ that fixes the origin. Then $\|C_\varphi|zH^2\| = 1$ if and only if φ is inner.*

Based on the theorem above, one can prove the following spectral property.

Proposition 3.5. *Composition operators C_φ induced by non-inner symbols φ fixing a point $p \in \mathbb{U}$ have disconnected spectra contained in the closed unit disk. More exactly, for each such operator, there is some $0 \leq r < 1$ so that*

$$(18) \qquad\qquad \{0,1\} \subseteq \sigma(C_\varphi) \subseteq r\overline{\mathbb{U}} \cup \{1\}.$$

For all non–automorphic symbols fixing a point in $\mathbb{U}$, 1 is an eigenvalue of multiplicity 1 of both C_φ and C_φ^*.

PROOF. Clearly $\{0,1\} \subseteq \sigma(C_\varphi)$ since C_φ is non–invertible and any composition operator satisfies $C_\varphi 1 = 1$. If $\varphi(0) = 0$ then it is straightforward to see that $C_\varphi^* 1 = 1$, hence the subspaces $\mathbb{C}$ and zH^2 are reducing subspaces of C_φ, for which reason, $\sigma(C_\varphi) = \sigma(C_\varphi|zH^2) \cup \{1\}$. Thus C_φ satisfies (18) with $r = \|C_\varphi|zH^2\|$ and $0 < r < 1$, by Theorem 3.4. By relation (17), any composition operator whose symbol is non-inner and fixes a point in $\mathbb{U}$ is similar to a composition operator induced by a non-inner selfmap of $\mathbb{U}$ fixing the origin. Thus (18) holds in general, for all composition operators with non-inner symbol fixing a point in $\mathbb{U}$. The multiplicity of the eigenvalue 1 is 1. Indeed, if $\varphi(0) = 0$, then $\|(C_\varphi|zH^2)^n\| = \|(C_\varphi|zH^2)^{*n}\| \to 0$. Thus $\mathbb{C}$ is the eigenspace associated to 1 in both the case of C_φ and C_φ^*. Given the operator similarity (17), the same remains true if the fixed point is not the origin. If φ is inner non–automorphic, the fact that the multiplicity of 1 is 1 for both C_φ and C_φ^* is a consequence of the Wold decomposition of composition isometries. Indeed, forward shifts have no eigenvalues and contractions have the same invariant vectors as their adjoints. These facts, combine as above with the operator similarity (17) to prove that the multiplicity of the eigenvalue 1 is 1 in the case of an arbitrary non-automorphic, inner symbol with a fixed point. □

In Koenigs's theorem ([12] or [17, Section 6.1]) it is proved that the multiplicity of the eigenvalue 1 is 1 if $\varphi'(p) \neq 0$ and φ is not a conformal automorphism (where p is the fixed point of φ). According to Proposition 3.5, the restriction $\varphi'(p) \neq 0$

is unnecessary. In the particular case of a composition operator whose symbol has orthogonal powers, the quantity $r = \|C_\varphi|zH^2\|$ in the proof above can be calculated with the formula $\|C_\varphi|zH^2\| = \|\varphi\|$, an equality very easy to establish, given that $\sqrt{C_\varphi^* C_\varphi}$ is the diagonal operator with diagonal entries $\{\|\varphi\|^n\}$. This is a particular case of [13, Theorem 7]. We also refer the reader to [13] for an alternative short proof of Theorem 3.4 in the current paper.

We have already described and used the Wold decomposition $H^2 = \mathbb{C} \oplus zH^2$ of a non–unitary composition isometry as found by Nordgren in [14]. Isometries are particular contractions and forward shifts are particular completely non–unitary contractions, that is contractions whose restrictions to any nonzero reducing subspace are not unitary. Actually, any Hilbert–space contraction has a "Wold decomposition" (sometimes called the canonical decomposition of a Hilbert–space contraction). This means that there is a unique representation of the whole space as the direct sum of two reducing subspaces, so that the restriction of the given contraction to the first subspace is unitary, whereas its restriction to the second is completely non–unitary [20, Theorem 3.2]. Does Nordgren's representation of the Wold decomposition of a non-unitary composition isometry extend to any non–unitary composition contraction? The answer is affirmative.

Proposition 3.6. *The canonical decomposition of a non–unitary composition contraction is* $H^2 = \mathbb{C} \oplus zH^2$.

PROOF. We already noted that composition operators whose symbols fix the origin are contractions. On the other hand, the well–known estimate $1/\sqrt{1 - |\varphi(0)|^2} \leq \|C_\varphi\|$, valid for the norm of any composition operator, shows that the composition contractions are exactly the composition operators whose symbols fix the origin. We need to address only the case when φ is not inner. We already noted above that the subspace $\mathbb{C}$ is reducing for C_φ and $C_\varphi|\mathbb{C}$ is the identity on $\mathbb{C}$, a unitary operator, in the case of any composition contraction. On the other hand, if $f \in zH^2$ and $f \neq 0$, then, by Theorem 3.4, $\|C_\varphi f\| < \|f\|$, which proves that the restriction of C_φ to any nonzero subspace of zH^2 is non–isometric, hence $C_\varphi|zH^2$ is a completely non–unitary contraction. Given the uniqueness of the canonical decomposition, the proof is over.   □

As a concluding comment, we note that any composition operator whose symbol has a fixed point $p \in \mathbb{U}$ and is conjugated via (17) to a symbol with orthogonal powers, has spectrum and essential spectrum determined by our main results, Theorems 2.6 and 2.7.

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