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Weighted Composition Operators on H^2 and Applications

Valentin Matache

Abstract. Operators on function spaces acting by composition to the right with a fixed selfmap φ of some set are called composition operators of symbol φ . A weighted composition operator is an operator equal to a composition operator followed by a multiplication operator. We summarize the basic properties of bounded and compact weighted composition operators on the Hilbert Hardy space on the open unit disk and use them to study composition operators on Hardy–Smirnov spaces.

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1. Introduction

A composition operator C_φ is an operator on a space \mathcal{S} consisting of functions defined on the same set G , acting by composition to the right with a chosen selfmap φ of G , i.e.

$$C_\varphi f = f \circ \varphi \quad f \in \mathcal{S}.$$

The map φ is called the *symbol* of the composition operator C_φ . A class of spaces where composition operators have been intensely studied are Hardy spaces $H^p(\mathbb{U})$, $0 < p < +\infty$, over the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, that is the spaces of all functions f analytic in \mathbb{U} satisfying the mean growth condition

$$\|f\|_p := \sup_{0 < r < 1} \left(\int_{\partial\mathbb{U}} |f(r\zeta)|^p dm(\zeta) \right)^{1/p} < +\infty. \quad (1)$$

In (1), the symbol m denotes the normalized arc-length Lebesgue measure. The quantity $\|\cdot\|_p$ is a Banach space norm on $H^p(\mathbb{U})$, $1 \leq p < +\infty$, respectively a complete F -norm, if $0 < p < 1$, [11]. In the case of $p = 2$, the norm in (1) is even

a Hilbert space norm with the alternative description

$$\|f\|_2 = \sqrt{\sum_{n=0}^{\infty} |c_n|^2}, \quad (2)$$

where $\{c_n\}$ is the sequence of Maclaurin coefficients of f . For each $f \in H^p(\mathbb{U})$, the radial limit $\lim_{r \rightarrow 1^-} f(r\zeta)$ exists for almost all $\zeta \in \partial\mathbb{U}$. The radial limit function of f , will be denoted by the same symbol as the function itself, relying on context to distinguish between the two notions. It is well known that the radial limit function of f is a L^p -function whose L^p -norm coincides to $\|f\|_p$.

The space of bounded analytic functions on \mathbb{U} endowed with the supremum norm $\|\cdot\|_{\infty}$ is denoted $H^{\infty}(\mathbb{U})$.

The fact that any analytic selfmap φ of \mathbb{U} induces a bounded composition operator C_{φ} on any of the spaces $H^p(\mathbb{U})$ is a consequence of a basic function theory principle known as the Littlewood Subordination Principle, [11, Theorem 1.7].

Riemann's well known conformal equivalence theorem provides two natural ways of constructing Hardy spaces over proper, simply connected domains G . Both are described in [11, Chapter 10]. The first method produces conformal copies of the Hardy spaces $H^p(\mathbb{U})$, (i.e., if γ is a conformal isomorphism, (or Riemann map) of \mathbb{U} onto G , then $f \in H^p(G)$ if and only if $f \circ \gamma \in H^p(\mathbb{U})$). Most major problems on composition operators on such spaces can be reduced to the corresponding problems on $H^p(\mathbb{U})$.

The second method presented in [11, Chapter 10] produces spaces of a more exotic nature, more exactly, $H^p(G)$ is constructed as follows. Let γ be a conformal isomorphism of \mathbb{U} onto G . For each $0 < p < \infty$, the Hardy-Smirnov space $H^p(G)$ is by definition, the collection of all functions f analytic in G that satisfy the condition

$$\sup_{0 < r < 1} \left(\int_{\Gamma_r} |f(z)|^p |dz| \right)^{1/p} < +\infty, \quad (3)$$

where, for each r , Γ_r is the image under γ of the circle of radius r about the origin. Although condition (3) seems to produce spaces that depend on the conformal isomorphism γ , it is shown in [11, Theorem 10.1] that, for each $0 < p < +\infty$, $H^p(G)$ depends only on G . Some authors refer to the spaces constructed by both these methods, simply as Hardy spaces over G . To avoid confusion and acknowledge Smirnov's contribution to the study of the spaces obtained by the second method, we chose to call them Hardy-Smirnov spaces. See also the notes to Chapter 10 in [11].

We studied composition operators on such spaces constructed over half-planes, the main result being that they do not support compact composition operators, [17].

That result was strongly generalized in [23] where the following is proved.

Theorem 1 ([23, Main Theorem]). *The necessary and sufficient condition that the Hardy–Smirnov spaces $H^p(G)$ support compact composition operators is*

$$\gamma' \in H^1(\mathbb{U}). \quad (4)$$

An immediate consequence is

Corollary 1. *Hardy–Smirnov spaces over unbounded domains cannot support compact composition operators.*

Indeed, a well known result of Hardy space theory, [11, Theorem 3.11] says that, if (4) holds, then γ has a continuous extension to the closure of \mathbb{U} and hence, G must be bounded.

The criteria for boundedness known for composition operators on Hardy–Smirnov spaces over half–planes are in terms of Carleson measures induced by their symbols. As is observed in [17], they are hard to use in practical problems. This author concludes [17] by raising the problem of describing the γ – conformal conjugates $\varphi = \gamma^{-1} \circ \phi \circ \gamma$ of any given analytic selfmap ϕ of a half–plane inducing bounded composition operators on the Hardy–Smirnov spaces over that half–plane. In section 4 we are able to solve this problem, using recent results on local Dirichlet spaces and their composition operators due to D. Sarason and Nuno O. Silva, [21] and the fact that the study of boundedness and compactness of Hardy–Smirnov–space composition operators can be reduced to the study of associated weighted composition operators on $H^2(\mathbb{U})$.

A multiplication operator on a linear space \mathcal{L} consisting of functions on the same set S is an operator of the form

$$M_\psi f = \psi f \quad f \in \mathcal{L}.$$

The function ψ is called the symbol of the multiplication operator. A weighted composition operator is a composition operator followed by a multiplication operator, that is an operator of the form

$$T_{\psi, \varphi} f = \psi(f \circ \varphi) \quad f \in \mathcal{L}.$$

To avoid triviality we will always assume that ψ is not the null function. According to [19], composition operators appeared implicitly in works dealing with theoretical mechanics published in the nineteen thirties by B. O. Koopman. However they began being called composition operators and being studied explicitly in the late sixties, [18]. Weighted composition operators appeared in about the same period and a first result supporting their importance is the fact that Hardy–space isometries are necessarily weighted composition operators [13], (if $p \neq 2$).

The main purpose of this introductory section is setting up the notation and presenting a brief outline of the paper. Section 3 is dedicated to the study of bounded and compact weighted composition operators on $H^2(\mathbb{U})$. We collect the basic facts including original results, results by others, or results known as "folklore" in a brief study. It mainly represents a survey on the topic of weighted composition operators on $H^2(\mathbb{U})$, which in our opinion is needed at this time,

when the results are scattered in the literature and sometimes incompletely or improperly formulated. In section 4 we study bounded and compact composition operators on Hardy–Smirnov spaces. Most of the results in that section are new.

One of our tools in both section 3 and 4 is the theory of angular derivatives. We dedicate the first part of the next section to reviewing the main concepts and results related to it. Section 2 also contains a new approach to the existence of angular derivatives of an analytic selfmap φ based on what we call *tangential approach regions*. It has applications to some of the results in section 4.

2. Angular Derivatives and Tangential Contact

For each $\alpha > 1$ and each $\omega \in \partial\mathbb{U}$, we consider the boundary approach regions

$$R_{M,\alpha}(\omega) = \left\{ z \in \mathbb{U} : \frac{|\omega - z|^\alpha}{1 - |z|^2} < M \right\} \quad M > 0 \quad (5)$$

and call them *tangential approach regions* that make α – contact with the unit circle at ω . Some tangential approach regions making $3/2$ – contact with the unit circle at 1 are graphed in Figure 1.

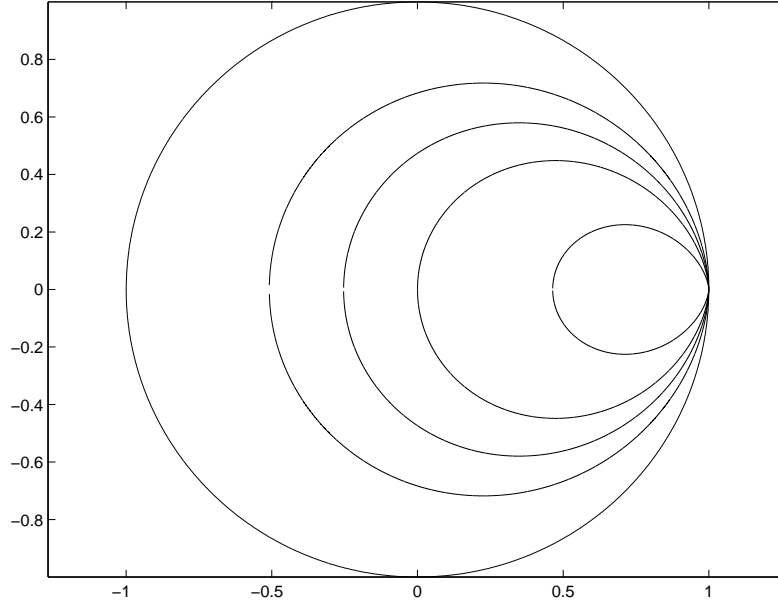


Figure 1: Nested tangential approach regions making $3/2$ – contact at 1.

If, in (5) one took $\alpha = 1$, one would produce substitutes of the regions

$$\Gamma_M(\omega) = \left\{ z \in \mathbb{U} : \frac{|\omega - z|}{1 - |z|} < M \right\} \quad M > 1$$

called *nontangential approach regions* with vertex at ω . Some are graphed in Figure 2.

Note that the regions $R_{M,\alpha}(\omega)$ exhaust \mathbb{U} as the constant M ranges from 0 to ∞ .

The tangential approach regions making contact of order 2 are special, since they are circular. We call them horodisks tangent at ω , and call their boundaries horocycles. The Julia–Carathéodory theorem is a geometric function theory result saying, among other things, that analytic functions having an angular derivative at a boundary point $\omega \in \partial\mathbb{U}$ map horodisks tangent at ω into horodisks tangent at, a (possibly different) boundary point $\eta \in \partial\mathbb{U}$.

Definition 1. *An analytic selfmap φ of \mathbb{U} has an angular derivative at a boundary point $\omega \in \partial\mathbb{U}$ if there is some $\eta \in \partial\mathbb{U}$ and some $c \in \mathbb{C}$, so that, for each $M > 1$,*

$$\frac{\eta - \varphi(z)}{\omega - z} \rightarrow c \quad \text{as } z \rightarrow \omega \quad \text{inside } \Gamma_M(\omega).$$

In that case, the value c is called the angular derivative of φ at ω , and we denote $c = \varphi'(\omega)$. Clearly η is the angular limit of φ at ω , i.e. the limit of $\varphi(z)$ as $z \rightarrow \omega$ inside each region $\Gamma_M(\omega)$.

With this terminology we state the following.

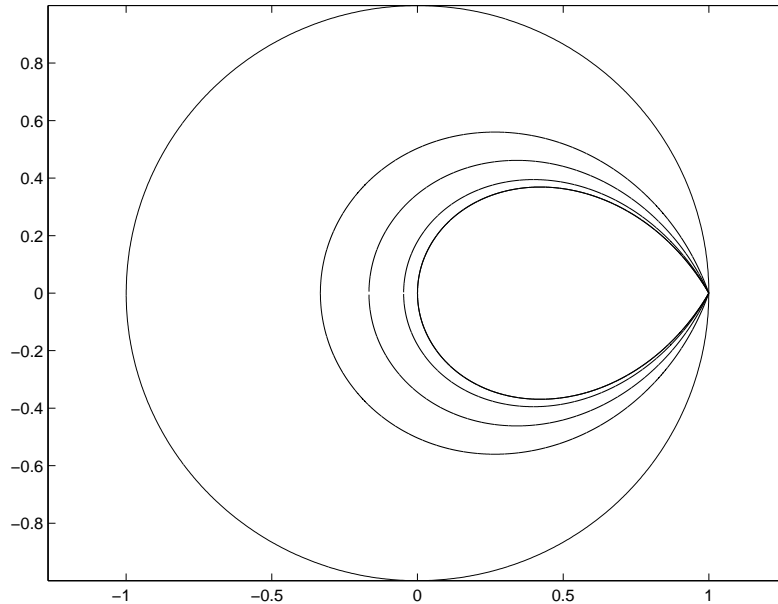


Figure 2: Nested nontangential approach regions at 1.

Theorem 2 (The Julia–Carathéodory Theorem). *For each analytic selfmap φ of \mathbb{U} and each pair ω, η of unimodular numbers, the ratio $(\eta - \varphi(z))/(\omega - z)$ has a (possibly infinite) nontangential limit at ω . The situation when the limit is finite corresponds to the case when φ has an angular derivative at ω . Each of the following is an alternative necessary and sufficient condition for that to happen,*

$$\beta = \sup \left\{ \frac{|\eta - \varphi(z)|^2}{1 - |\varphi(z)|^2} / \frac{|\omega - z|^2}{1 - |z|^2} : z \in \mathbb{U} \right\} < +\infty \quad (6)$$

$$\liminf_{z \rightarrow \omega} \frac{1 - |\varphi(z)|}{1 - |z|} < +\infty. \quad (7)$$

If φ has an angular derivative at ω and if the angular limit at ω is η , then the angular derivative can be calculated with the formula

$$\varphi'(\omega) = \eta\beta\bar{\omega}. \quad (8)$$

Also, the connection between the quantity in (7), and the angular derivative is

$$|\varphi'(\omega)| = \beta = \liminf_{z \rightarrow \omega} \frac{1 - |\varphi(z)|}{1 - |z|}. \quad (9)$$

Finally, the function $\varphi'(z)$ has angular limit $\varphi'(\omega)$ at ω .

For the proof of this theorem, we refer the reader to [1], [8], or [22].

Borrowing heavily from that proof, as presented in [1], we wish to note that, even when $\alpha \neq 2$, the fact that φ maps nontangential approach regions making α – contact into similar regions at a constant rate implies the existence of angular derivatives.

Theorem 3. *Let $\omega, \eta \in \partial\mathbb{U}$, and $\alpha > 1$ be fixed. Let φ be an analytic selfmap of \mathbb{U} . If*

$$\beta_\alpha = \sup \left\{ \frac{|\eta - \varphi(z)|^\alpha (1 - |z|^2)}{|\omega - z|^\alpha (1 - |\varphi(z)|^2)} : z \in \mathbb{U} \right\} < +\infty,$$

then the angular derivative $\varphi'(\omega)$ exists, the angular limit of φ at ω being η , and the following inequality holds

$$|\varphi'(\omega)|^{\alpha-1} \leq \beta_\alpha.$$

Proof. Let us note that the following string of inequalities is valid for each $0 < r < 1$.

$$\begin{aligned} \beta_\alpha &\geq \frac{|\eta - \varphi(r\omega)|^\alpha}{1 - |\varphi(r\omega)|^2} \frac{1 - r^2}{|\omega - r\omega|^\alpha} = \frac{|\eta - \varphi(r\omega)|^\alpha}{1 - |\varphi(r\omega)|^2} \frac{1 + r}{(1 - r)^{\alpha-1}} \geq \quad (10) \\ &\left(\frac{|\eta - \varphi(r\omega)|}{1 - r} \right)^{\alpha-1} \frac{1 + r}{1 + |\varphi(r\omega)|} \geq \left(\frac{1 - |\varphi(r\omega)|}{1 - r} \right)^{\alpha-1} \frac{1 + r}{1 + |\varphi(r\omega)|}. \end{aligned}$$

Note also that, by the first inequality above,

$$|\eta - \varphi(r\omega)|^\alpha \leq \beta_\alpha \frac{(1 - r)^{\alpha-1}}{1 + r}$$

so, if β_α is finite, then $\varphi(r\omega) \rightarrow \eta$ when $r \rightarrow 1^-$. Keeping this in mind and letting $r \rightarrow 1^-$ in (10) one obtains

$$\begin{aligned} \liminf_{z \rightarrow \omega} \frac{1 - |\varphi(z)|}{1 - |z|} &\leq \liminf_{r \rightarrow 1^-} \frac{1 - |\varphi(r\omega)|}{1 - r} = \\ \liminf_{r \rightarrow 1^-} \frac{1 - |\varphi(r\omega)|}{1 - r} \left(\frac{1 + r}{1 + |\varphi(r\omega)|} \right)^{1/(\alpha-1)} &\leq \beta_\alpha^{1/(\alpha-1)}, \end{aligned}$$

which, by the Julia–Carathéodory Theorem, proves our claims. \square

The geometric interpretation of the result above is that, if tangential approach regions making α – contact with $\partial\mathbb{U}$ at some point ω are always mapped by some φ , “at a constant rate”, into the same kind of regions making contact at a (possibly different) point $\eta \in \partial\mathbb{U}$, then the angular derivative $\varphi'(\omega)$ needs to exist, that is, more formally: if there is $C > 0$ such that

$$\varphi(R_{M,\alpha}(\omega)) \subseteq R_{MC,\alpha}(\eta) \quad M > 0,$$

then $\varphi'(\omega)$ exists and η is the angular limit of φ at ω .

Let us note that the boundedness–condition in Theorem 3 holds globally if and only if it holds locally about ω , that is:

Remark 1. *Let φ , α , ω , and η be as in Theorem 3. If there is some $\delta > 0$ such that*

$$\sup \left\{ \frac{|\eta - \varphi(z)|^\alpha (1 - |z|^2)}{|\omega - z|^\alpha (1 - |\varphi(z)|^2)} : z \in \mathbb{U}, |\omega - z| < \delta \right\} < +\infty,$$

then $\beta_\alpha < +\infty$.

Indeed, if arguing by contradiction, one assumes that $\beta_\alpha = +\infty$, then there must exist a sequence $\{z_n\}$ in \mathbb{U} tending to a boundary point $u \in \partial\mathbb{U}$, $u \neq \omega$ such that

$$\lim_{n \rightarrow +\infty} \frac{|\eta - \varphi(z_n)|^\alpha (1 - |z_n|^2)}{|\omega - z_n|^\alpha (1 - |\varphi(z_n)|^2)} = +\infty.$$

Since the quantity $\frac{|\eta - \varphi(z_n)|^\alpha}{|\omega - z_n|^\alpha}$ is bounded, one deduces that $\liminf_{z \rightarrow u} \frac{1 - |\varphi(z)|}{1 - |z|} = 0$, which is impossible, since, as is well known [8, Corollary 2.40], the quantity $\frac{1 - |\varphi(z)|}{1 - |z|}$ is bounded away from 0. \square

Note that the boundedness away from 0 of the ratio $(1 - |\varphi(z)|)/(1 - |z|)$, used in the proof above, combines with formula (9) and leads to the conclusion that an angular derivative cannot be null.

The reciprocal of Theorem 3 holds only for $1 < \alpha \leq 2$.

Theorem 4. *If the angular derivative $\varphi'(\omega)$ exists and η denotes the angular limit of φ at ω then $\beta_\alpha < +\infty$ for any $1 < \alpha \leq 2$. Consequently φ has an angular derivative at ω with angular limit η if and only if*

$$\limsup_{z \rightarrow \omega} \frac{|\eta - \varphi(z)|^\alpha (1 - |z|^2)}{|\omega - z|^\alpha (1 - |\varphi(z)|^2)} < +\infty \quad (11)$$

for some $1 < \alpha \leq 2$.

Proof. Let $\rho(z, a)$, $z, a \in \mathbb{U}$ denote the pseudohyperbolic distance on \mathbb{U} , that is

$$\rho(z, a) = \left| \frac{a - z}{1 - \bar{a}z} \right| \quad z, a \in \mathbb{U}.$$

By the Schwarz–Pick Lemma, [22, section 4.3], one has

$$\frac{1 - \rho^2(a, z)}{1 - \rho^2(\varphi(a), \varphi(z))} \leq 1 \quad a, z \in \mathbb{U}.$$

It is well known and straightforward to prove that

$$|1 - \bar{a}z| = \frac{(1 - |a|^2)^{1/2}(1 - |z|^2)^{1/2}}{(1 - \rho^2(a, z))^{1/2}} \quad a, z \in \mathbb{U}.$$

Therefore

$$\begin{aligned} & \frac{|1 - \overline{\varphi(a)}\varphi(z)|^\alpha(1 - |z|^2)}{|1 - \bar{a}z|^\alpha(1 - |\varphi(z)|^2)} = \tag{12} \\ & \left(\frac{1 - |\varphi(a)|^2}{1 - |a|^2} \right)^{\alpha/2} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{1-\alpha/2} \left(\frac{1 - \rho^2(a, z)}{1 - \rho^2(\varphi(a), \varphi(z))} \right)^{\alpha/2} \leq \\ & M^{1-\alpha/2} \left(\frac{1 - |\varphi(a)|^2}{1 - |a|^2} \right)^{\alpha/2}, \end{aligned}$$

where

$$M = \left(\sup \left\{ \frac{1 - |z|^2}{1 - |\varphi(z)|^2} : z \in \mathbb{U} \right\} \right)^{1-\alpha/2}.$$

By Julia’s Lemma, [22, section 4.4], one can choose a sequence $\{a_n\}$ in \mathbb{U} so that $a_n \rightarrow \omega$, $\varphi(a_n) \rightarrow \eta$, and $(1 - |\varphi(a_n)|)/(1 - |a_n|) \rightarrow |\varphi'(\omega)|$. Substitute in (12) a by a_n , then let $n \rightarrow +\infty$. One gets $\beta_\alpha \leq M^{1-\alpha/2}|\varphi'(\omega)|^{\alpha/2} < +\infty$. The fact that (11) is equivalent to the existence of the angular derivative follows now as a direct consequence of Theorem 3 and Remark 1. \square

Our next remark emphasizes the fact that the reciprocal of Theorem 3, holds in select cases, when $\alpha > 2$.

Remark 2. *If $\limsup_{z \rightarrow \omega} |\eta - \varphi(z)|/|\omega - z| < +\infty$, then $\beta_\alpha < +\infty$ for any $1 < \alpha < +\infty$.*

Indeed, if $\limsup_{z \rightarrow \omega} |\eta - \varphi(z)|/|\omega - z| < +\infty$, then the angular derivative of φ at ω exists and the angular limit of φ at ω is η as we show below.

$$\frac{1 - |\varphi(r\omega)|}{1 - |r\omega|} \leq \frac{|\eta - \varphi(r\omega)|}{|\omega - r\omega|} \quad 0 < r < 1$$

so

$$\liminf_{z \rightarrow \omega} \frac{1 - |\varphi(z)|}{1 - |z|} \leq \limsup_{z \rightarrow \omega} \left| \frac{\eta - \varphi(z)}{\omega - z} \right| < +\infty.$$

Thus $\varphi'(\omega)$ exists. The fact that, $\beta_\alpha < +\infty$ for any $1 < \alpha < +\infty$ is a consequence of the identity

$$\frac{|\eta - \varphi(z)|^\alpha (1 - |z|^2)}{1 - |\varphi(z)|^2 |\omega - z|^\alpha} = \frac{|\eta - \varphi(z)|^2 (1 - |z|^2)}{1 - |\varphi(z)|^2 |\omega - z|^2} \left| \frac{\eta - \varphi(z)}{\omega - z} \right|^{\alpha-2} \quad z \in \mathbb{U}$$

condition (6), Theorem 4, and Remark 1. \square

On the other hand, it is perfectly possible that the angular derivative of some analytic selfmap φ of \mathbb{U} exist at some $\omega \in \partial\mathbb{U}$, but $\beta_\alpha = +\infty$ for any value $\alpha > 2$. To give such an example, let us recall first that convergent Blaschke products B with infinitely many factors and the property $B(0) \neq 0$ are the analytic selfmaps of the form

$$B(z) = \lambda \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z} \quad z \in \mathbb{U}$$

where $\{z_n\}$ is a sequence in $\mathbb{U} \setminus \{0\}$ satisfying the condition $\sum_{n=1}^{\infty} (1 - |z_n|) < +\infty$, (necessary for the convergence of the product above) and λ is a unimodular constant. Clearly $\{z_n\}$ is the sequence of zeros of the product B .

Example 1. For each $n = 1, 2, 3, \dots$ choose $z_n \in \mathbb{U} \setminus \{0\}$ with the properties

$$\frac{|1 - z_n|^2}{1 - |z_n|^2} = n^2 \quad \text{and} \quad |1 - z_n| \leq \exp(-n) \quad n = 1, 2, \dots$$

Let φ denote the Blaschke product of zeros $\{z_n\}$. Then φ is a convergent Blaschke product which has an angular derivative at $\omega = 1$, but $\beta_\alpha = +\infty$ for any $\alpha > 2$.

Proof. First note that, for each positive integer n , one can choose z_n with the properties above because $|1 - z|^2 = n^2(1 - |z|^2)$ is the equation of a horocycle tangent at 1. By the way this choice was made, we note that

$$\sum_{n=1}^{\infty} (1 - |z_n|) \leq 2 \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|1 - z_n|^2} < +\infty$$

and so, the Blaschke product under consideration is convergent and has an angular derivative at 1, [20, Theorem 3.1, Proposition 3.5]. On the other hand, denoting by η , the angular limit of φ at 1, one has that

$$\beta_\alpha \geq \frac{|\eta - \varphi(z_n)|^\alpha (1 - |z_n|^2)}{1 - |\varphi(z_n)|^2 |1 - z_n|^\alpha} = \frac{1}{n^2 |1 - z_n|^{\alpha-2}} \geq \frac{\exp((\alpha - 2)n)}{n^2} \rightarrow +\infty.$$

\square

3. Weighted Composition Operators on $H^2(\mathbb{U})$

In the following we will prove the basic facts on bounded and compact weighted composition operators on $H^2(\mathbb{U})$ following the similar results known for composition operators. Although there is an extensive literature on weighted composition operators, we have seen this job only partially done, often in particular cases. Besides results by others, this section contains new results too. We will indicate carefully when a result exists, even in a particular form, in other papers.

3.1. Boundedness

We begin by making an elementary remark partially contained in [16].

If $T_{\psi,\varphi}(H^2(\mathbb{U})) \subseteq H^2(\mathbb{U})$, then $T_{\psi,\varphi}1 = \psi \in H^2(\mathbb{U})$. It is clear that $T_{\psi,\varphi}$ can be bounded on $H^2(\mathbb{U})$ only if both ψ and φ are analytic. Further, from the Closed Graph Principle, $T_{\psi,\varphi}$ is bounded if and only if $T_{\psi,\varphi}$ maps $H^2(\mathbb{U})$ into itself. In particular, if $\psi \in H^\infty(\mathbb{U})$, then $T_{\psi,\varphi}$ is bounded.

The space $H^2(\mathbb{U})$ is a reproducing kernel Hilbert space. This means that, for each $w \in \mathbb{U}$ the function $K_w(z) = 1/(1 - \bar{w}z)$, $z \in \mathbb{U}$ has *the reproducing property*:

$$\langle f, K_w \rangle = f(w) \quad w \in \mathbb{U}, f \in H^2(\mathbb{U}).$$

Thus the kernel-function K_w has norm $1/(\sqrt{1 - |w|^2})$. Since, weak convergence in $H^2(\mathbb{U})$ is equivalent to norm-boundedness plus uniform convergence on compacts, an important observation for the study of compact composition operators is the fact that the normalized kernels $k_w = \sqrt{1 - |w|^2}K_w$ tend weakly to 0 if $|w| \rightarrow 1^-$.

The key ingredient necessary in order to obtain information on compact composition operators, based on the observation above is the fact, initially proved by Caughran and Schwartz [4], that adjoints of composition operators leave invariant the set of kernel-functions. More formally, the following is known as the Caughran-Schwartz equation

$$C_\varphi^* K_w = K_{\varphi(w)} \quad w \in \mathbb{U}.$$

The Caughran-Schwartz equation can be easily adapted to weighted composition operators. Here are the details.

Theorem 5. *If $T_{\psi,\varphi}$ is a bounded weighted composition operator on $H^2(\mathbb{U})$, then*

$$T_{\psi,\varphi}^* K_w = \overline{\psi(w)} K_{\varphi(w)} \quad w \in \mathbb{U}. \quad (13)$$

Proof. Using the reproducing property, one can write

$$\begin{aligned} \langle T_{\psi,\varphi}^* K_w, f \rangle &= \langle K_w, \psi(f \circ \varphi) \rangle = \overline{\psi(w)} \overline{f(\varphi(w))} = \\ &= \overline{\psi(w)} \langle K_{\varphi(w)}, f \rangle \quad f \in H^2(\mathbb{U}). \end{aligned}$$

□

The theorem above, with the same proof, is presented in [23], for a particular class of weight-functions ψ and is probably well-known as “folklore”. In the section

4.1, it will turn out that the following immediate consequence is important for understanding the boundedness of composition operators acting on Hardy–Smirnov spaces.

Corollary 2. *If $T_{\psi,\varphi}$ is a bounded weighted composition operator on $H^2(\mathbb{U})$, then necessarily*

$$B := \sup \left\{ \frac{|\psi(w)|^2(1-|w|^2)}{1-|\varphi(w)|^2} : w \in \mathbb{U} \right\} < +\infty. \quad (14)$$

Proof. Relation (14) is the direct consequence of equation (13) and the fact that, for each $w \in \mathbb{U}$, one has $\|T_{\psi,\varphi}^* k_w\|_2^2 \leq \|T_{\psi,\varphi}\|^2$. \square

The authors of [7] prove that, if φ is a finite Blaschke product, then $T_{\psi,\varphi}$ is bounded if and only if $\psi \in H^\infty(\mathbb{U})$. We prove that this happens if and only if condition (14) holds, and in the process give a new, alternative proof to the result in [7] referred above.

Recall that finite Blaschke products are finite products of disk automorphisms, that is of maps of the form

$$\lambda(z) = e^{i\theta_0} \frac{a-z}{1-\bar{a}z} \quad z \in \mathbb{U}, \quad (15)$$

where a is a fixed element of \mathbb{U} and $e^{i\theta_0}$ a unimodular constant.

Lemma 1. *If $\varphi = \varphi_1\varphi_2 \dots \varphi_N$ is a finite Blaschke product, with automorphic factors φ_j , $j = 1, \dots, N$, then*

$$\frac{1-|\varphi(z)|^2}{1-|z|^2} \leq \sum_{j=1}^N \frac{1+|\varphi_j(0)|}{1-|\varphi_j(0)|} \quad z \in \mathbb{U}. \quad (16)$$

Proof. We begin by proving (16) in the particular case when φ has a single factor, that is φ has the form (15). In such a case

$$1-|\varphi(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2} \leq \frac{(1-|z|^2)(1+|a|)}{1-|a|},$$

that is

$$\frac{1-|\varphi(z)|^2}{1-|z|^2} \leq \frac{1+|\varphi(0)|}{1-|\varphi(0)|} \quad z \in \mathbb{U}. \quad (17)$$

In the general case when $\varphi = \varphi_1\varphi_2 \dots \varphi_N$ has N factors, note that the representation

$$1-|\varphi(z)|^2 = 1-|\varphi_1(z)|^2 + |\varphi_1(z)|^2(1-|\varphi_2(z)|^2) + |\varphi_1(z)\varphi_2(z)|^2(1-|\varphi_3(z)|^2) + \dots + |\varphi_1(z)\varphi_2(z) \dots \varphi_{N-1}(z)|^2(1-|\varphi_N(z)|^2)$$

combined with relation (17) establishes (16). \square

We are able now to prove the result announced.

Theorem 6. *Let φ be a finite Blaschke product, then the following statements are equivalent:*

$$T_{\psi, \varphi} \text{ is bounded.} \quad (18)$$

$$\text{Condition (14) holds.} \quad (19)$$

$$\psi \in H^\infty(\mathbb{U}). \quad (20)$$

Proof. For any analytic selfmap φ of \mathbb{U} , the quantity $\frac{1-|\varphi(z)|^2}{1-|z|^2}$ is bounded away from 0, [8, Corollary 2.40]. In the particular case when φ is a finite Blaschke product, the same quantity is bounded above, according to Lemma 1. The consequence is that, conditions (14) and (20) are equivalent if φ is a finite Blaschke product. Now, by Corollary 2, condition (14) is necessary for the boundedness of $T_{\psi, \varphi}$ whereas condition (20) is sufficient for it. This ends the proof. \square

Actually, the paper [7] contains an extra fact besides what we mentioned above, namely: finite Blaschke products are the only symbols φ for which ψ needs to be bounded in order to have $T_{\psi, \varphi}$ bounded.

We continue by a norm-computation.

Let

$$P(z, u) = \Re \frac{u+z}{u-z} \quad u \in \partial\mathbb{U}, z \in \mathbb{U}$$

be the usual Poisson kernel. Recall that, for each measurable, essentially bounded function ϕ on $\partial\mathbb{U}$, the Toeplitz operator of symbol ϕ is the operator

$$T_\phi f = P(\phi f) \quad f \in H^2,$$

where P is the orthogonal projection of $L^2_{\partial\mathbb{U}}$ onto $H^2(\mathbb{U})$. The authors of [2] proved that $C_\varphi^* C_\varphi$ is the Toeplitz operator having symbol $P(\varphi(0), u)$, $u \in \partial\mathbb{U}$, if φ is an inner function, that is an analytic selfmap of \mathbb{U} with the property $|\varphi(u)| = 1$ a.e. on $\partial\mathbb{U}$. We are able to extend that result in the following theorem. More notation needs to be introduced first. For any Borel measure μ on $\partial\mathbb{U}$ and any inner φ , $\mu\varphi^{-1}$ denotes the pull-back of μ under φ , that is the measure described by the equality $\mu\varphi^{-1}(E) = \mu(\varphi^{-1}(E))$, for each measurable $E \subseteq \partial\mathbb{U}$. If $f \in L^1_{\partial\mathbb{U}}$, $f \geq 0$ a.e., the notation μ_f denotes the measure described by the equality $\mu_f(E) = \int_E f d\mu$.

Theorem 7. *If φ is inner and $\psi \in H^2(\mathbb{U})$ then $T_{\psi, \varphi}$ is bounded, if and only if the Nykodim derivative $dm_{|\psi|^2}\varphi^{-1}/dm$ is essentially bounded. In that case*

$$\|T_{\psi, \varphi}\| = \left\| \sqrt{dm_{|\psi|^2}\varphi^{-1}/dm} \right\|_\infty. \quad (21)$$

If $T_{\psi, \varphi}$ is bounded then the operator $T_{\psi, \varphi}^ T_{\psi, \varphi}$ is a Toeplitz operator if and only if φ is an inner function. If φ is a conformal disk automorphism and $\psi \in H^\infty(\mathbb{U})$ then equation (21) has the form*

$$\|T_{\psi, \varphi}\| = \|\psi \circ \varphi^{-1}(u) \sqrt{P(\varphi(0), u)}\|_\infty. \quad (22)$$

Proof. Assume first that $T_{\psi,\varphi}$ is bounded. To show that $T_{\psi,\varphi}^* T_{\psi,\varphi}$ is Toeplitz when φ is inner, we check the equivalent condition $M_z^* T_{\psi,\varphi}^* T_{\psi,\varphi} M_z = T_{\psi,\varphi}^* T_{\psi,\varphi}$. The condition holds, since

$$\begin{aligned} \langle M_z^* T_{\psi,\varphi}^* T_{\psi,\varphi} M_z f, g \rangle &= \langle \psi f \circ \varphi, \psi g \circ \varphi \rangle = \langle \psi f \circ \varphi, \psi g \circ \varphi \rangle = \\ &= \langle T_{\psi,\varphi}^* T_{\psi,\varphi} f, g \rangle \quad f, g \in H^2(\mathbb{U}). \end{aligned}$$

To show that $T_{\psi,\varphi}^* T_{\psi,\varphi}$ is Toeplitz only if φ is inner, assume that $T_{\psi,\varphi}$ is bounded and $T_{\psi,\varphi}^* T_{\psi,\varphi}$ is a Toeplitz operator. Therefore

$$\langle M_z^* T_{\psi,\varphi}^* T_{\psi,\varphi} M_z(1), 1 \rangle = \langle T_{\psi,\varphi}^* T_{\psi,\varphi}(1), 1 \rangle,$$

that is $\|\psi\varphi\|_2 = \|\psi\|_2$. Hence

$$\int_{\partial\mathbb{U}} (|\psi(u)|^2 - |\varphi(u)\psi(u)|^2) dm(u) = 0$$

which implies $|\varphi(u)| = 1$ a.e. because $|\psi(u)|^2 - |\varphi(u)\psi(u)|^2 \geq 0$ a.e., (since φ is a selfmap of \mathbb{U}) and $\psi(u) \neq 0$ a.e.

If φ is a disk automorphism, then ψ needs to be bounded in order that $T_{\psi,\varphi}$ be bounded. In this particular case, the symbol of the Toeplitz operator $T_{\psi,\varphi}^* T_{\psi,\varphi}$ is easy to determine. By a formula established by Nordgren in [18],

$$\int_{\partial\mathbb{U}} f \circ \varphi(u) dm(u) = \int_{\partial\mathbb{U}} f(u) P(\varphi(0), u) dm(u) \quad f \in L^1_{\partial\mathbb{U}}$$

if φ is in an inner function. Thus, it is legitimate to write

$$\langle T_{\psi,\varphi}^* T_{\psi,\varphi} f, g \rangle = \langle T_{\psi,\varphi} f, T_{\psi,\varphi} g \rangle = \quad (23)$$

$$\begin{aligned} \int_{\partial\mathbb{U}} \psi(u) f \circ \varphi(u) \overline{\psi(u) g \circ \varphi(u)} dm(u) &= \int_{\partial\mathbb{U}} (|\psi \circ \varphi^{-1}|^2 f \bar{g}) \circ \varphi(u) dm(u) \\ &= \int_{\partial\mathbb{U}} |\psi \circ \varphi^{-1}(u)|^2 P(\varphi(0), u) f(u) \bar{g}(u) dm(u). \end{aligned}$$

Therefore $T_{\psi,\varphi}^* T_{\psi,\varphi}$ is the Toeplitz operator of symbol $|\psi \circ \varphi^{-1}(u)|^2 P(\varphi(0), u)$. The fact that $\|T_{\psi,\varphi}^* T_{\psi,\varphi}\| = \|T_{\psi,\varphi}\|^2$ combined with the formula for the norm of a Toeplitz operator, [9, Corollary 7.8], prove (22).

To finish the proof, note that by (23), we can only write

$$\begin{aligned} \langle T_{\psi,\varphi}^* T_{\psi,\varphi} f, g \rangle &= \int_{\partial\mathbb{U}} (f \bar{g}) \circ \varphi(u) |\psi(u)|^2 dm(u) = \int_{\partial\mathbb{U}} f(u) \overline{g(u)} dm_{|\psi|^2} \varphi^{-1}(u) \\ &= \int_{\partial\mathbb{U}} (dm_{|\psi|^2} \varphi^{-1} / dm)(u) f(u) \overline{g(u)} dm(u) \end{aligned}$$

if φ is an arbitrary inner function, $\psi \in H^2(\mathbb{U})$, and $T_{\psi,\varphi}$ is bounded. Above, the relation $dm_{|\psi|^2} \varphi^{-1}(u) \ll dm(u)$ was used. That relation is an immediate consequence of Nordgren's formula, mentioned earlier in this proof. Since $T_{\psi,\varphi}^* T_{\psi,\varphi}$ is a Toeplitz operator, its symbol must then be $dm_{|\psi|^2} \varphi^{-1} / dm$ and (21) follows then exactly like (22). Conversely, if $dm_{|\psi|^2} \varphi^{-1} / dm$ is essentially bounded, then

the Toeplitz operator of symbol $\sqrt{dm_{|\psi|^2}\varphi^{-1}/dm}$ is bounded and, by writing the last string of inequalities in reverse order for $f = g \in H^2(\mathbb{U})$, one has that

$$\|T_{\sqrt{dm_{|\psi|^2}\varphi^{-1}/dm}}f\|_2^2 = \|T_{\psi,\varphi}f\|_2^2 \quad f \in H^2(\mathbb{U}),$$

which proves that $T_{\psi,\varphi}$ is bounded if $dm_{|\psi|^2}\varphi^{-1}/dm$ is essentially bounded. \square

Finding the norm of a weighted composition operator is not an easy problem. To see that, note that, if ψ is an inner function, then M_ψ is an isometry and hence $\|T_{\psi,\varphi}\| = \|C_\varphi\|$. The norm of a composition operator is known in very few cases.

3.2. Compactness

It is well-known that the Caughran–Schwartz equation combines with the weak convergence of the normalized kernels referred above to show that compact composition operators C_φ have the property that the angular derivative of φ exists nowhere on $\partial\mathbb{U}$. The version of that result adapted to weighted composition operators is presented in the theorem below, which is partially contained in [25, Theorem 3].

Theorem 8. *If $T_{\psi,\varphi}$ is compact on $H^2(\mathbb{U})$, then, necessarily*

$$\lim_{|w| \rightarrow 1^-} \frac{|\psi(w)|^2(1 - |w|^2)}{1 - |\varphi(w)|^2} = 0. \quad (24)$$

Consequently, if the radial limit $\psi(u)$ exists and is nonzero at some $u \in \partial\mathbb{U}$, then the angular derivative of φ at u does not exist. Hence, the angular derivative of φ does not exist a.e. on $\partial\mathbb{U}$. If, in addition, ψ is bounded away from 0 in the proximity of the boundary, (that is there are constants $0 < \delta < 1, M > 0$ so that $M < |\psi(z)|$ if $1 - |z| < \delta$), then the angular derivative of φ exists nowhere and φ must have a fixed point.

Proof. By equation (13), the fact that $\|T_{\psi,\varphi}^*k_w\|^2 \rightarrow 0$ if $|w| \rightarrow 1^-$ is equivalent to (24), so, if $T_{\psi,\varphi}$ is compact, then (24) must hold.

If $\psi(u) \neq 0$, then

$$\lim_{r \rightarrow 1^-} \frac{1 - |\varphi(ru)|}{(1 - r)} = +\infty.$$

Hence

$$\lim_{r \rightarrow 1^-} \frac{|\eta - \varphi(ru)|}{|u - ru|} = +\infty, \quad \eta \in \partial\mathbb{U}$$

and therefore φ cannot have an angular derivative at u . Since ψ is an $H^2(\mathbb{U})$ -function, other than the null function, the angular derivative of φ cannot exist a.e. on $\partial\mathbb{U}$. Clearly that angular derivative cannot exist anywhere if φ is bounded away from 0 in the proximity of the boundary, and so, by the Denjoy–Wolff Theorem, (Theorem 16 in this paper), φ must have a fixed point. \square

The existence of a fixed point if ψ is bounded away from 0 on $\partial\mathbb{U}$ was proved by similar methods for a wider category of spaces than $H^2(\mathbb{U})$ in [14].

Recall that, on any Hilbert space \mathcal{H} , the Hilbert–Schmidt norm $\|A\|_{\text{HS}}$ of an operator A is defined as follows,

$$\|A\|_{\text{HS}} = \sqrt{\sum_{n=0}^{\infty} \|Ae_n\|^2}, \quad (25)$$

where $\{e_n\}$ is an orthonormal basis of \mathcal{H} . The quantity in (25) does not depend on the orthonormal basis chosen [10], hence the Hilbert–Schmidt norm is larger than the operator norm. The characterization of Hilbert–Schmidt composition operators is easy to prove [24]. Its adaptation to weighted composition operators is equally easy.

Theorem 9. *The operator $T_{\psi,\varphi}$ is Hilbert–Schmidt on $H^2(\mathbb{U})$ if and only if*

$$\|T_{\psi,\varphi}\|_{\text{HS}}^2 = \int_{\partial\mathbb{U}} \frac{|\psi(\zeta)|^2}{1 - |\varphi(\zeta)|^2} dm(\zeta) < \infty. \quad (26)$$

Proof. To calculate the Hilbert–Schmidt norm of $T_{\psi,\varphi}$ we use the standard orthonormal basis $\{1, z, z^2, z^3, \dots\}$ of $H^2(\mathbb{U})$ and write

$$\begin{aligned} \|T_{\psi,\varphi}\|_{\text{HS}}^2 &= \sum_{n=0}^{\infty} \|T_{\psi,\varphi}(z^n)\|_2^2 = \sum_{n=0}^{\infty} \int_{\partial\mathbb{U}} |\psi(\zeta)|^2 |\varphi(\zeta)|^{2n} dm(\zeta) = \\ &= \int_{\partial\mathbb{U}} \frac{|\psi(\zeta)|^2}{1 - |\varphi(\zeta)|^2} dm(\zeta). \end{aligned}$$

□

Corollary 3. *If $\psi \in H^2(\mathbb{U})$ and $\|\varphi\|_{\infty} < 1$, then*

$$\|T_{\psi,\varphi}\| \leq \frac{\|\psi\|_2}{\sqrt{1 - \|\varphi\|_{\infty}^2}}$$

and equality holds if and only if φ is constant.

Proof. Since $|\varphi(\zeta)| \leq \|\varphi\|_{\infty}$ a.e., the integral in (26) is smaller than $\|\psi\|_2^2 / (1 - \|\varphi\|_{\infty}^2)$, which proves the upper estimate above for the norm of $\|T_{\psi,\varphi}\|$. If the inequality above is an equality then $0 = \|T_{\psi,\varphi}\|_e < \|T_{\psi,\varphi}\| = \|T_{\psi,\varphi}\|_{\text{HS}}$, where $\| \cdot \|_e$ denotes the essential norm. In that case, $T_{\psi,\varphi}$ is a norm–attaining operator [15, Proposition 2.2], (that is $\|T_{\psi,\varphi}\| = \|T_{\psi,\varphi}f\|$ for some norm–one f). By formula (25), the equality $\|T_{\psi,\varphi}\| = \|T_{\psi,\varphi}\|_{\text{HS}}$ implies that $T_{\psi,\varphi}$ is a rank one operator, which can happen if and only if φ is constant.

Conversely, if $\varphi \equiv p$, for some $p \in \mathbb{U}$, then, using the reproducing property, one can write

$$\begin{aligned} \|T_{\psi,\varphi}\| &= \sup\{\|T_{\psi,\varphi}f\|_2 : \|f\|_2 = 1\} = \\ &= \sup\{|\langle \psi, f \rangle| : \|f\|_2 = 1\} = \frac{\|\psi\|_2}{\sqrt{1 - |p|^2}}. \end{aligned}$$

□

Condition (26) cannot hold unless $|\varphi| < 1$ a.e. This is known to be necessary for the compactness of a composition operator. The same is true for weighted composition operators. The proof appears in [23]. We include it for completeness.

Theorem 10. *If the operator $T_{\psi,\varphi}$ is compact, then $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$.*

Proof. Arguing by contradiction, assume that there is a measurable set $E \subseteq \partial\mathbb{U}$ having positive measure, on which φ is unimodular. The sequence $\{z^n\}$ of $H^2(\mathbb{U})$ tends weakly to 0, but $\|T_{\psi,\varphi}(z^n)\|_2$ does not tend to 0, since

$$\|T_{\psi,\varphi}(z^n)\|_2^2 = \int_{\partial\mathbb{U}} |\psi(\zeta)|^2 |\varphi(\zeta)|^{2n} dm(\zeta) \geq \int_E |\psi(\zeta)|^2 dm(\zeta)$$

and ψ is an $H^2(\mathbb{U})$ -function, other than the null function. \square

A standard source of Hilbert–Schmidt composition operators is the, so called, *Polygonal Range Theorem*, [22], saying that composition operators on $H^2(\mathbb{U})$ whose symbol has range contained in a polygon are Hilbert–Schmidt. This theorem can be extended to weighted composition operators as follows.

Theorem 11. *If $\varphi(\mathbb{U})$ is contained in the interior of a convex polygon inscribed in $\partial\mathbb{U}$ having smallest angle of measure π/s , $s > 1$, and $\psi \in H^{2p}(\mathbb{U})$, for some $p > 1$ such that $q = (1 - 1/p)^{-1} < s$, then $T_{\psi,\varphi}$ is a Hilbert–Schmidt operator.*

Proof. Assume initially that $\varphi(\mathbb{U}) \subseteq \Gamma_M(1)$, for some $M > 1$. In that case

$$1 - |\varphi(z)| \leq |1 - \varphi(z)| \leq M(1 - |\varphi(z)|) \quad z \in \mathbb{U},$$

and hence, condition (26) is equivalent to

$$\int_{\partial\mathbb{U}} \frac{|\psi(\zeta)|^2}{|1 - \varphi(\zeta)|} dm(\zeta) < +\infty.$$

One can choose constants $s > 1$ and $C > 0$ so that $S(z) = 1 - C(1 - z)^{1/s}$, $z \in \mathbb{U}$ have the property $\Gamma_M(1) \subseteq S(\mathbb{U}) \subseteq \mathbb{U}$. Note that the aperture angle of $\Gamma_M(1)$ should be at most π/s . Thus $S^{-1} \circ \varphi = \sigma$, is an analytic selfmap of \mathbb{U} and so C_ψ is Hilbert–Schmidt because

$$\begin{aligned} \int_{\partial\mathbb{U}} \frac{|\psi(\zeta)|^2}{|1 - \varphi(\zeta)|} dm(\zeta) &= \int_{\partial\mathbb{U}} \frac{|\psi(\zeta)|^2}{|1 - S \circ \sigma(\zeta)|} dm(\zeta) \leq \\ \|\psi^2\|_p \left\| C_\sigma \left(\frac{1}{1 - S} \right) \right\|_q &\leq \|\psi\|_{2p}^2 \left(\frac{1 + |\sigma(0)|}{1 - |\sigma(0)|} \right)^{1/q} \left\| \frac{1}{1 - S} \right\|_q < +\infty, \end{aligned}$$

by Hölder's inequality. Above we used the fact that q , the Hölder conjugate of p , is less than s , to draw the conclusion $\left\| \frac{1}{1 - S} \right\|_q < +\infty$. The well known estimate

$\|C_\sigma\| \leq \left(\frac{1 + |\sigma(0)|}{1 - |\sigma(0)|} \right)^{1/q}$ for the norm of C_σ as an operator on $H^q(\mathbb{U})$ was also used.

In the general case when the range of φ is inside a polygon, the argument above can be repeated in the neighborhood of each vertex provided that one uses

for p a value large enough that $(1 - 1/p)^{-1} < s$, where π/s is the measure of the smallest angle of the polygon. \square

Example 2.3 in [23] contains a principle that deserves generalization: $T_{\psi,\varphi}$ is compact if ψ is “essentially small” where φ is “essentially large”. Here are the details.

Theorem 12. *If $\psi \in H^2(\mathbb{U})$ and*

$$\lim_{\delta \rightarrow 0^+} (\text{ess sup } \{|\psi(\zeta)|^2 : \zeta \in \mathbb{T}, |\varphi(\zeta)| \geq 1 - \delta\}) = 0, \quad (27)$$

then $T_{\psi,\varphi}$ is compact.

Proof. Consider a sequence $\{f_n\}$ in $H^2(\mathbb{U})$, tending weakly to 0, with the intention of showing $\|T_{\psi,\varphi}f_n\|_2 \rightarrow 0$. Recall that weakly convergent sequences are bounded so, one can assume without loss of generality that $\{f_n\}$ is in the unit ball of $H^2(\mathbb{U})$. For arbitrary $\epsilon > 0$, choose $0 < \delta < 1$ small enough that the essential supremum of the set

$$E_\delta := \{|\psi(\zeta)|^2 : \zeta \in \mathbb{T}, |\varphi(\zeta)| \geq 1 - \delta\}$$

be less than $\epsilon/2\|C_\varphi\|^2$. One has then

$$\int_{E_\delta} |\psi(\zeta)|^2 |f_n \circ \varphi(\zeta)|^2 dm(\zeta) \leq \frac{\epsilon}{2\|C_\varphi\|^2} \|f_n \circ \varphi\|_2^2 < \frac{\epsilon}{2} \quad n \geq 1 \quad (28)$$

Also

$$\int_{E_\delta^c} |\psi(\zeta)|^2 |f_n \circ \varphi(\zeta)|^2 dm(\zeta) \leq \sup\{|f_n(z)|^2 : z \in (1 - \delta)\mathbb{U}\} \|\psi\|_2^2.$$

Since weak convergence in $H^2(\mathbb{U})$ implies uniform convergence on compact subsets, one can find an integer N , large enough that

$$\int_{E_\delta^c} |\psi(\zeta)|^2 |f_n \circ \varphi(\zeta)|^2 dm(\zeta) < \frac{\epsilon}{2} \quad n \geq N. \quad (29)$$

Putting (28) and (29) together, one gets that $\|T_{\psi,\varphi}f_n\|_2 \rightarrow 0$. \square

A common situation when the assumptions in Theorem 12 are satisfied is the following.

Theorem 13. *Let φ be an analytic selfmap of \mathbb{U} continuously extendable on the boundary, with the property $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$. Let ψ be an $H^2(\mathbb{U})$ -function such that there is a union S of open arcs of $\partial\mathbb{U}$ containing the set $E = \{\zeta \in \mathbb{U} : |\varphi(\zeta)| = 1\}$ with the properties that the nontangential limit-function of ψ exists and is continuous on \overline{S} , and $\psi(\zeta) = 0$, for each $\zeta \in E$. Under these assumptions, $T_{\psi,\varphi}$ is compact.*

Proof. The quantity $M(\delta) = \text{ess sup } \{|\psi(\zeta)|^2 : \zeta \in \mathbb{T}, |\varphi(\zeta)| \geq 1 - \delta\}$ decreases as $\delta \rightarrow 0$. Therefore, Theorem 13 will follow as a consequence of Theorem 12 if we are able to prove that for each $\epsilon > 0$ there is some $0 < \delta_\epsilon < 1$ such that $M(\delta_\epsilon) < \epsilon$.

We do so in the following. Since ψ is uniformly continuous on S , one can consider some $0 < \delta < 1$ so that $|\psi(u) - \psi(v)| < \sqrt{\epsilon/2}$ if $u, v \in S$ and $|u - v| < \delta$. On the other hand, $m(E) = 0$, so one can consider a union B of open arcs such that $E \subseteq B \subseteq S$, each connected component of B meets E and has measure less than δ . For that reason, for each $u \in B$, one can find $v \in E$ within δ from u and hence $|\psi(u)|^2 \leq \epsilon/2$, for all $u \in B$. If there is δ_ϵ such that $\{|\psi(\zeta)|^2 : \zeta \in \mathbb{T}, |\varphi(\zeta)| \geq 1 - \delta_\epsilon\} \subseteq B$, the proof is over. Arguing by contradiction, let us assume the opposite. In that case one can find a sequence $\{\zeta_n\}$ in $\partial\mathbb{U} \setminus B$ such that $|\varphi(\zeta_n)| \rightarrow 1$. One does not reduce generality by assuming that $\zeta_n \rightarrow \zeta$ for some $\zeta \in \partial\mathbb{U} \setminus B$ because $\partial\mathbb{U} \setminus B$ is compact. By the continuity of φ one gets $|\varphi(\zeta)| = 1$, a contradiction, since $E \subseteq B$. \square

Example 2. Let $\psi(z) = 1 - z$ and $\varphi(z) = (1 + z)/2$. The operator $T_{\psi, \varphi}$ is compact.

The interesting facts about Example 2 are the following. It shows $T_{\psi, \varphi}$ may be compact even if C_φ is not. Indeed, the angular derivative of φ at 1 exists. It proves that, in Theorem 8 the term ‘‘a.e.’’ cannot be dropped. The composition operators whose symbol has range making α -contact with $\partial\mathbb{U}$ at some point, for some $\alpha > 1$ are known to be noncompact, [22]. This criterion does not extend to weighted composition operators, since, in the example above, $\varphi(\mathbb{U})$ is a horodisk. Finally, although $T_{\psi, \varphi}$ is compact, φ does not fix any point in \mathbb{U} , so the hypothesis of ψ being bounded away from zero in the proximity of the unit circle cannot be dropped in Theorem 8.

4. Composition Operators on Hardy–Smirnov Spaces

4.1. Bounded Composition Operators

We investigate now which conditions are necessary for the boundedness of composition operators C_ϕ on the Hardy–Smirnov spaces $H^p(G)$ over some simply connected domain G obtained as the image $G = \gamma(\mathbb{U})$ of some Riemann map γ . We begin by recalling some facts proved in [23].

Theorem 14 ([23, Section 2.1 and Theorem 2.7]). *For any proper simply connected domain G , it is true that an analytic selfmap ϕ of G simultaneously induces bounded composition operators on all spaces $H^p(G)$, $0 < p < \infty$. This happens if and only if the following weighted composition operator is bounded*

$$Tf = \sqrt{\frac{\gamma'}{\gamma' \circ \phi}} C_\phi f \quad f \in H^2(\mathbb{U}),$$

since there is an onto isometry V so that $T = V^{-1}C_\phi V$.

Above, φ is the γ -conformal conjugate of ϕ , i.e. $\phi := \gamma \circ \varphi \circ \gamma^{-1}$.

An immediate consequence is the following.

Remark 3. *If C_ϕ is bounded on $H^2(G)$ then necessarily*

$$\sqrt{\frac{\gamma'}{\gamma' \circ \varphi}} \in H^2(\mathbb{U}). \tag{30}$$

$$B := \sup \left\{ \frac{(1 - |z|^2)|\gamma'(z)|}{(1 - |\varphi(z)|^2)|\gamma' \circ \varphi(z)|} : z \in \mathbb{U} \right\} < +\infty. \tag{31}$$

Also, C_ϕ is bounded on $H^2(G)$ if and only if C_φ leaves invariant the space $\sqrt{\frac{1}{\gamma'}} H^2(\mathbb{U})$. If φ is a finite Blaschke product then C_ϕ is bounded if and only if $\frac{\gamma'}{\gamma' \circ \varphi} \in H^\infty(\mathbb{U})$.

These are direct consequences of Corollary 2, Theorem 6, and Theorem 14, obtained by applying to T the results on weighted composition operators contained in the previous section. □

To illustrate the utility of the remark above, let us determine the bounded automorphic composition operators, (i.e. bounded composition operators whose symbol is a univalent analytic map of G onto itself), for certain domains G . Consider for instance $\gamma_1(z) = z - .5z^2$, $\gamma_2(z) = \log((1 + z)/(1 - z))$, and the Koebe map $\gamma_3(z) = z(1 - z)^{-2}$. Each of these maps is a Riemann map transforming \mathbb{U} into the interior of the cardioid $G_1 = \gamma_1(\mathbb{U})$ having cusp at $1/2$, the horizontal strip $G_2 = \gamma_2(\mathbb{U}) = \{z \in \mathbb{C} : -\pi/2 < \Im z < \pi/2\}$, respectively the slit plane $G_3 = \gamma_3(\mathbb{U}) = \mathbb{C} \setminus \{z \in \mathbb{C} : -\infty < \Re z < -1/4, \Im z = 0\}$, [12]. One can characterize the bounded, automorphic composition operators as follows.

Example 3. *A conformal automorphism ϕ induces a bounded composition operator on the spaces $H^p(G_i)$, $i = 1, 2, 3$ if and only if*

- (i) $\phi(1/2) = 1/2$ for $i = 1$,
- (ii) $\phi(\infty) = \infty$ for $i = 2$,

respectively

- (iii) $\phi(-1/4) = -1/4$ and $\phi(\infty) = \infty$ for $i = 3$.

Proof. To prove (i), note that $\gamma_1'(z) = 1 - z$, so, if φ is the γ_1 -conformal conjugate of ϕ , then by Remark 3, $(1 - z)/(1 - \varphi(z))$ must be bounded, which happens if and only if $\varphi(1) = 1$. Since γ_1 transforms 1 into $1/2$, the inner cusp of the cardioid ∂G_1 , the proof of (i) is over. The statements (ii) and (iii) have similar proofs, which are left to the reader. □

As another application of Remark 3, let us consider the case of the upper half-plane $\Pi^+ = \{z = x + iy \in \mathbb{C} : y > 0\}$, obtained by transforming \mathbb{U} under the

action of $\gamma(z) = i(1+z)/(1-z)$. Note that $\gamma'(z) = 2i/(1-z)^2$. In order to completely characterize the conformal conjugates of symbols of bounded composition operators on Hardy–Smirnov spaces over half–planes we will consider Riemann maps with property $\gamma'(z) = g(z)/(\omega - z)^\alpha$, where $\omega \in \partial\mathbb{U}$, $\alpha > 1$, and g is analytic.

Proposition 1. *Let γ be a Riemann map that transforms \mathbb{U} onto G . If ϕ is an analytic selfmap of G which induces a bounded composition operator on the spaces $H^p(G)$, and the γ -conformal conjugate φ of ϕ has the property that there exist boundary points $\omega, \eta \in \partial\mathbb{U}$ and the constants $\alpha > 1$, $c, \delta > 0$ such that*

$$|\eta - \varphi(z)|^\alpha |\gamma' \circ \varphi(z)| \leq c |\omega - z|^\alpha |\gamma'(z)| \quad |z - \omega| < \delta, \quad (32)$$

then φ has an angular derivative $\varphi'(\omega)$ at ω with angular limit η . In particular, if $\gamma'(z) = g(z)/(\omega - z)^\alpha$, where $\omega \in \partial\mathbb{U}$, $\alpha > 1$, and g is an analytic function with the property

$$\limsup_{z \rightarrow \omega} \frac{|g \circ \varphi(z)|}{|g(z)|} < +\infty, \quad (33)$$

then ω must be a boundary fixed point of φ where the angular derivative $\varphi'(\omega)$ exists.

Proof. Combining conditions (31) and (32), one gets

$$\sup \left\{ \frac{|\eta - \varphi(z)|^\alpha (1 - |z|^2)}{|\omega - z|^\alpha (1 - |\varphi(z)|^2)} : z \in \mathbb{U}, |\omega - z| < \delta \right\} \leq Bc < +\infty.$$

By Remark 1, $\beta_\alpha < +\infty$, so the conclusion of Theorem 3 holds. \square

Recall that the local Dirichlet space $D(\delta_\omega)$, associated to the Dirac unit mass δ_ω , for some fixed $\omega \in \partial\mathbb{U}$, is the set of all analytic functions f on \mathbb{U} with the property

$$D_{\delta_\omega}(f) = \int_{\mathbb{U}} |f'|^2(z) \Re \left(\frac{\omega + z}{\omega - z} \right) dA(z) < +\infty,$$

where A is the normalized area measure on \mathbb{U} . It is shown in [20] that an analytic function f on \mathbb{U} belongs to $D(\delta_\omega)$ if and only if there exist a constant c and an $H^2(\mathbb{U})$ -function g such that f can be represented as

$$f(z) = c + (\omega - z)g(z) \quad z \in \mathbb{U}. \quad (34)$$

If representation (34) holds then the horocyclic limit of f at ω exists and equals c . This statement means that $f(z)$ tends to c as z tends to ω inside each horodisk tangent to the unit circle at ω .

Theorem 15. *Let ϕ be an analytic selfmap of a half–plane $G = \gamma(\mathbb{U})$ and $\omega \in \partial\mathbb{U}$ the point transformed by γ into the point at infinity. Let φ be the γ -conformal conjugate of ϕ . The operator C_ϕ is bounded on $H^2(G)$ if and only if ω is a boundary fixed point of φ where the angular derivative $\varphi'(\omega)$ exists.*

Proof. For simplicity we will work on the case $G = \Pi^+$, so $\gamma(z) = i(1+z)/(1-z)$, $\gamma'(z) = 2i/(1-z)^2$, and $\omega = 1$. If C_ϕ is bounded then 1 is a boundary fixed point of φ where the angular derivative exists, by Proposition 1.

For the proof of the sufficiency of those two conditions now, assume $\varphi'(1)$ exists and $\varphi(1) = 1$, then by the estimate $D_{\delta_1}(\varphi) \leq |\varphi'(1)|$, [20, Proposition 3.5], one deduces $D_{\delta_1}(\varphi) < +\infty$. Thus φ is an element of $D(\delta_1)$ having horocyclic limit equal to 1 at 1. Furthermore, φ induces a bounded composition operator on $D(\delta_1)$, [21, Theorem 2].

Because φ has an angular derivative at 1, it maps horodisks tangent at 1 into, (possibly larger or smaller) horodisks tangent at 1, (by the Julia–Carathéodory theorem). Therefore, for each f in $(1-z)H^2(\mathbb{U})$, $C_\varphi f$ is an element of $D(\delta_1)$ having null horocyclic limit at 1, i.e., by (34), C_φ leaves $(1-z)H^2(\mathbb{U})$ invariant. By Remark 3, it follows that C_ϕ is bounded on $H^2(\Pi^+)$. \square

Remark 4. *The γ -conformal conjugates φ of the symbols ϕ that induce bounded composition operators on Hardy–Smirnov spaces over a half-plane are exactly those φ that fix ω and induce bounded composition operators on $D(\delta_\omega)$ where ω is the point transformed by γ into the point at infinity.*

Corollary 4. *If C_ϕ is a bounded automorphic composition operator on $H^2(\Pi^+)$ and φ is the conformal conjugate of ϕ , then*

$$\|C_\phi\| = \frac{\sqrt{1 - |\varphi^{-1}(0)|^2}}{|1 - \varphi^{-1}(0)|} \quad (35)$$

Proof. By Theorem 15, $\varphi = \lambda(a-z)/(1-\bar{a}z)$, $a \in \mathbb{U}$, is the conformal conjugate of selfmap ϕ of Π^+ inducing a bounded composition operator on $H^2(\Pi^+)$ if and only if $\varphi(1) = 1$, that is, if and only if $\lambda = (1-\bar{a})/(a-1)$. Using formula (22), one gets, after a straightforward computation that

$$\begin{aligned} \|C_\phi\| &= \max_{u \in \partial\mathbb{U}} \left| \frac{1-u}{1-\varphi^{-1}(u)} \right| \sqrt{P(\lambda a, u)} = \\ &= \max_{u \in \partial\mathbb{U}} \left| \frac{1-\bar{a}\lambda u}{1-a} \right| \frac{\sqrt{1-|a|^2}}{|u-\lambda a|} = \frac{\sqrt{1-|a|^2}}{|1-a|}. \end{aligned}$$

\square

Recently, the authors of [5] have characterized the isometric composition operators on $H^2(\Pi^+)$ as follows.

Proposition 2 ([5, Proposition 2.1]). *Let ϕ be an analytic selfmap of Π^+ and φ its conformal conjugate. The operator C_ϕ is an isometry of $H^2(\Pi^+)$ if and only if φ is an inner function with the property that $\frac{1-\varphi(z)}{1-z}$ is a norm-one $H^2(\mathbb{U})$ -function.*

This leaves open the question: *Which inner functions φ have the property that $\frac{1-\varphi(z)}{1-z}$ is a norm-one $H^2(\mathbb{U})$ -function?* We complete the description of the isometric composition operators on $H^2(\Pi^+)$ by answering it. First recall a classical geometric-function theory result.

Theorem 16 (The Denjoy–Wolff Theorem). *If φ is an analytic selfmap of \mathbb{U} not the identity or an elliptic disk automorphism, then there is a remarkable point $\omega \in \mathbb{U} \cup \partial\mathbb{U}$ toward which the sequence $\varphi^{[n]} = \varphi \circ \dots \circ \varphi$, (n times), of iterates of φ converges uniformly on compacts. That point is called the Denjoy–Wolff point of φ . If φ has no fixed point in \mathbb{U} , then ω is a boundary fixed point of φ where the angular derivative $\varphi'(\omega)$ exists, namely, the only boundary fixed point where the condition $\varphi'(\omega) \leq 1$ is satisfied.*

We refer to [8], [22], or [3] for this theorem. Analytic selfmaps φ with the property that the Denjoy–Wolff point ω is on $\partial\mathbb{U}$ and the equality $\varphi'(\omega) = 1$ holds are called selfmaps of *parabolic* type. We refer the reader to [22] for more explanations on the terminology, which is motivated by linear fractional models for analytic selfmaps of \mathbb{U} , see also [3, section 2.5].

Since isometries are particular kinds of contractions, we begin by examining contractive composition operators on Hardy–Smirnov spaces of a half-plane giving a simple generalization to [5, Lemma 5.2].

Proposition 3. *If $G = \gamma(\mathbb{U})$ is a half-plane, ω is the point transformed by γ into the point at infinity, and ϕ is an analytic selfmap of G inducing a bounded composition operator on $H^2(G)$ then $\sqrt{|\varphi'(\omega)|} \leq \|C_\phi\|$, where φ is the γ -conformal conjugate of ϕ . Hence, if C_ϕ is a contraction, then ω needs to be the Denjoy–Wolff point of φ .*

Proof. For simplicity we will write the proof in the case $G = \Pi^+$. Note, that in the proof of Corollary 2, we established the inequality

$$\sup \left\{ \frac{|\psi(z)|^2(1-|z|^2)}{1-|\varphi(z)|^2} : z \in \mathbb{U} \right\} \leq \|T_{\psi,\varphi}\|^2,$$

valid for any weighted composition operator on $H^2(\mathbb{U})$. Applying this inequality to the weighted composition operator $T_{\sqrt{|\varphi'|/|\varphi' \circ \varphi|}, \varphi}$, one gets

$$|\varphi'(1)| = \sup \left\{ \frac{|1-\varphi(z)|^2(1-|z|^2)}{(1-|\varphi(z)|^2)|1-z|^2} : z \in \mathbb{U} \right\} \leq \|C_\phi\|^2.$$

□

Note that there are bounded composition operators C_ϕ on $H^2(\Pi^+)$ with ϕ conjugated to φ so that C_ϕ has norm larger than 1 although 1 is the Denjoy–Wolff of φ . To give an example, consider the family of hyperbolic disk automorphisms $\varphi(z) = (r+z)/(1+rz)$, $0 < r < 1$, with attractive fixed point 1, and note that the norm of a composition operator on $H^2(\Pi^+)$ induced by the conformal conjugate of an automorphism like that can be greater than 1, by formula (35).

To finish the characterization of isometric composition operators on Hardy–Smirnov spaces over a half–plane, we prove the following.

Proposition 4. *Let ϕ be an analytic selfmap of a half–plane $G = \gamma(\mathbb{U})$ and φ its conformal conjugate. The operator C_ϕ is an isometry of $H^2(G)$ if and only if φ is an inner function of parabolic type whose Denjoy–Wolff point ω is the point transformed by γ into the point at infinity.*

Proof. Like before, we present the proof in the case $G = \Pi^+$. According to (34) and the discussion related to it, $D_{\delta_1}(\varphi) < +\infty$ if and only if the horocyclic limit $\varphi(1)$ exists and $\frac{\varphi(1)-\varphi(z)}{1-z} \in H^2(\mathbb{U})$. Therefore $\frac{1-\varphi(z)}{1-z} \in H^2(\mathbb{U})$ if and only if $D_{\delta_1}(\varphi) < +\infty$ and 1 is the horocyclic limit of φ at 1, (in particular 1 must be a boundary fixed point of φ). On the other hand, the authors of [20] prove the formula $D_{\delta_1}(\varphi) = \left\| \frac{\varphi(1)-\varphi(z)}{1-z} \right\|_2$, [20, Proposition 2.2], thus C_ϕ is isometric if and only if φ is inner with horocyclic limit 1 at 1 and $D_{\delta_1}(\varphi) = 1$. But for inner functions, the authors of [20] establish the formula $|\varphi'(1)| = D_{\delta_1}(\varphi)$, [20, Proposition 3.5], which completes the proof. \square

The Denjoy–Wolff Theorem allows one to study the sequence of powers $\{C_\phi^n\}$ of a composition operator.

Proposition 5. *Let ϕ be an analytic selfmap of G with a fixed point $p \in G$, not the identity or the conformal conjugate of an elliptic disk automorphism. If $H^2(G)$ does not support compact composition operators and $\|C_\phi\| < +\infty$, then $\|C_\phi^n\| \rightarrow +\infty$. If ϕ is an analytic selfmap of G with property $d(\phi(G), \partial G) > 0$, then necessarily ϕ has a fixed point p and if, in addition, C_ϕ is bounded, then $H^2(G)$ supports compact composition operators and $\{C_\phi^n\}$ tends weakly to C_p , the composition operator of constant symbol p . Hence, the sequence $\{C_\phi^n\}$ is norm–bounded.*

Proof. Assume $H^2(G)$ does not support compact composition operators and $\|C_\phi\| < +\infty$, but, arguing by contradiction, there is some norm–bounded subsequence $\{C_\phi^{n_k}\}$. Without loss of generality one can assume that $\{C_\phi^{n_k}\}$ is the whole sequence $\{C_\phi^n\}$. This means that, for some $M > 0$ one has that

$$\left\| T \sqrt{\frac{\gamma'}{\gamma' \circ \varphi^{[n]}}} \right\| \leq M \quad \text{hence} \quad \left\| \frac{\gamma'}{\gamma' \circ \varphi^{[n]}} \right\|_1 \leq M \quad n \geq 1.$$

Let $q = \gamma^{-1}(p)$, the fixed point of φ , the γ –conformal conjugate of ϕ . Applying Fatou’s Lemma in integration theory and the Denjoy–Wolff Theorem, one gets

$$\frac{1}{|\gamma'(q)|} \int_{\partial\mathbb{U}} |\gamma'(r\zeta)| dm(\zeta) \leq \liminf_{n \rightarrow +\infty} \left\| \frac{\gamma'}{\gamma' \circ \varphi^{[n]}} \right\|_1 \leq M \quad 0 < r < 1,$$

a contradiction, by Theorem 1.

For the second part of the proposition now, note that $\|C_\phi\| < +\infty$ implies that $\gamma'/\gamma' \circ \varphi \in H^1(\mathbb{U})$. The condition $d(\phi(G), \partial G) > 0$ is equivalent to $\|\varphi\|_\infty < 1$

and so, $\gamma' \circ \varphi$ is bounded away from 0. Thus $\gamma' \in H^1(\mathbb{U})$, hence $H^2(G)$ supports compact composition operators. The condition $\|\varphi\|_\infty < 1$ implies that the Denjoy–Wolff point of φ cannot be on $\partial\mathbb{U}$ that is φ (and hence ϕ too) must have a fixed point. On the other hand, the sequence $\{\gamma' \circ \varphi^{[n]}\}$ is uniformly bounded away from 0, that is $|\gamma' \circ \varphi^{[n]}(z)| \geq 1/M$ for all $z \in \mathbb{U}$ and all $n \geq 0$, where M is a positive constant. Keeping this in mind, note that

$$\|f \circ \varphi^{[n]}\|_\infty \leq \frac{\|f\|_2}{\sqrt{1 - \|\varphi\|_\infty^2}} \quad f \in H^2,$$

which is an immediate consequence of the inequality

$$|f(z)| \leq \frac{\|f\|_2}{\sqrt{1 - |z|^2}} \quad z \in \mathbb{U}, f \in H^2(\mathbb{U})$$

obtained by the Cauchy–Schwarz inequality combined with the reproducing property. Thus, for each $f \in H^2$ one has

$$\begin{aligned} \left\| T \sqrt{\frac{\gamma'}{\gamma' \circ \varphi^{[n]}}} f \right\|_2 &= \left\| \sqrt{\frac{\gamma'}{\gamma' \circ \varphi^{[n]}}} f \circ \varphi^{[n]} \right\|_2 \leq \\ \|f \circ \varphi^{[n]}\|_\infty \left\| \frac{\gamma'}{\gamma' \circ \varphi^{[n]}} \right\|_1 &\leq M \|\gamma'\|_1 \frac{\|f\|_2}{\sqrt{1 - \|\varphi\|_\infty^2}} \quad n \geq 1. \end{aligned}$$

We proved that the sequence $\left\{ T \sqrt{\frac{\gamma'}{\gamma' \circ \varphi^{[n]}}} f \right\}$ is norm–bounded. By the Denjoy–

Wolff Theorem, that sequence tends uniformly on compacts to the limit $\sqrt{\frac{\gamma'}{\gamma'(q)}} f(q)$ where like above, $q = \gamma^{-1}(p)$. In $H^2(\mathbb{U})$, norm boundedness plus uniform convergence on compacts is equivalent to weak convergence. Thus, we proved that $\left\{ T \sqrt{\frac{\gamma'}{\gamma' \circ \varphi^{[n]}}} \right\}$ tends weakly to $T \sqrt{\frac{\gamma'}{\gamma'(q), q}}$. Since those weighted composition operators are unitarily conjugated to C_ϕ and C_p , respectively, the proof is over. \square

We conclude our discussion of the boundedness of composition operators by mentioning the following result in [23].

Theorem 17 ([23, Theorem 6.1]). *All holomorphic selfmaps of G induce bounded composition operators on $H^p(G)$ if and only if γ' is both bounded and bounded away from zero.*

4.2. Compact Composition Operators

Clearly, investigating which symbols ϕ induce compact composition operators on the Hardy–Smirnov spaces over a proper, simply connected domain G obtained as the range of the Riemann map γ is interesting only if $\gamma' \in H^1(\mathbb{U})$ or, equivalently if the spaces $H^p(G)$ contain the constant functions. We will work under this assumption throughout this section without mentioning it each time.

The symbols inducing compact composition operators on the Hardy spaces over \mathbb{U} are fully characterized in terms of the Nevanlinna counting function of the symbol [8], [22], the Aleksandrov measures induced by the symbol [6], or alternatively, some Carleson measures associated to the symbol, [8].

Borrowing on some of the techniques and ideas present in the proof of Theorem 1, as given in [23], one gets the following.

Theorem 18. *If an analytic selfmap ϕ of G induces a compact composition operator on $H^p(G)$, then φ , the γ -conformal conjugate of ϕ must satisfy the condition*

$$\lim_{|z| \rightarrow 1^-} \left| \frac{z - \varphi(z)}{1 - \bar{z}\varphi(z)} \right| = 1. \quad (36)$$

If φ has boundary fixed points, then the angular derivative of φ cannot exist at any of those points.

Proof. According to the Koebe distortion theorem, [12, Theorem 2.5]

$$\left(\frac{1 - |z|}{1 + |z|} \right)^2 \leq \frac{|\gamma'(z)|(1 - |z|^2)}{|\gamma'(0)|} \quad z \in \mathbb{U}. \quad (37)$$

Since the inequality above is valid for any univalent, analytic map on \mathbb{U} , it is valid for $\gamma \circ \alpha_a$, where α_a is the selfinverse conformal disk automorphism $\alpha_a(z) = (a - z)/(1 - \bar{a}z)$, with $a \in \mathbb{U}$ arbitrary and fixed. A straightforward computation shows that

$$\frac{|(\gamma \circ \alpha_a)'(z)|(1 - |z|^2)}{|(\gamma \circ \alpha_a)'(0)|(1 - |a|^2)} = \frac{|\gamma'(\alpha_a(z))|}{|\gamma'(a)|(1 - |a|^2)} (1 - |\alpha_a(z)|^2), \quad (38)$$

where the identity $1 - |\alpha_a(z)|^2 = (1 - |z|^2)(1 - |a|^2)/|1 - \bar{a}z|^2$ was used.

Substitution of z by $\alpha_a(z)$ in (38) combined with (37) leads to

$$\left(\frac{1 - |\alpha_a(z)|}{1 + |\alpha_a(z)|} \right)^2 \leq \frac{|\gamma'(z)|(1 - |z|^2)}{|\gamma'(a)|(1 - |a|^2)} \quad z, a \in \mathbb{U}. \quad (39)$$

Taking $a = \varphi(z)$ in (39) and letting $|z| \rightarrow 1^-$ one gets that (36) holds as a consequence of Theorem 8.

Arguing by contradiction, assume now that $\omega \in \partial\mathbb{U}$ is a boundary fixed point of φ where the angular derivative $\varphi'(\omega)$ exists. Let $z = r\omega$, $0 < r < 1$. Note that

$$\frac{z - \varphi(z)}{1 - \bar{z}\varphi(z)} = \omega \frac{\frac{\omega - \varphi(r\omega)}{\omega - r\omega} - 1}{1 + r \frac{\omega - \varphi(r\omega)}{\omega - r\omega}}.$$

Letting $r \rightarrow 1^-$ one gets, that, by condition (36)

$$\left| \frac{\varphi'(\omega) - 1}{\varphi'(\omega) + 1} \right| = 1.$$

Since, by formula (8), the angular derivative at a boundary fixed point is a non-negative real number, one gets

$$\frac{\varphi'(\omega) - 1}{\varphi'(\omega) + 1} = \pm 1.$$

Both of the equalities above are contradictory, since an angular derivative cannot be null. \square

We obtain as a corollary a fact proved in [23] by different methods.

Corollary 5. *If an analytic selfmap ϕ of G induces a compact composition operator on $H^p(G)$, then ϕ has a fixed point.*

Proof. Indeed if we assume that ϕ is fixed point free, then so is its γ -conformal conjugate φ , in which case, the Denjoy–Wolff Theorem implies that φ has a boundary fixed point where the angular derivative exists, a contradiction. \square

Note also that the main result in [17], namely the fact that Hardy–Smirnov spaces over half-planes cannot support compact operators, can be obtained as an immediate consequence of Theorem 18 and Theorem 15. The authors of [7] have also reproved that result by alternative methods.

Another consequence of Theorem 18 is the following.

Corollary 6. *If an analytic selfmap ϕ of G induces a compact composition operator on $H^p(G)$, then φ , the γ -conformal conjugate of ϕ , must satisfy the condition*

$$\liminf_{z \rightarrow \omega} \frac{|\omega - z|^\alpha |\gamma'(z)|}{|\omega - \varphi(z)|^\alpha |\gamma' \circ \varphi(z)|} = 0 \quad \omega \in \partial\mathbb{U}, \alpha > 1. \quad (40)$$

Proof. Indeed, if for some $\alpha > 1$ and $\omega \in \partial\mathbb{U}$ condition (40) fails, then condition (32) holds with $\omega = \eta$ for some constants $c, \delta > 0$, which implies the contradictory fact that ω is a boundary fixed point of φ where the angular derivative exists. \square

The compactness of composition operators on Hardy–Smirnov spaces $H^p(G)$ depends a lot on the shape of G , ranging from no compacts, when $\gamma' \notin H^1(G)$, to a lot of compacts, e.g. if G satisfies the assumptions of Theorem 17, when C_ϕ is compact on $H^2(G)$ if and only if C_φ is compact on $H^2(\mathbb{U})$, (as usual, φ is the γ -conformal conjugate of ϕ). Indeed, this is an immediate consequence of Theorem 14, the fact that the multiplication operators $M_{\sqrt{\gamma'/\gamma' \circ \varphi}}$ and $M_{\sqrt{\gamma' \circ \varphi/\gamma'}}$ are bounded, and the fact that the set of all compacts is a bilateral ideal of the algebra of all bounded operators.

To illustrate the statement that the status of compact composition operators can vary a lot with the shape of the domain under consideration, we give a last example. Note that the cardioid-shaped domain $G = \gamma(\mathbb{U})$ that appears in Example 3 supports compact composition operators, since $\gamma(z) = z - .5z^2$, hence $\gamma'(z) = 1 - z \in H^1(\mathbb{U})$, but not all analytic selfmaps of G induce bounded composition operators, since γ' is not bounded away from 0.

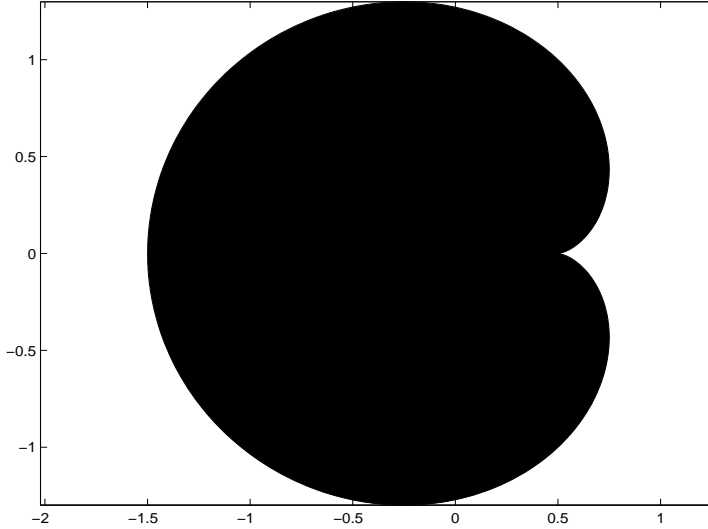


Figure 3: The cardioid-shaped range of $z - .5z^2$

One can easily give examples of compact and bounded composition operators on $H^2(G)$. Indeed:

Proposition 6. *If ϕ is an analytic selfmap of G , then C_ϕ is bounded on $H^2(G)$ whenever $1/2$ is not a cluster-point of $\phi(G)$ and, in that case, C_ϕ is compact if C_φ is compact on $H^2(\mathbb{U})$. If, in addition, ϕ maps a neighborhood of $1/2$ relative to G onto a subset of G situated at a positive distance of ∂G , then the compactness of C_ϕ is equivalent to that of C_φ .*

Proof. Indeed, the fact that $1/2$ is not a cluster-point of $\phi(G)$ is equivalent to the fact that 1 is not a cluster point of the range of its γ -conformal conjugate φ . Thus $M_{\sqrt{\gamma'/\gamma'\circ\varphi}} = M_{\sqrt{(1-z)/(1-\varphi(z))}}$ is a bounded operator, hence so is C_ϕ , and in case C_φ is compact, then $T_{\sqrt{\gamma'/\gamma'\circ\varphi}}\varphi$ is compact, hence C_ϕ is compact. The converse implication can be proved if ϕ maps a neighborhood of $1/2$ relative to G onto a subset of G situated at a positive distance of ∂G . The consequence of this fact is that there is some $0 < c < 1$ and an open arc A of $\partial\mathbb{U}$ containing 1 , so that $|\varphi(\zeta)| \leq c$ a.e. on A .

Consider now a sequence $\{f_n\}$ in $H^2(\mathbb{U})$ tending weakly to 0 and an arbitrary $\epsilon > 0$. Since $\{f_n\}$ tends to 0 uniformly on compacts, there is some positive integer n_1 so that $|f_n(\varphi(\zeta))| < \epsilon/2$ a.e. if $n \geq n_1$ and $\zeta \in A$. Thus

$$\int_A |f_n(\varphi(\zeta))|^2 dm(\zeta) < \frac{\epsilon}{2} \quad n \geq n_1.$$

On the other hand, $T_{\sqrt{\gamma'/\gamma'\circ\varphi}}$, φ is compact, hence

$$\frac{\max\{|1-\zeta| : \zeta \in A^c\}}{2} \int_{A^c} |f_n(\varphi(\zeta))|^2 dm(\zeta) \leq \int_{A^c} \left| \frac{1-\zeta}{1-\varphi(\zeta)} \right| |f_n(\varphi(\zeta))|^2 dm(\zeta) < \frac{\epsilon \max\{|1-\zeta| : \zeta \in A^c\}}{4} \quad n \geq n_2$$

for some positive integer n_2 . □

So the inward cusp of the cardioid above makes quite a difference between it and the unit disk. As we noted, unbounded composition operators on $H^2(G)$ exist and the compactness of an operator C_ϕ on $H^2(G)$ is not always equivalent to that of the operator C_φ on $H^2(\mathbb{U})$, as one can see by taking $\varphi(z) = i(z+1)/2$ and setting ϕ to be the selfmap of G which is γ -conformally conjugate to φ . Arguing like in Example 2, one shows that C_ϕ is compact, but C_φ is not.

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