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Abstract

Composition operators on the Hilbert Hardy space of the unit disk are considered. The shape of their numerical range is determined in the case when the symbol of the composition operator is a monomial or an inner function fixing 0. Several results on the numerical range of composition operators of arbitrary symbol are obtained. It is proved that 1 is an extreme boundary point if and only if 0 is a fixed point of the symbol. If 0 is not a fixed point of the symbol 1 is shown to be interior to the numerical range. Some composition operators whose symbol fixes 0 and has infinity norm less than 1 have closed numerical ranges in the shape of a cone-like figure, i.e. a closed convex region with a corner at 1, 0 in its interior, and no other corners. Compact composition operators induced by a univalent symbol whose fixed point is not 0 have numerical ranges without corners, except possibly a corner at 0.

1 Introduction

If $\mathcal{H}$ is a complex Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ is a bounded, linear operator on $\mathcal{H}$, the set $W(T) = \{< Th, h >: h \in \mathcal{H}, ||h|| = 1\}$ is called the numerical range of T. It is well known that the numerical range is a bounded, convex subset of the

complex plane $\mathbb{C}$. This set is not necessarily closed. The problem of determining the bounded, convex, subsets of $\mathbb{C}$ which are numerical ranges of operators is still unsolved. Our intention is to determine $W(T)$ for $T = C_\phi$ a composition operator acting on $\mathcal{H} = H^2$, the Hilbert Hardy space over the open unit disk, $\mathbb{D}$, i.e. the space of all complex functions analytic on $\mathbb{D}$ with square-summable Taylor coefficients. Recall that for each holomorphic selfmap ϕ of $\mathbb{D}$, the composition operator of symbol ϕ is the (necessarily bounded, [1]) operator

$$C_\phi f = f \circ \phi \qquad f \in H^2.$$

In the second section we start this job, and are able to complete it for composition operators acting on H^2, and having symbol ϕ a monomial and for inner symbols fixing 0. We begin with the simplest possible cases, (e.g when the operator is diagonal), we continue by proving that composition operators with constant symbol have elliptic numerical ranges, unless the constant is 0, when the numerical range is a line segment. When the monomial has degree bigger than 1 and its coefficient has absolute value strictly less than 1, the numerical range has the shape of the convex hull of a set consisting of a circle and a point situated outside the circle. The purpose of this second section is to provide examples, and show that even in the case of such simple symbols the shape of the numerical range can be rather diverse. In the last section we prove that composition operators with symbol ϕ such that $\|\phi\|_\infty < 1$ and $\phi(0) = \phi'(0) = 0$ have closed numerical ranges whose shape is a cone-like region with vertex at 1 i.e. the numerical range of such operators is a closed convex region with a corner at 1, 0 in its interior, and no other corners. We also prove that 1 is either an extreme boundary point or an interior point of the numerical range. The former situation occurs if and only if $\phi(0) = 0$. Finally we prove that if ϕ is univalent, induces a compact composition operator, and has nonzero fixed point, then the closure $\overline{W(C_\phi)}$ of $W(C_\phi)$ cannot have corners, except possibly 0.

In the sequel we will use more than once the elliptic range theorem, i.e. the description of the numerical range of a complex 2×2 matrix. Such a matrix with distinct eigenvalues λ and μ is known to have numerical range in the shape of a closed

elliptical disk of foci λ and μ. The major and minor axis of this elliptical disk are given by the following equalities.

$$\text{Major Axis} = \frac{|\lambda - \mu|}{\sqrt{1 - |<f,g>|^2}}$$

$$\text{minor axis} = \frac{|<f,g>||\lambda - \mu|}{\sqrt{1 - |<f,g>|^2}}$$

where f and g are unital eigenvectors corresponding to λ and μ respectively. Observe that the ellipse is degenerated, i.e. reduced to its focal axis if and only if f and g are orthogonal. If the eigenvalues are not distinct, the numerical range is a closed circular disk, [5].

2 The Numerical Range When the Symbol is a Monomial

Clearly the easiest case is the case of C_ϕ, $\phi(z) = z$ when, trivially the numerical range is the singleton $\{1\}$. The first step from this trivial setting is determining $W(C_\phi)$ for $\phi(z) = \lambda z$, $|\lambda| = 1$, $\lambda \neq 1$. With this assumptions for ϕ and λ, one easily obtains the following.

Proposition 2.1 *If λ is a primitive root of 1 of order $n \geq 2$ then $W(C_\phi)$ is the closed segment $[-1, 1]$ if $n = 2$, respectively $W(C_\phi)$ is the closed, regular polygonal region with n sides and a vertex at 1, inscribed in the unit circle if $n > 2$. If λ is not a root of 1 then $W(C_\phi) = \mathbb{D} \cup \{\lambda^n : n \geq 0\}$.*

Proof. Suppose first that λ is a primitive root of 1 of order $n \geq 2$. In that case C_ϕ is a diagonal operator with cyclic diagonal, more precisely the diagonal consists of the finite sequence 1, λ, λ^2, λ^3, ..., λ^{n-1}, repeated infinitely many times. The diagonal entries are eigenvalues, hence they belong to $W(C_\phi)$. This last set, being convex will contain the convex hull of the set $V = \{1, \lambda, \ldots, \lambda^{n-1}\}$, which is the closed polygonal

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region whose boundary is the regular polygon with n sides, inscribed in the unit circle, having one of the vertices at 1. Let's denote the latter set by P. We proved $P \subseteq W(C_\phi)$. The converse inclusion follows easily from the fact that C_ϕ is a diagonal operator, hence a normal operator. It is well-known that the closure of the numerical range of a normal operator equals the convex hull of its spectrum. In our case the spectrum of C_ϕ is equal to V, so $P = \overline{W(C_\phi)}$, which concludes the proof, for the case when λ is a root of 1. Needless to say that in the particular case $n = 2$, P is reduced to the segment $[-1, 1]$.

We consider now the case when λ is not a root of 1. In that case the set $V = \{\lambda^n : n \geq 0\}$ is a dense subset of the unit circle $\mathbb{T}$. On the other hand V is a subset of the point spectrum of C_ϕ, hence V is a subset of $W(C_\phi)$. Again by the convexity of the numerical range the convex hull of V is contained by $W(C_\phi)$. By the fact that V is a dense subset of $\mathbb{T}$ we easily can see that the convex hull of V is $\mathbb{D} \cup V$. On the other hand, using as above the fact that C_ϕ is diagonal, we get $\overline{W(C_\phi)} = \overline{\mathbb{D}}$. To see that, note also that the spectrum of C_ϕ equals the closure of the diagonal of C_ϕ which is $\mathbb{T}$. So the only thing to decide now is which points of $\mathbb{T}$ besides those in V can belong to $W(C_\phi)$. The answer is none, which will conclude the proof. To see that, suppose there is some unimodular complex number $e^{i\tau} \in \mathbb{T} \setminus V$ and some $f \in H^2$, $f(z) = \sum_{n=0}^{\infty} c_n z^n$, $\sum_{n=0}^{\infty} |c_n|^2 = 1$ such that $e^{i\tau} = <C_\phi f, f>$. By the Cauchy-Schwartz inequality $1 \leq ||C_\phi f|| ||f|| = 1$. We used the fact that C_ϕ is a diagonal operator having unimodular entries on the diagonal, for which reason C_ϕ is a unitary operator. The Cauchy-Schwartz inequality becomes an equality if and only if the two vectors involved in it are linearly dependent, hence $C_\phi f = \mu f$ i.e. f must be an eigenfunction of C_ϕ corresponding to some eigenvalue μ. The eigenvalues of C_ϕ are exactly the values on its diagonal. So $\mu = \lambda^m$, for some $m \geq 0$. The equality $C_\phi f = \lambda^m f$ implies $<C_\phi f, f> = \lambda^m$, a contradiction. $\qquad \square$

The next thing is to consider C_ϕ with ϕ of the following form $\phi(z) = az$, $|a| < 1$. In that case the description of the numerical range is as follows.

Proposition 2.2 *If $a \notin \mathbb{R}$, then $W(C_\phi)$ is a closed polygonal region, whose vertices form a finite subset of the set $\{a^n : n \geq 0\}$. If $a \in \mathbb{R}$, then $W(C_\phi) = (0,1]$ if $a > 0$, respectively $W(C_\phi) = [a,1]$ if $a \leq 0$.*

Proof. Consider first $a \in \mathbb{D} \setminus \mathbb{R}$. C_ϕ is in this case a diagonal operator whose diagonal is the sequence $(a^n)_n$. Since $|a| < 1$ this sequence tends to zero. Therefore the spectrum of this operator is $\sigma(C_\phi) = \{0\} \cup \{a^n : n \geq 0\}$, and since the operator is normal $\overline{W(C_\phi)}$ will equal the convex hull of $\sigma(C_\phi)$. This last set is a polygonal region whose vertices belong to the set $V = \{a^n : n \geq 0\}$, because the argument of a is not 0 or π, so we have at least three non colinear points in V. On the other hand, the fact that $a^n \to 0$ implies that there is some positive integer N such that a^n belongs to the convex hull of the set $\{1, a, a^2, \ldots, a^N\}$, which is a closed polygonal region containing 0 in the interior, $\forall n \geq N$. The vertices of the aforementioned region are eigenvalues of C_ϕ, which implies that the region must be contained by $W(C_\phi)$. This concludes the proof for the case $a \notin \mathbb{R}$. If $a \in \mathbb{R}$, repeating the same proof, and observing that, in this case all the points of the sequence $(a^n)_n$ are on the real axis, one easily gets $W(C_\phi) = [a,1]$ if $a \leq 0$, respectively $(0,1] \subseteq W(C_\phi) \subseteq \overline{W}(C_\phi) = [0,1]$, if $a > 0$. In this last case we get $W(C_\phi) = (0,1]$ by observing that $0 \notin W(C_\phi)$. Indeed if $f = \sum_n c_n z^n$ is in H^2 and $||f|| = 1$, then clearly $< C_\phi f, f >= \sum_n a^n |c_n|^{2n} > 0$ if $a > 0$. $\qquad \square$

Figure 1 represents the polygonal numerical ranges of the composition operators with symbols $\phi(z) = \lambda e^{i\pi/3} z$, for λ equal to 1, 1/1.1, 1/1.2, and 1/2. The numerical ranges are nested and the previous ordering of the values of λ corresponds to the polygons in Figure 1 going from exterior toward interior i.e. the bigger the value of λ the larger the polygon.
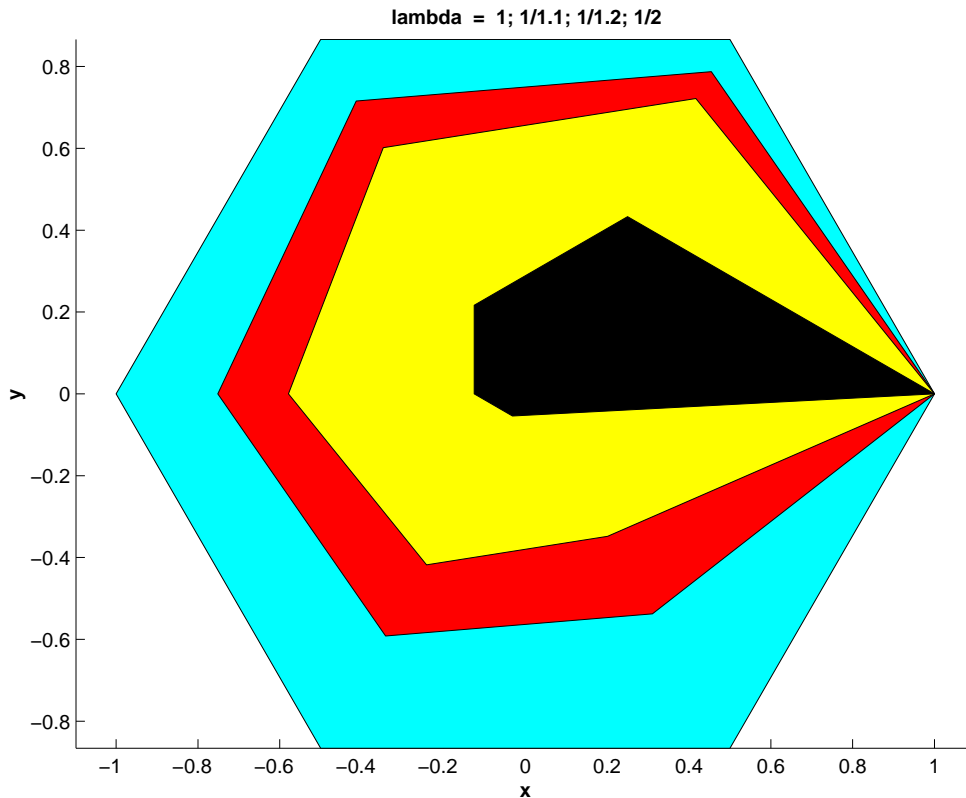
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Figure 1: The numerical range of C_ϕ for $\phi(z) = \lambda e^{\frac{i\pi}{3}} z$

Observe now that if $\phi(z) = cz^k$, $|c| = 1$, $k \geq 2$, then C_ϕ is necessarily a non unitary isometry.

Proposition 2.3 *If ϕ is an inner function other than a disk automorphism and $\phi(0) = 0$ the numerical range of C_ϕ is given by the equality $W(C_\phi) = \mathbb{D} \cup \{1\}$.*

Proof. It is shown in [7] that under the hypothesis above C_ϕ is an isometry with Wold decomposition $H^2 = \mathbb{C} \oplus zH^2$, i.e. the spaces in the previous direct sum are reducing subspaces of C_ϕ, $C_\phi|\mathbb{C}$ is a unitary operator, and $C_\phi|zH^2$ a unilateral forward shift. Obviously $W(C_\phi|\mathbb{C}) = \{1\}$ and the numerical range of any unilateral shift operator is $\mathbb{D}$. To find the numerical range of the direct sum of the two operators above one takes the convex hull of the union of their numerical ranges obtaining $W(C_\phi) = \mathbb{D} \cup \{1\}$. The statement that unilateral shifts have numerical range $\mathbb{D}$ is a direct consequence of the rather well-known fact that any $\lambda \in \mathbb{D}$ is an eigenvalue of any backward unilateral shift and that unilateral shifts have no eigenvalues on the unit circle. On the other hand eigenvalues are in the numerical range of an operator

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and if an operator of norm 1 has unimodular numbers in the numerical range, then by the Cauchy-Schwartz inequality those numbers must be eigenvalues of the given operator. $\square$

An immediate consequence of Proposition 2.3 is the following.

Corollary *If $\phi(z) = cz^k$, $|c| = 1$, $k \geq 2$, then $W(C_\phi) = \mathbb{D} \cup \{1\}$.*

To finish the job for the case when the symbol of the composition operator is a monomial we should now study symbols of the form $\phi(z) = az^k$, $k \geq 2$, $|a| < 1$, $a \neq 0$, and the case when the symbol is a nonzero constant function. We begin with the latter case. In the following we consider $a \in \mathbb{D}$, $a \neq 0$ and the constant function ϕ, taking the value a at each $z \in \mathbb{D}$. If we denote the 1-dimensional subspace of the constant functions by $\mathbb{C} \subseteq H^2$, one can easily see that $C_\phi H^2 \subseteq \mathbb{C}$ so C_ϕ is a rank 1 operator, hence C_ϕ is compact, and hence $0 \in \sigma(C_\phi)$ where $\sigma(C_\phi)$ is the spectrum of C_ϕ. Because C_ϕ is compact, the nonzero values in its spectrum must be eigenvalues. Since obviously $C_\phi 1 = 1$ we deduce $1 \in \sigma(C_\phi)$, and it is trivial to see that $C_\phi f = \lambda f$ is equivalent to $f(a) = \lambda f$. So if we claim that f is not the zero function and $\lambda \neq 0$ we deduce that f must be constant and λ must equal 1. This proves that for each choice of $a \in \mathbb{D}$, $\sigma(C_\phi) = \{0, 1\}$. We will see that the numerical ranges for different values of a can be different, and all are significantly larger than the spectrum. For each $z_0 \in \mathbb{C}$ and each $r > 0$, $D(z_0, r)$ denotes the open disk centered at z_0 of radius r, and $\overline{D}(z_0, r)$ is its closure. The exact description of the numerical range of the composition operator of constant symbol $\phi \equiv a$, $0 < |a| < 1$ is contained by the following theorem.

Theorem 2.4 $W(C_\phi)$ *is a closed elliptical disk whose boundary is the ellipse of foci 0 and 1, having horizontal axis of length* $\frac{1}{\sqrt{1-|a|^2}}$.

Proof. Consider the functions $f(z) = 1$, $k_a(z) = 1/(1 - \bar{a}z)$, and $g = 1/|a| - ((1 - |a|^2)/|a|)k_a)$. It is easy to check that $C_\phi 1 = 1$, and $C_\phi g = 0$, so f and g are norm 1, eigenvectors of C_ϕ corresponding to the eigenvalues 1 and 0 respectively. Hence

the 2-dimensional subspace M of H^2 spanned by f and g is invariant for C_ϕ. To see this subspace is even reducing, observe that any function h in its orthocomplement is perpendicular to k_a, for which reason, $C_\phi h = h(a) = <h, k_a> = 0$. The last equality is the consequence of the fact that k_a is a reproducing kernel function of H^2, (see [5], Ch. 4). Therefore if T is the restriction of C_ϕ to M, C_ϕ can be represented as $T \oplus 0$. For this reason, the numerical range of C_ϕ will be the convex hull of the union of the singleton $\{0\}$ and $W(T)$. By the elliptic range theorem, since M is 2-dimensional, $W(T)$ will be a closed elliptical disk of foci, the eigenvalues of T, i.e. 0 and 1 and having major axis $|1 - 0| / \sqrt{1 - |<f, g>|^2} = 1/\sqrt{1 - |a|^2}$. □
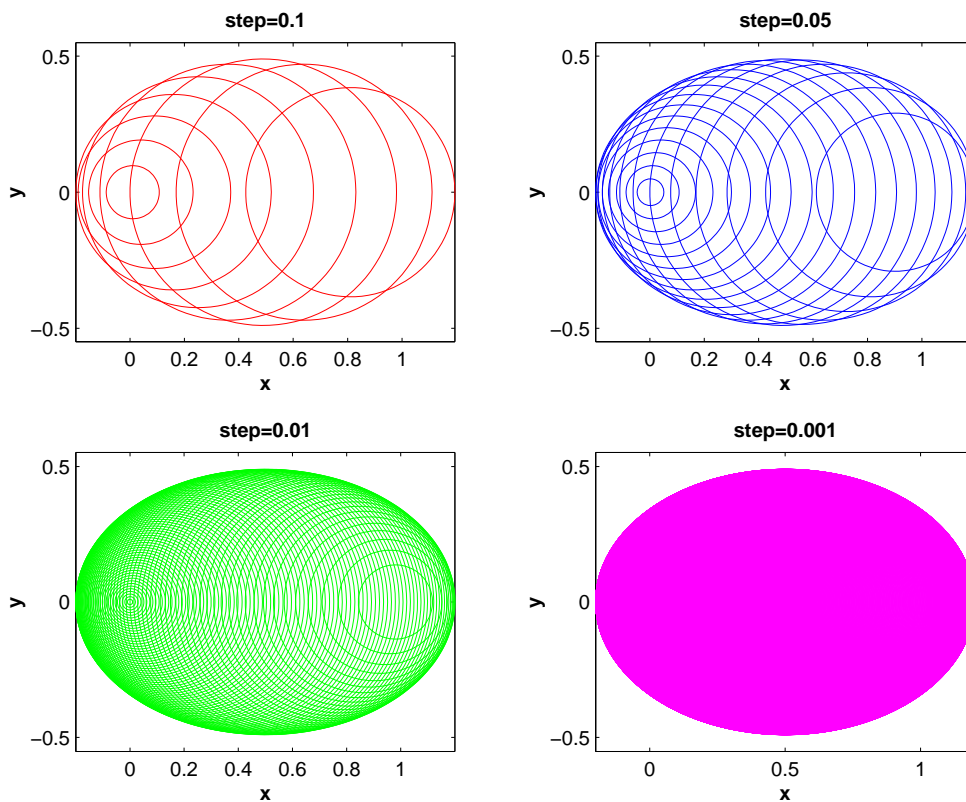

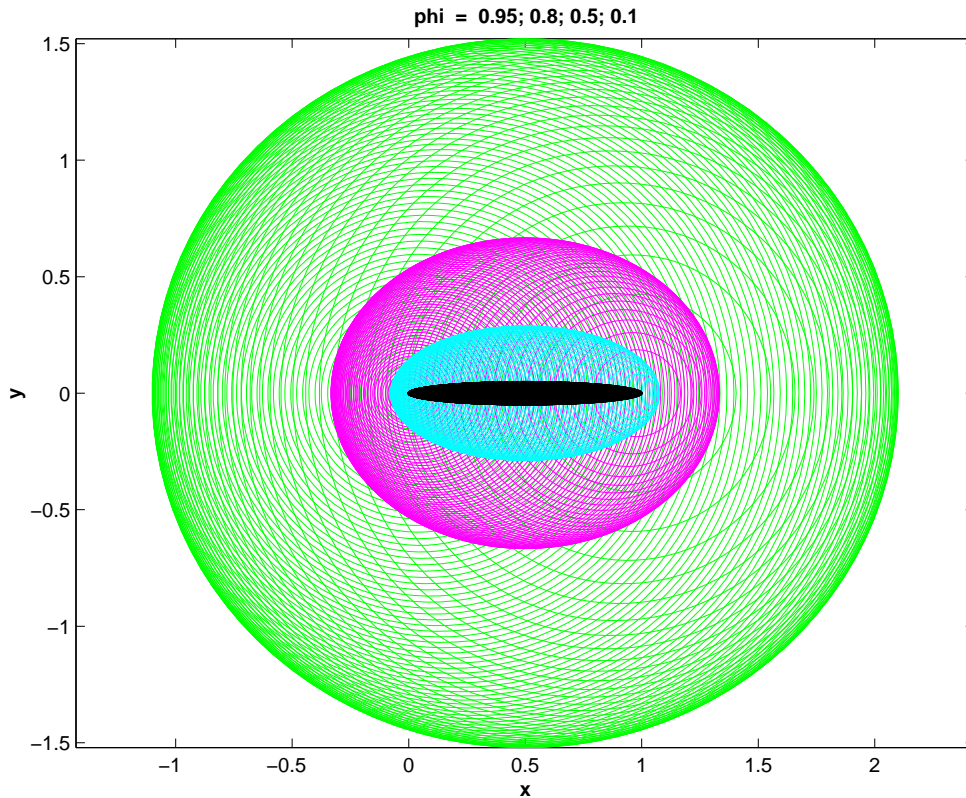Figure 2: The numerical range of C_ϕ for $\phi \equiv 0.7$

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Figure 3: The numerical range of C_ϕ for $\phi \equiv 0.1$, 0.5, 0.8, 0.95

Figure 3 represents the nested numerical ranges of 4 composition operators with constant symbols, equal respectively to 0.1, 0.5, 0.8, and 0.95. Observe that if $|a| \to 0$ the corresponding ellipses tend to become flat and overlap the segment $[0, 1]$, whereas if $|a| \to 1$, the ellipses increase in size unrestrictedly (indeed $1/\sqrt{1-|a|^2} \to \infty$ if $|a| \to 1$!), and tend to become circles, i.e. the quotient of their two axes tends to 1. We wrote a short MATLAB program in order to generate these figures. The technique was based on an earlier proof of the previous theorem in which we represented the numerical range of C_ϕ as the union of uncountably many closed disks $D_t = \overline{D}(t^2, t\sqrt{1-t^2}\frac{|a|}{\sqrt{1-|a|^2}})$, $t \in [0, 1]$. The program graphs finitely many of the boundary circles ∂D_t corresponding to as many values of t, obtained by starting with $t = 0$ and successively adding a fixed increment called "step". The smaller the

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"step" the larger the number of circles graphed.

The only case we did not treat completely is when the symbol has the form $\phi(z) = az^k$, $0 < |a| < 1$, and $k \geq 2$. The cases $a = 0$ and $|a| = 1$ have already been treated.

We'll use the non conventional terminology of "ice-cream cone" to designate the convex hull of a disk and a point situated outside the disk. We'll denote by $\mathbb{Z}$ the set of all integers, and by $\mathbb{N}$ the subset of $\mathbb{Z}$ consisting of all integers bigger than or equal to 0. For any sequence $(a_n)_{n \in \mathbb{Z}}$ of scalars, and any $l \in \mathbb{N}$, $S_l(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$ will denote the circularily symmetric functions used by Q.F.Stout in [10]; we prefer to refer the reader to that paper rather than reproduce the definitions in the text, since they are rather lengthy. With these preliminaries we can prove the following.

Theorem 2.5 *For each ϕ of the form $\phi(z) = az^k$, $0 < |a| < 1$, and $k \geq 2$, $W(C_\phi)$ is a closed "ice-cream cone" with vertex at 1 and disk centered at 0. The radius of this disk is $1/\sqrt{t}$, where t is the smallest positive root of the entire function*

$$(8) \qquad F(z) = \sum_{l=0}^{\infty} S_l(\ldots, a_{-2}^2, a_{-1}^2, a_0^2, a_1^2, a_2^2, \ldots)(-1/4)^l z^l$$

where $S_l(\ldots, a_{-2}^2, a_{-1}^2, a_0^2, a_1^2, a_2^2, \ldots)$ corresponds to the weight sequence $(a_n)_{n \in \mathbb{Z}}$, $a_n = |a|^{k^n}$, if $n \geq 0$, and $a_n = 0$ if $n < 0$.

Proof. Observe that if P is the orthogonal projection onto the subspace $\mathbb{C}$ of constant functions, then $PC_\phi f = C_\phi P f = f(0)$ for any $f \in H^2$, which proves that $\mathbb{C}$ is a reducing subspace of C_ϕ. The restriction of C_ϕ to $\mathbb{C}$ is the identity operator on $\mathbb{C}$, so its numerical range will be the singleton $\{1\}$. Let's denote by A the restriction of C_ϕ to zH^2, the orthocomplement of $\mathbb{C}$. If we show that $W(A)$ is the closed disk centered at 0 of radius $1/\sqrt{t}$ we are done because the numerical range of a direct sum of two operators equals the convex hull of the union of the numerical ranges of those operators. To find $W(A)$ we observe that A is unitarily equivalent to a direct sum of countably many forward weighted shifts. Let E be the set of all integers bigger than or equal to 1 which are not multiples of k. For each fixed $m \in E$ we denote by E_m the set $E_m = \{mk^n : n \in \mathbb{N}\}$. The obvious relation

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$C_\phi z^{mk^n} = a^{mk^n} z^{mk^{n+1}}$, $\forall n \in \mathbb{N}$, shows that if $\mathcal{S}_m$ denotes the closed subspace of zH^2 spanned by all the functions z^s, $s \in E_m$, then $C_\phi|\mathcal{S}_m$, is a forward weighted shift of weight sequence $u_n = a^{mk^n}$, and is therefore unitarily equivalent to a forward weighted shift of weight sequence $w_n = |a|^{mk^n}$, [9]. This last weighted shift will be denoted by T_m. The fact that $(E_m)_{m\in E}$ is a partition of $\mathbb{N} \setminus \{0\}$ shows that A is unitarily equivalent to $\sum_{m\in E} \oplus T_m$, so $W(A)$ will be the convex hull of $\cup_{m\in E} W(T_m)$. But for each m, $W(T_m)$ is a disk centered at 0, [10]; therefore $W(A)$ will be a disk centered at 0 of radius $\sup_{m\in E} w(T_m)$, where for each operator T we denote by $w(T)$ its numerical radius. If $(W_n)_n$ is the weight sequence of the weighted shift T_1, and for an arbitrary fixed $m \in E$, $(w_n)_n$ is the weight sequence of T_m, then the obvious relation $W_n \geq w_n$, $\forall n \in \mathbb{N}$, implies $w(T_1) \geq w(T_m)$, hence $w(A) = w(T_1)$. Clearly T_1 has the same numerical range as the bilateral, forward weighted shift with weight sequence $(a_n)_{n\in\mathbb{Z}}$, $a_n = W_n$, if $n \in \mathbb{N}$, $a_n = 0$, if $n < 0$. To see that, recall that if $0 \in W(T)$, then $W(T) = \{< Tx, x >: ||x|| \leq 1\}$, ([5], solution of Problem 213). The weight sequence $(a_n)_{n\in\mathbb{Z}}$ is square-summable so the aforementioned bilateral weighted shift is Hilbert-Schmidt, and by [10], Theorem 3, its numerical radius is equal to $1/\sqrt{t}$. Finally, the numerical range of A is a closed disk, because it contains 0 and A is compact [2]. The compactness of A can be proved in several ways, for instance observe that C_ϕ is compact because $||\phi||_\infty < 1$, [1] or [8]. $\square$

Figure 4 shows the numerical range of C_ϕ for $\phi(z) = z^2/2$. It was generated by using a similar technique as the one used for Figures 2 and 3. The numerical range of a composition operator with symbol $\phi(z) = az^k$, $0 < |a| < 1$, can be represented as a union of disks as follows.

$$W(C_\phi) = \bigcup_{t\in[0,1]} (t + (1 - t)W(A))$$

where A is the restriction of the composition operator under consideration to the reducing subspace zH^2, i.e. an operator whose numerical range is a closed disk, according to the proof of the previous theorem. The term step which appears in Figure 4 has the same meaning as in the case of Figure 2.
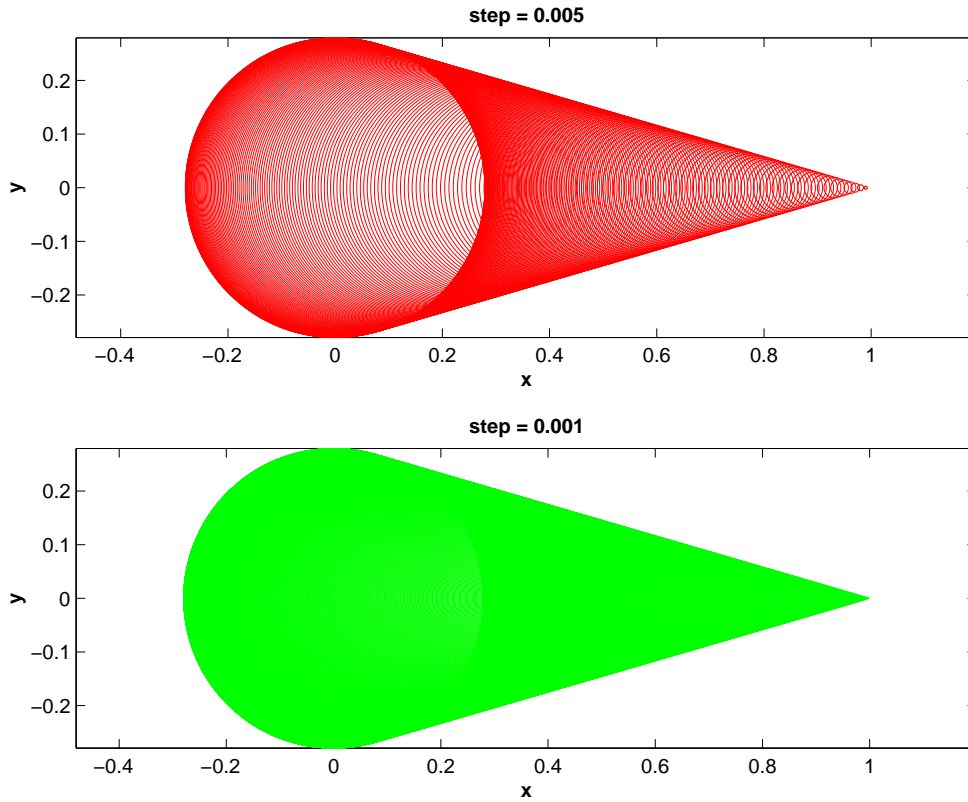
Figure 4: The numerical range of C_ϕ for $\phi(z) = \frac{z^2}{2}$

3   More Properties

We call a point z in the closure of a convex subset $C \subseteq \mathbb{C}$, an extreme point of C if it is not in the interior of a line segment with endpoints in C. A point z in the closure of C is a corner of C if there is a closed angle with vertex at z and aperture angle smaller than π containing $\overline{C}$. A support line L of C at z is any line through z such that C lies in one of the closed half-planes determined by that line.

All the symbols of the composition operators in the previous section, with the exception of the nonzero, constant ones, had the same, interior fixed point, namely 0. As we saw, the shapes of the numerical ranges were rather diverse. So, one cannot expect to show that all symbols having the same fixed, interior point, induce composition operators with numerical ranges of the same shape. However some

general, common properties can be proved. For instance, in most of our examples contained by the previous section, 0 was an interior point of the numerical range. Straight-forward considerations about the spectra of compositions operators lead to the following.

Remark *Zero is an interior point of $W(C_\phi)$ in each of the following situations.*

(i)   *ϕ has an interior fixed point $a \in \mathbb{D}$, and $\phi'(a) \notin \mathbb{R}$*

(ii)   *ϕ is univalent, but not a disk automorphism, has an interior fixed point, and C_ϕ is not essentially quasinilpotent (i.e. has positive essential spectral radius).*

(iii)   *ϕ, not an inner function, is analytic in a neighborhood of the closed unit disk, has an interior fixed point, and C_ϕ is not essentially quasinilpotent.*

(iv)   *ϕ is inner, not a disk automorphism.*

Proof. Under the assumptions in (i), the spectrum of C_ϕ contains the values $(\phi'(a))^n$, $n \geq 0$, the closure of the numerical range will contain the convex hull of this set of values, and since $\phi'(a) \notin \mathbb{R}$, this convex set will be a polygon with 0 in its interior. (See [1], Theorem 7.32). Under the assumptions in each of the following statements, $\sigma(C_\phi)$ contains an open disk centered at 0. (See [1], Chapter 7). $\square$

One can use the examples in the previous section to see that the conditions above are not necessary in order to have $0 \in \text{Int}(W(C_\phi))$. For instance, in Theorem 2.5 we have symbols ϕ with the properties $\phi(0) = 0$ and $\phi'(0) = 0$, hence $\phi'(0) \in \mathbb{R}$, but still 0 is in the interior of $W(C_\phi)$. Also observe that the operators in Theorem 2.5 are compact.

Theorem 3.1 *If C_ϕ is a compact composition operator, $\phi(0) = 0$, $\phi'(0) = 0$, and ϕ is not the null function, then 0 is an interior point of $W(C_\phi)$, and 1 is the only possible corner of $W(C_\phi)$.*

Proof. Observe that $\sigma(C_\phi) = \{0, 1\}$ [8], and that, as in the proof of Theorem 2.5, $\mathbb{C}$ is a reducing subspace for C_ϕ, and C_ϕ can be represented as $C_\phi = 1 \oplus A$ with respect to the direct sum decomposition $H^2 = \mathbb{C} \oplus zH^2$. A denotes of course the

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restriction to zH^2. Following M. Embry's notation [3] we denote for each complex λ, $M_\lambda = \{f \in H^2 :< C_\phi f, f >= \lambda||f||^2\}$. $0 \in W(C_\phi)$ because $\phi(z) = z^k \varphi(z)$, $k \geq 2$, $\varphi(0) \neq 0$, and $< C_\phi z^n, z^n >=< z^{kn} \varphi^n(z), z^n >=< z^{n(k-1)} \varphi^n(z), 1 >= 0$. Among other things, the previous computation shows that $z^n \in M_0$, $\forall n \geq 1$. According to [3], 0 is an extreme boundary point of $W(C_\phi)$ if and only if M_0 is a linear subspace. Since M_0 is always closed this last thing would imply $zH^2 \subseteq M_0$ and hence $A = 0$, a contradiction, since ϕ is not the null function. If we suppose that 0 is a non extreme boundary point of $W(C_\phi)$, then one can consider a support line L of $W(C_\phi)$ at 0, such that a segment of this line is contained in the boundary of $W(C_\phi)$, 0 belongs to this segment, and is not one of the endpoints. The linear space spanned by M_0, SpM_0 is closed and can be represented in the form Sp$M_0 = \cup_{\lambda \in L} M_\lambda$, [3]. We will show this generates a contradiction. Indeed $zH^2 \subseteq$ SpM_0, so $W(A) \subseteq L$. If $W(A)$ is a segment, (necessarily containing 0), of a different line than the real line, then $W(C_\phi)$ will be a triangle, since it is the convex hull of this segment and the singleton $\{1\}$, because $C_\phi = 1 \oplus A$. The corners of this triangle would be in the spectrum of C_ϕ, [4], 1.5-6., a contradiction. If $W(A)$ is a sub segment of the real line, the convex hull of this segment and $\{1\}$ has to be a segment of the real line containing 0 in its interior, so one of the endpoints of this segment would be neither 0 nor 1. By the same result in [4], we used above this endpoint would be in $\sigma(C_\phi)$, which is a contradiction. We admit therefore that 0 is an interior point of $W(C_\phi)$. Any corner of $W(C_\phi)$ has to belong to the spectrum, so only 1, could be a corner. $\square$

Examples in section 2 show that, under the assumptions above 1 can be a corner of $W(C_\phi)$ or $W(C_\phi)$ can be without corners. Denoting as in the proof above by A the restriction of C_ϕ to zH^2, we wish to remark the following.

Remark *If ϕ satisfies the hypothesis above and $||A|| < 1$, then $W(C_\phi)$ is a closed cone-like figure i.e. a closed, convex set with nonempty interior and only one corner. The corner is necessarily at 1. This is always the case when, $\phi(0) = \phi'(0) = 0$, and $||\phi||_\infty < 1$.*

Proof. Indeed, the inequality $||A|| < 1$ implies that $W(A)$ is the subset of a disk centered at zero of radius less than 1, and by the theorem above $W(A)$ will be a convex set containing a disk centered at 0. Since $W(C_\phi)$ is the convex hull of $W(A)$ and $\{1\}$, it will be a closed, convex set with boundary consisting of two line segments forming an angle with vertex at 1, one contained in the closed upper half-plane the other in the closed lower half-plane, (each segment not a sub segment of the real line), and a smooth curve joining the endpoints different from 1 of the two segments. Clearly this happens if $||\phi||_\infty < 1$, because in that case C_ϕ is compact [8], and $||A|| < 1$, since for each $f \in H^2$, $||f|| \leq 1$, we can write $||A(zf(z))|| = \int_\mathbb{T} |\phi|^2 |f \circ \phi|^2 \leq ||\phi||_\infty^2 ||C_\phi f||^2 \leq ||\phi||_\infty^2 < 1$. We took into consideration that C_ϕ is a contraction, because $\phi(0) = 0$, ([1], or [8]). Above we made the statement that the numerical range was closed. Recall that compact operators with numerical range containing zero have closed numerical ranges [2]. □

Another general fact related to the shape of the particular numerical ranges described in Theorem 2.5 is the following.

Theorem 3.2 *The point 1 is an extreme boundary point of $W(C_\phi)$ if and only if $\phi(0) = 0$.*

Proof. In the following 1 will be regarded as both the scalar 1 and the constant function $1 \in H^2$. The obvious relation $C_\phi 1 = 1$ shows that 1 is an eigenvalue of any composition operator, and hence belongs to the numerical range of any composition operator. If $\phi(0) = 0$, C_ϕ is a contraction, [1] or [8], and hence 1 is an extreme boundary point of $W(C_\phi)$. Conversely, if we know that 1 is an extreme boundary point of $W(C_\phi)$, then, using M. Embry's notation, already introduced in the proof of Theorem 3.1, we have that M_1 is linear and hence the intersection of all maximal linear subspaces of M_1 containing 1 is M_1 itself. It is our intention to use Theorem 2 in [3]. In that theorem the previously described intersection is denoted K_1. Clearly $1 \in M_1$, and if L is a support line of $W(C_\phi)$ at 1, then $M_1 \subseteq \cup_{w \in L} M_w = N$. So $C_\phi 1 = 1 \in N$. Therefore, by Theorem 2 in [3], $C_\phi^* 1 = 1$. If for each $a \in \mathbb{D}$, k_a is the

Szegö kernel at a, i.e. the function $k_a(z) = 1/(1 - \bar{a}z)$, then $k_0 = 1$, and by a well-known property of composition operators, $C_\phi^* k_0 = k_{\phi(0)}$. We have $1 = 1/(1 - \bar{\phi}(0)z)$, which is possible only if $\phi(0) = 0$. $\quad\square$

So theoretically 1 can be a non extreme boundary point or an interior point of $W(C_\phi)$ if 0 is not a fixed point of ϕ. In connection with this comment we can prove the following.

Theorem 3.3 *If $\phi(0) \neq 0$ then 1 is an interior point of $W(C_\phi)$.*

Proof. Consider the compression of C_ϕ to the two-dimensional subspace $\mathcal{L}$ of H^2 spanned by the functions 1 and z. Denote this operator by A. Clearly one eigenfunction of A is the constant function 1 corresponding to the eigenvalue 1. Observe that under our assumptions $\phi'(0) \neq 1$. Indeed, we have the obvious string of inequalities

$$|\phi'(0)| \leq ||\phi||_2 \leq ||\phi||_\infty \leq 1.$$

Therefore, if we suppose $\phi'(0) = 1$ we get that ϕ must be the identity function, which is not possible since $\phi(0) \neq 0$. We can consider then the function

$$v = -\frac{\phi(0)}{1 - \phi'(0)} + z.$$

It is easy to check that v is an eigenfunction of A corresponding to the eigenvalue $\phi'(0)$. Observe that

$$< 1, v > = -\frac{\phi(0)}{1 - \phi'(0)} \neq 0$$

so, by the elliptic range theorem $W(A)$ will contain a non degenerated, closed elliptic disk with foci 1 and $\phi'(0)$. Since $W(A) \subseteq W(C_\phi)$, the proof is over. $\quad\square$

Some compact composition operators, like those having symbols of the form $\phi(z) = \lambda z^k$, $|\lambda| < 1$, $k \geq 1$ have numerical ranges with corners. Some others, like those with nonzero, constant symbol have numerical ranges without corners. Under certain assumptions the numerical ranges of compact composition operators are sets with nonempty interior and smooth boundaries.

Theorem 3.4 *If ϕ a univalent function induces a compact composition operator and the fixed point $a \in \mathbb{D}$ of ϕ is not 0 then $W(C_\phi)$ is a set with nonempty interior and the only corner $\overline{W(C_\phi)}$ can have is 0. If in addition, $\phi'(a) \notin \mathbb{R}$, then $\overline{W(C_\phi)}$ has no corners.*

Proof. Theorem 2.4 covers the case when ϕ is constant. Consider the case when ϕ is not constant. Let σ be the Königs map of C_ϕ (see [1] or [8] for the definition of this notion). Under the assumptions above we wish to show that $\sigma(0) \neq 0$. Suppose the contrary. Denote $\phi^{(n)} = \phi \circ \phi \circ \cdots \circ \phi$, n times. Since $C_\phi^n \sigma = (\phi'(a))^n \sigma$, we deduce $\sigma(\phi^{(n)}(0)) = 0$, $\forall n \geq 1$. This implies that $\sigma = 0$, because $\phi^{(n)}(0) \to a \in \mathbb{D}$ and under our assumptions the sequence $(\phi^{(n)}(0))_n$ is not stationary. This is a contradiction since σ is an eigenfunction of C_ϕ, [1] or [8]. To see $(\phi^{(n)}(0))_n$ is not a stationary sequence, observe that if $\phi^{(n)}(0) = a$ for some $n \geq 1$, one has $\phi^{(n)}(0) = \phi^{(n)}(a)$ which implies $a = 0$, a contradiction. The fact that 0 is not a fixed point of σ implies that σ^n is not orthogonal to 1, $\forall n \geq 1$. Under our assumptions, the spectrum of C_ϕ consists of 0, 1, and the eigenvalues $(\phi'(a))^n$, which correspond to the eigenfunctions σ^n, $\forall n \geq 1$, respectively. Recall that $\phi'(a) \neq 1$, [8], page 90). For any fixed n, if 1 and $\phi'(a)^n$ are distinct, one can apply the elliptic range theorem to the restriction of C_ϕ to the subspace spanned by the eigenfunctions 1 and σ^n getting that $W(C_\phi)$ contains a closed, non degenerated elliptical disk of foci 1 and $(\phi'(a))^n$. Since the corners of $\overline{W(C_\phi)}$ are in the spectrum of the operator, [4], only 0 could be a corner. Under the supplementary assumption $\phi'(a) \notin \mathbb{R}$, 0 is an interior point of $W(C_\phi)$, by (i), so the closure of the numerical range has boundary without corners in this case. $\square$

The results in this paper circulated in preprint form [6].

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