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CONVERGENT SEQUENCES OF COMPOSITION OPERATORS

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Abstract. Composition operators C_φ on the Hilbert Hardy space H^2 over the unit disk are considered. We investigate when convergence of sequences $\{\varphi_n\}$ of symbols, (i.e. of analytic selfmaps of the unit disk) towards a given symbol φ, implies the convergence of the induced composition operators, $C_{\varphi_n} \to C_\varphi$. If the composition operators C_{φ_n} are Hilbert-Schmidt operators, we prove that convergence in the Hilbert-Schmidt norm, $\|C_{\varphi_n} - C_\varphi\|_{\mathrm{HS}} \to 0$ takes place if and only if the following conditions are satisfied. $\|\varphi_n - \varphi\|_2 \to 0$, $\int 1/(1-|\varphi|^2) < \infty$, and $\int 1/(1-|\varphi_n|^2) \to \int 1/(1-|\varphi|^2)$. The convergence of the sequence of powers of a composition operator is studied.

1. Introduction

In this paper, $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk, and $\mathbb{T}$ its boundary, the unit circle. By m we denote the normalized arc-length measure on $\mathbb{T}$. We consider the Hilbert Hardy space H^2, consisting of all analytic functions f on $\mathbb{U}$ for which

$$\|f\|_2 := \sup_{0<r<1} \left(\int_{\mathbb{T}} |f(r\zeta)|^2 \, dm(\zeta) \right)^{1/2} < \infty. \tag{1}$$

The quantity in (1) is the norm of H^2 and has the alternative description

$$\|f\|_2 = \sqrt{\sum_{n=0}^{\infty} |c_n|^2}, \tag{2}$$

where $\{c_n\}$ is the sequence of the Taylor coefficients of f.

Any H^2-function f has a radial limit function defined as follows

$$f(\zeta) = \lim_{r \to 1^-} f(r\zeta) \qquad \zeta \in \mathbb{T}.$$

It is well-known that the radial limit function is defined m-a.e. on $\mathbb{T}$. Throughout this paper, it will be denoted by the same symbol f as the function itself. The H^2-norm of any H^2-function equals the L^2-norm of its radial limit function.

The space H^∞ is the space of all bounded analytic functions f on $\mathbb{U}$ endowed with the norm

$$\|f\|_\infty := \sup_{|z|<1} |f(z)|. \tag{3}$$

Let $\mathcal{S}$ denote the subset of H^∞ consisting of the analytic selfmaps of $\mathbb{U}$. Each function $\varphi \in \mathcal{S}$, induces a bounded composition operator C_φ, defined as follows

$$C_\varphi f := f \circ \varphi \qquad f \in H^2,$$

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and referred to as *the composition operator of symbol* φ.

In this paper, our basic problem can be formulated as follows. Consider all composition operators on the Hilbert Hardy space H^2. For a sequence of symbols $\{\varphi_n\}$ assume that there is some $\varphi \in \mathcal{S}$ so that $\{\varphi_n\}$ converges in some sense to φ. We will investigate under what circumstances one can deduce that $C_{\varphi_n} \to C_\varphi$ in the sense of some usual convergence-concept for sequences of operators.

The problem under investigation is not interesting if the space $\mathcal{L}(H^2)$ of all linear bounded operators on H^2 is endowed with the uniform operator-topology, unless φ satisfies the condition

$$|\varphi(\zeta)| < 1 \quad m-\text{a.e.} \quad \text{on} \quad \mathbb{T}. \tag{4}$$

Indeed, a result by Berkson [1] states that, if $E_\varphi = \{\zeta \in \mathbb{T} : |\varphi(\zeta)| = 1\}$ is such that $m(E_\varphi) > 0$, then C_φ is an isolated element of the space of all composition operators in $\mathcal{L}(H^2)$ endowed with the uniform operator topology. Hence, we are interested mainly in symbols $\varphi \in \mathcal{S}$ with the property (4).

Howard Schwartz was the first to consider the problem of relating convergence of symbols and convergence of the induced composition operators. In [6] he settled the cases of weak and strong operator–convergence, proving that $C_{\varphi_n} \to C_\varphi$ in the weak operator topology if and only if $\varphi_n \to \varphi$ weakly in H^2, respectively that $C_{\varphi_n} \to C_\varphi$ strongly if and only if $\|\varphi_n - \varphi\|_2 \to 0$. He obtained partial results on uniform convergence too. According to [6, Theorem 4.5], $\|C_{\varphi_n} - C_\varphi\| \to 0$ if $\varphi_n \to \varphi$ a.e.,

$$\int_{\mathbb{T}} \frac{dm}{1-|\varphi_n|^2} < \infty \qquad n = 1, 2, \dots,$$

and

$$\int_{\mathbb{T}} \frac{dm}{1-|\varphi_n|^2} \to \int_{\mathbb{T}} \frac{dm}{1-|\varphi|^2} < \infty.$$

The operators involved in the theorem above are Hilbert–Schmidt composition operators. It is natural to ask if this theorem holds in the stronger sense of Hilbert–Schmidt norm–convergence.

In Section 2 we show that this is the case and, besides improving Schwartz's result, obtain a necessary and sufficient characterization of the situation when a sequence of Hilbert-Schmidt composition operators $\{C_{\varphi_n}\}$ converges in the Hilbert-Schmidt norm to some composition operator C_φ. More exactly we prove that this happens if and only if

$$\int_{\mathbb{T}} \frac{dm}{1-|\varphi|^2} < \infty, \quad \|\varphi_n - \varphi\|_2 \to 0, \quad \text{and} \quad \int_{\mathbb{T}} \frac{dm}{1-|\varphi_n|^2} \to \int_{\mathbb{T}} \frac{dm}{1-|\varphi|^2}.$$

Recent results of [5] are also extended in Section 2. In Section 3 we consider the sequence $\{C_\varphi^n\}$ for some non–inner $\varphi \in \mathcal{S}$ having a fixed point $w \in \mathbb{U}$. We prove that $\|C_\varphi^n - C_w\| \to 0$. Since C_w is a rank–one idempotent, hence, in particular, a Hilbert–Schmidt operator, it is normal to ask if the previous result can be improved to Hilbert–Schmidt norm convergence, in select cases. We show that this happens if and only if C_φ^k is Hilbert–Schmidt for some k.

In section 4 we prove the norm estimate

$$\|C_\varphi - C_\psi\| \le 2\sqrt{\int_{\mathbb{T}} \frac{|\varphi(u)-\psi(u)|}{(1-|\varphi(u)|)(1-|\psi(u)|)}\, dm(u)}$$

and deduce that $\|C_{\varphi_n} - C_\varphi\| \to 0$ if

$$\int_{\mathbb{T}} \frac{|\varphi_n - \varphi|}{(1-|\varphi|)(1-|\varphi_n|)}\, dm \to 0,$$

which relates to a result in Section 2, where it is proved that, if the condition above holds and C_φ is Hilbert–Schmidt, then C_{φ_n} tends to C_φ in the Hilbert–Schmidt norm.

As a final remark in this introductory section, we would like to observe that, if a sequence of composition operators tends weakly to an operator, then that operator too must be a composition operator. Indeed:

Remark 1 ([6]). *The set of all composition operators is weakly sequentially compact.*

2. Hilbert-Schmidt Norm Convergence

On any Hilbert space $\mathcal{H}$, the Hilbert-Schmidt norm $\|T\|_{\mathrm{HS}}$ of an operator T is defined as follows,

$$\|T\|_{\mathrm{HS}} = \sqrt{\sum_{n=0}^{\infty} \|Te_n\|^2}, \tag{5}$$

where $\{e_n\}$ is an orthonormal basis of $\mathcal{H}$. The quantity in (5) does not depend on the orthonormal basis chosen [4], thus it is larger than or equal the operator norm $\|T\|$ of T. Therefore, if we can prove that, under certain assumptions on $\{\varphi_n\}$, one has that $\|C_{\varphi_n} - C_\varphi\|_{\mathrm{HS}} \to 0$, then we can deduce $\|C_{\varphi_n} - C_\varphi\| \to 0$.

Recall that a Hilbert-space operator is called a Hilbert-Schmidt operator if it has finite Hilbert-Schmidt norm. It is well-known [7], that a composition operator C_φ on H^2 has Hilbert-Schmidt norm given by

$$\|C_\varphi\|_{\mathrm{HS}} = \sqrt{\int_{\mathbb{T}} \frac{dm}{1-|\varphi|^2}}. \tag{6}$$

Lemma 1. *The sequence $\{C_{\varphi_n}\}$ of Hilbert-Schmidt composition operators tends toward the composition operator C_φ in the Hilbert-Schmidt norm if $\varphi_n \to \varphi$, a.e. on $\mathbb{T}$,*

$$\int_{\mathbb{T}} \frac{dm}{1-|\varphi|^2} < \infty, \tag{7}$$

and

$$\int_{\mathbb{T}} \frac{dm}{1-|\varphi_n|^2} \to \int_{\mathbb{T}} \frac{dm}{1-|\varphi|^2}. \tag{8}$$

Proof. Using formula (5) and the standard orthonormal basis $\{1, z, z^2, z^3, \dots\}$ of H^2 , one gets

$$\|C_{\varphi_n} - C_\varphi\|_{\mathrm{HS}}^2 = \int_{\mathbb{T}} \left(\frac{1}{1-|\varphi_n|^2} + \frac{1}{1-|\varphi|^2} - 2\Re \frac{1}{1-\overline{\varphi_n}\varphi} \right) dm = \tag{9}$$

$$\int_{\mathbb{T}} \frac{1}{1-|\varphi_n|^2}\, dm + \int_{\mathbb{T}} \frac{1}{1-|\varphi|^2}\, dm - 2\Re \int_{\mathbb{T}} \frac{1}{1-\overline{\varphi_n}\varphi}\, dm.$$

By hypothesis,

$$\int_{\mathbb{T}} \frac{dm}{1-|\varphi_n|^2} \to \int_{\mathbb{T}} \frac{dm}{1-|\varphi|^2}.$$

Since

$$\left|\frac{1}{1-\overline{\varphi_n}\varphi}\right| \leq \frac{1}{1-|\varphi|} \qquad m-a.e.,$$

the a.e. convergence hypothesis and the dominated convergence theorem combine to show that

$$\int_{\mathbb{T}} \frac{1}{1-\overline{\varphi_n}\varphi}\, dm \to \int_{\mathbb{T}} \frac{1}{1-|\varphi|^2}\, dm.$$

By (9), it follows that $\|C_{\varphi_n} - C_\varphi\|_{\mathrm{HS}} \to 0$. □

The lemma above is an improvement in the framework of H^2 of [6, Theorem 4.5] and allows us to prove the main result of this section.

Theorem 1. *If C_φ is a Hilbert–Schmidt operator, then $\|C_{\varphi_n} - C_\varphi\|_{\mathrm{HS}} \to 0$ if and only if $\|C_{\varphi_n}\|_{\mathrm{HS}} \to \|C_\varphi\|_{\mathrm{HS}}$ and $\|\varphi_n - \varphi\|_2 \to 0$.*

Proof. First note that if $\|C_{\varphi_n} - C_\varphi\|_{\mathrm{HS}} \to 0$ then clearly $\|C_{\varphi_n}\|_{\mathrm{HS}} \to \|C_\varphi\|_{\mathrm{HS}}$ and

$$\|\varphi_n - \varphi\|_2 = \|C_{\varphi_n}(z) - C_\varphi(z)\|_2 \leq \|C_{\varphi_n} - C_\varphi\|_{\mathrm{HS}} \to 0.$$

Conversely, assume by way of contradiction that $\|C_{\varphi_n}\|_{\mathrm{HS}} \to \|C_\varphi\|_{\mathrm{HS}} < \infty$, $\|\varphi_n - \varphi\|_2 \to 0$, but $\|C_{\varphi_n} - C_\varphi\|_{\mathrm{HS}} \nrightarrow 0$. Then one can find some $\epsilon_0 > 0$ and a subsequence $\{C_{\varphi_{n_k}}\}$ such that

$$\|C_{\varphi_{n_k}} - C_\varphi\|_{\mathrm{HS}} \geq \epsilon_0. \tag{10}$$

Since, $\|\varphi_{n_k} - \varphi\|_2 \to 0$, there is a subsequence of $\{\varphi_{n_k}\}$ that converges a.e. to φ. Applying Lemma 1 to that subsequence one gets a contradiction to (10). □

Corollary 1. *Let C_φ be a Hilbert–Schmidt composition operator and $\{\varphi_n\}$ a sequence in $\mathcal{S}$. If*

$$\int_{\mathbb{T}} \frac{|\varphi_n - \varphi|}{(1-|\varphi|)(1-|\varphi_n|)}\, dm \to 0 \tag{11}$$

then $\|C_{\varphi_n} - C_\varphi\|_{\mathrm{HS}} \to 0$.

Proof. Clearly

$$\|\varphi_n - \varphi\|_1 \leq \int_{\mathbb{T}} \frac{|\varphi_n - \varphi|}{(1-|\varphi|)(1-|\varphi_n|)}\, dm \qquad n = 1, 2, 3, \ldots$$

so $\|\varphi_n - \varphi\|_2 \to 0$, since $\|\varphi_n - \varphi\|_2^2 \leq 2\|\varphi_n - \varphi\|_1$. Also

$$\left|\, \|C_{\varphi_n}\|_{\mathrm{HS}}^2 - \|C_\varphi\|_{\mathrm{HS}}^2 \right| = \left| \int_{\mathbb{T}} \frac{|\varphi_n|^2 - |\varphi|^2}{(1-|\varphi|^2)(1-|\varphi_n|^2)}\, dm \right| \leq$$

$$2\int_{\mathbb{T}} \frac{|\varphi_n - \varphi|}{(1-|\varphi|)(1-|\varphi_n|)}\, dm \to 0.$$

□

As another application, we prove the following "dominated convergence principle" for Hilbert-Schmidt norm-convergence of composition operators, which improves a result in [6], where uniform convergence is proved under assumptions slightly more restrictive that the ones below.

Theorem 2. *Let $\varphi, \varphi_n \in \mathcal{S}$, $n = 1, 2, \ldots$ If there exists a measurable function $\chi : \mathbb{T} \to [0, \infty]$ so that, for each n*

$$|\varphi_n| \leq \chi \leq 1 \qquad m - a.e.,$$

$$\int_{\mathbb{T}} \frac{dm}{1 - |\chi|^2} < \infty,$$

and $\|\varphi_n - \varphi\|_2 \to 0$, then $\|C_{\varphi_n} - C_\varphi\|_{\mathrm{HS}} \to 0$.

Proof. Observe that

$$\frac{1}{1 - |\varphi_n|^2} \leq \frac{1}{1 - |\chi|^2}, \qquad m\text{-a.e.} \qquad n = 1, 2, \ldots$$

Since $\{\varphi_n\}$ has a subsequence $\{\varphi_{n_k}\}$ that converges a.e. to φ, Lebesgue's dominated convergence theorem leads to

$$\|C_{\varphi_{n_k}}\|_{\mathrm{HS}} \to \|C_\varphi\|_{\mathrm{HS}} \leq \int_{\mathbb{T}} \frac{dm}{1 - |\chi|^2} < \infty.$$

Hence, by Theorem 1, $\|C_{\varphi_{n_k}} - C_\varphi\|_{\mathrm{HS}} \to 0$. Based on that, one can prove by contradiction that $\|C_{\varphi_n} - C_\varphi\|_{\mathrm{HS}} \to 0$ exactly as in the proof of that theorem. □

As a corollary, we obtain Theorem 1 of [5]:

Corollary 2. *The map $\varphi \to C_\varphi$ is continuous from the open unit ball of H^∞ endowed with $\| \quad \|_\infty$ into the set of Hilbert–Schmidt composition operators.*

Proof. Indeed, one may choose a positive constant r so that $\|\varphi\|_\infty + r < 1$, set $\chi := \|\varphi\|_\infty + r$, and apply the previous theorem, (which is possible, since $\|\varphi_n - \varphi\|_\infty \to 0$ implies that $\|\varphi_n\|_\infty \leq \|\varphi\|_\infty + r$ for all values of n large enough). □

Actually, in [5], the statement above is deduced as a consequence of the fact that the map $\varphi \to C_\varphi$ is Lipschitz continuous from each ball of H^∞ of radius r, $0 < r < 1$, endowed with $\| \quad \|_\infty$ into the set of Hilbert–Schmidt composition operators, (a fact the authors of [5] establish). As a last remark in this section, we would like to note that a simple upper norm estimate for the Hilbert–Schmidt norm of a difference of two composition operators proves that the map $\varphi \to C_\varphi$ is Lipschitz continuous on subsets of the unit ball of H^∞ larger than the balls above.

Remark 2. *For any pair of distinct symbols $\varphi, \psi \in \mathcal{S}$, let $\chi := \max\{|\varphi|, |\psi|\}$. The following upper estimate of $\|C_\varphi - C_\psi\|_{\mathrm{HS}}$ holds*

$$\|C_\varphi - C_\psi\|_{\mathrm{HS}} \leq \sqrt{\int_{\mathbb{T}} \frac{1 + \chi^2}{(1 - \chi^2)^3}\, dm}\, \|\varphi - \psi\|_\infty. \tag{12}$$

Hence for each $R > 0$, the map $\varphi \to C_\varphi$ is Lipschitz continuous on $\mathcal{S}_R := \{\varphi \in \mathcal{S} : \int_{\mathbb{T}} dm/(1 - |\varphi|)^3 \leq R\}$, that is there is some $M > 0$ such that

$$\|C_\varphi - C_\psi\|_{\mathrm{HS}} \leq M \|\varphi - \psi\|_\infty \qquad \varphi, \psi \in \mathcal{S}_R.$$

Proof. By [3, pp. 339], one can write

$$\|C_\psi - C_\varphi\|_{\mathrm{HS}}^2 = \int_{\mathbb{T}} \left| \frac{\psi-\varphi}{1-\overline{\psi}\varphi} \right|^2 \left(\frac{1}{1-|\varphi|^2} + \frac{1}{1-|\psi|^2} - 1 \right) dm \leq$$
$$\int_{\mathbb{T}} |\varphi - \psi|^2 \frac{1+\chi^2}{(1-\chi^2)^3}\, dm,$$

hence (12) holds. □

3. The Powers of a Composition Operator

In this section we treat the norm convergence of the operator sequence $\{C_\varphi^n\}$, where $\varphi \in \mathcal{S}$ has a fixed point $w \in \mathbb{U}$ and is not an inner function. Recall that an inner function is an analytic selfmap of $\mathbb{U}$ whose radial limit-function is unimodular m-a.e. on $\mathbb{T}$.

For each $w \in \mathbb{U}$, C_w denotes the composition operator of constant symbol w. Denote $\varphi^{[n]} = \varphi \circ \cdots \circ \varphi$, n times for each $n = 1, 2, \ldots$. Clearly $C_{\varphi^{[n]}} = C_\varphi^n$.

Theorem 3. *Let $\varphi \in \mathcal{S}$ be a non-inner symbol. If for some $w \in \mathbb{U}$, $\varphi(w) = w$, then $\|C_\varphi^n - C_w\| \to 0$.*

Proof. Assume first that $w = 0$. Let $H_0^2 = \{f \in H^2 : f(0) = 0\}$. Recall that $\|C_\varphi|H_0^2\| = \delta < 1$, [8]. Consider any $f \in H^2$, $\|f\|_2 = 1$, and note that $\|C_\varphi f - C_0 f\|_2 = \|C_\varphi(f - f(0))\|_2 \leq \delta \|f - f(0)\|_2$. Hence $\|C_\varphi^n f - C_0 f\|_2 = \|C_\varphi(f \circ \varphi^{[n-1]} - f(0))\|_2 \leq \delta \|f \circ \varphi^{[n-1]} - f(0)\|_2 = \delta \|C_\varphi^{n-1} f - C_0 f\|_2$. Iterating, one gets $\|C_\varphi^n - C_0\| \leq \delta^n \to 0$. A conformal conjugation argument takes care of the case $w \neq 0$. Indeed, consider the selfinverse conformal automorphism $\alpha_w(z) = (w-z)/(1-\overline{w}z)$, and set $\psi = \alpha_w \circ \varphi \circ \alpha_w$. Note that $\psi(0) = 0$, hence $\|C_\psi^m - C_\psi^n\| \to 0$ if $m, n \to \infty$. This fact implies $\|C_\varphi^m - C_\varphi^n\| \to 0$ if $m, n \to \infty$. Indeed, for each k one has $C_\varphi^k = C_{\alpha_w} C_\psi^k C_{\alpha_w}$ and hence $\|C_\varphi^m - C_\varphi^n\| \leq \|C_{\alpha_w}\|^2 \|C_\psi^m - C_\psi^n\|$, $m, n = 1, 2, \ldots$ We established that the sequence $\{C_\varphi^n\}$ is norm-convergent. Let T denote its limit. It is well-known that $\varphi^{[n]} \to w$ uniformly on compact subsets of $\mathbb{U}$, hence also weakly in H^2, (see [7, the Denjoy–Wolff Theorem]). Since $C_\varphi^n(z) = \varphi^{[n]}$, it follows by Remark 1 that $T = C_w$. □

The argument used to prove Theorem 3, occurs, with minor changes, in [2]. There it is used to show that the iterates of φ converge to w in the H^2 norm. We included the proof of Theorem 3 for the sake of completeness.

If φ is inner, then for any n: $\|C_\varphi^n - C_w\| \geq \|(C_\varphi^n - C_w)(z)\| \geq 1 - |w|$, so that Theorem 3 cannot be extended to this case. In fact, by a result of Berkson ([1], see also [9]) $\|C_\varphi^n - C_w\| \geq 1$ in this case.

The situation when $\|C_\varphi^n - C_w\|_{\mathrm{HS}} \to 0$ is characterized in the following.

Theorem 4. *Let φ be a non-inner function with a fixed point w in $\mathbb{U}$. Then $\|C_\varphi^n - C_w\|_{\mathrm{HS}} \to 0$ if and only if there is some positive integer k such that C_φ^k is a Hilbert-Schmidt operator.*

Proof. The necessity is evident, given that obviously C_w is Hilbert-Schmidt. To prove the sufficiency, assume first that $w = 0$. Note that $\varphi^{[n]} \to 0$, m-a.e. Indeed, using the notation in the proof of Theorem 3, observe that, in that proof, we obtained that

$$\|C_\varphi^n - C_0\| \le \delta^n \qquad n = 1, 2, 3, \ldots$$

hence

$$\sum_{n=1}^{\infty} \| \left(C_\varphi^n - C_0\right)(z)\|_2^2 < \infty,$$

that is

$$\sum_{n=1}^{\infty} \|\varphi^{[n]}\|_2^2 < \infty,$$

so, by Lebesgue's monotone convergence theorem,

$$\int_{\mathbb{T}} \left(\sum_{n=1}^{\infty} |\varphi^{[n]}|^2\right) dm < \infty,$$

which implies $\varphi^{[n]} \to 0$, m-a.e.

Now, by the Schwarz lemma in classical complex analysis,

$$|\varphi^{[n]}| \le |\varphi^{[k]}| \quad m-\text{a.e.} \quad n \ge k$$

so, setting $\chi := |\varphi^{[k]}|$ in Theorem 2 leads to the desired conclusion when $w = 0$. A standard conformal conjugation argument takes care of the general case like in the proof of Theorem 3. Indeed, for w arbitrary, one can associate to φ the conformal conjugate ψ as in that proof and note that $\|C_\varphi^m - C_\varphi^n\|_{\text{HS}} \le \|C_{\alpha_w}\|^2 \|C_\psi^m - C_\psi^n\|_{\text{HS}}$, $m, n = 1, 2, \ldots$, by [4, pp. 1012, Corollary 5]. Thus, by the first part of this proof, the sequence $\{C_{\varphi_n}\}$ tends to an operator T in the Hilbert–Schmidt norm. One shows that $T = C_w$ exactly as in the proof of Theorem 3. □

In the argument above we needed the fact that, if φ fixes a point w in $\mathbb{U}$ and is not an inner function, then its iterates tend a.e. to w. This was first established in [2]. For the sake of the self-sufficiency of the current paper, we decided to include the proof, rather than just refer the reader to [2].

The situation when the assumptions in Theorem 3 hold but those in Theorem 4 don't, may occur, as we show in the following.

Example 1. *Let $\varphi(z) = (z^3 + 1)/2$. This symbol satisfies the assumptions in* Theorem 3*, hence there is $w \in \mathbb{U}$ so that $\|C_\varphi^n - C_w\| \to 0$, but $\|C_\varphi^n - C_w\|_{\text{HS}} \nrightarrow 0$.*

Proof. Clearly φ is not inner. Indeed, by the triangle inequality, $|(z^3 + 1)/2| \le 1$, for all z in the closed unit disk, and equality occurs only if z is a cube root of 1. The fixed points of φ are the zeros of $z^3 - 2z + 1$, a polynomial that is real on the real line, positive at .5 and negative at .7. Therefore, φ has a fixed point $w \in \mathbb{U}$ and hence satisfies the assumptions in Theorem 3. On the other hand, all the iterates of φ have finite angular derivatives at 1. Thus, C_φ^n is not compact, $n = 1, 2, \ldots$, (see [3] or [7]), and hence, C_φ^n cannot be Hilbert-Schmidt. □

By Remark 1, the power-sequence $\{C_\varphi^n\}$ of a composition operator with symbol without fixed points in $\mathbb{U}$ is weakly divergent, since in that case, there is a unimodular constant function ω toward which $\{\varphi^{[n]}\}$ tends weakly, (by the Denjoy–Wolff theorem, [3], [7]).

4. Uniform Convergence

In this section we establish an upper norm estimate for the norm of a difference of two composition operators and show that if condition (11) holds, but one drops the requirement that C_φ be Hilbert–Schmidt, one can still prove that $\|C_{\varphi_n} - C_\varphi\| \to 0$. We begin with the norm estimate.

Theorem 5. *For any $\varphi, \psi \in \mathcal{S}$ the following inequality holds*

$$\|C_\varphi - C_\psi\| \le 2\sqrt{\int_{\mathbb{T}} \frac{|\varphi(u) - \psi(u)|}{(1-|\varphi(u)|)(1-|\psi(u)|)}\, dm(u)}. \tag{13}$$

Proof. First we prove a simple inequality involving the usual Poisson kernel $P(z, \zeta)$, $z \in \mathbb{U}$, $\zeta \in \mathbb{T}$, namely

$$|P(z,\zeta) - P(w,\zeta)| \le 2\frac{|z-w|}{|\zeta - z||\zeta - w|}, \qquad z, w \in \mathbb{U}, \zeta \in \mathbb{T}.$$

Indeed,

$$|P(z,\zeta) - P(w,\zeta)| = \left|\mathrm{Re}\left(\frac{\zeta+z}{\zeta-z} - \frac{\zeta+w}{\zeta-w}\right)\right| \le \left|\frac{\zeta+z}{\zeta-z} - \frac{\zeta+w}{\zeta-w}\right| = 2\frac{|z-w|}{|\zeta-z||\zeta-w|}.$$

Next, note that the above inequality can be used to show that

$$\begin{aligned} |f(z) - f(w)|^2 &\le 4|z-w| \sup_{\zeta \in \mathbb{T}} \left(\frac{1}{|\zeta - z|\,|\zeta - w|}\right) \|f\|_2^2 \\ &\le \frac{4|z-w|\|f\|_2^2}{(1-|z|)(1-|w|)} \qquad z, w \in \mathbb{U}, f \in H^2. \end{aligned} \tag{14}$$

Indeed, using the Cauchy-Schwartz inequality,

$$\begin{aligned} |f(z) - f(w)|^2 &\le \left(\int_{\mathbb{T}} |P(z,\zeta) - P(w,\zeta)|\, |f(\zeta)|\, dm(\zeta)\right)^2 \\ &\le \int_{\mathbb{T}} |P(z,\zeta) - P(w,\zeta)|^2 \; dm(\zeta) \|f\|_2^2 \\ &\le \sup_{\zeta\in\mathbb{T}} |P(z,\zeta) - P(w,\zeta)| \int_{\mathbb{T}} |P(z,\zeta) - P(w,\zeta)| \; dm(\zeta)\|f\|_2^2 \\ &\le 4|z-w| \sup_{\zeta\in\mathbb{T}} \left(\frac{1}{|\zeta - z|\,|\zeta - w|}\right) \|f\|_2^2 \\ &\le \frac{4|z-w|\|f\|_2^2}{(1-|z|)(1-|w|)} \qquad z, w \in \mathbb{U}, f \in H^2. \end{aligned}$$

Substitute z by $\varphi(u)$, w by $\psi(u)$, and integrate $dm(u)$ to obtain (13). □

Corollary 3. *If condition* (11) *holds then* $\|C_{\varphi_n} - C_\varphi\| \to 0$.

Clearly, inequality (13) is interesting only if $\varphi \ne \psi$, $|\varphi| < 1$, and $|\psi| < 1$, m-a.e. Indeed, the integral involved in it is infinite if $\varphi \ne \psi$ and any of these functions has unimodular radial function on a measurable subset of $\mathbb{T}$ having positive measure.

The paper [9] contains an upper norm–estimate for the difference of two composition operators. The methods used in [9, Theorem 3.2] can be adapted to show that, if the integral in estimate (13) is finite, then the operator $C_\varphi - C_\psi$ must be compact.

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