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CONVERGENT SEQUENCES OF COMPOSITION OPERATORS

VALENTIN MATACHE

Abstract. Composition operators $C_\varphi$ on the Hilbert Hardy space $H^2$ over the unit disk are considered. We investigate when convergence of sequences $\{\varphi_n\}$ of symbols, (i.e. of analytic selfmaps of the unit disk) towards a given symbol $\varphi$, implies the convergence of the induced composition operators, $C_{\varphi_n} \to C_\varphi$. If the composition operators $C_{\varphi_n}$ are Hilbert-Schmidt operators, we prove that convergence in the Hilbert-Schmidt norm, $\|C_{\varphi_n} - C_\varphi\|_{HS} \to 0$ takes place if and only if the following conditions are satisfied. $\|\varphi_n - \varphi\|_2 \to 0$, $\int 1/(1 - |\varphi|^2) < \infty$, and $\int 1/(1 - |\varphi_n|^2) \to \int 1/(1 - |\varphi|^2)$. The convergence of the sequence of powers of a composition operator is studied.

1. Introduction

In this paper, $U := \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk, and $T$ its boundary, the unit circle. By $m$ we denote the normalized arc-length measure on $T$. We consider the Hilbert Hardy space $H^2$, consisting of all analytic functions $f$ on $U$ for which

$$\|f\|_2 := \sup_{0<r<1} \left( \int_T |f(r\zeta)|^2 \, dm(\zeta) \right)^{1/2} < \infty. \quad (1)$$

The quantity in (1) is the norm of $H^2$ and has the alternative description

$$\|f\|_2 = \left( \sum_{n=0}^{\infty} |c_n|^2 \right)^{1/2}, \quad (2)$$

where $\{c_n\}$ is the sequence of the Taylor coefficients of $f$.

Any $H^2$-function $f$ has a radial limit function defined as follows

$$f(\zeta) = \lim_{r \to 1^-} f(r\zeta) \quad \zeta \in T.$$ 

It is well-known that the radial limit function is defined $m$-a.e. on $T$. Throughout this paper, it will be denoted by the same symbol $f$ as the function itself. The $H^2$-norm of any $H^2$-function equals the $L^2$-norm of its radial limit function.

The space $H^\infty$ is the space of all bounded analytic functions $f$ on $U$ endowed with the norm

$$\|f\|_\infty := \sup_{|z|<1} |f(z)|. \quad (3)$$

Let $S$ denote the subset of $H^\infty$ consisting of the analytic selfmaps of $U$. Each function $\varphi \in S$, induces a bounded composition operator $C_\varphi$, defined as follows

$$C_\varphi f := f \circ \varphi \quad f \in H^2,$$
and referred to as the composition operator of symbol $\varphi$.

In this paper, our basic problem can be formulated as follows. Consider all composition operators on the Hilbert Hardy space $H^2$. For a sequence of symbols $\{\varphi_n\}$ assume that there is some $\varphi \in \mathcal{S}$ so that $\{\varphi_n\}$ converges in some sense to $\varphi$. We will investigate under what circumstances one can deduce that $C_{\varphi_n} \to C_\varphi$ in the sense of some usual convergence-concept for sequences of operators.

The problem under investigation is not interesting if the space $L(H^2)$ of all linear bounded operators on $H^2$ is endowed with the uniform operator-topology, unless $\varphi$ satisfies the condition

\[ |\varphi(\zeta)| < 1 \quad m - \text{a.e. on } T. \]

Indeed, a result by Berkson [1] states that, if $E_\varphi = \{\zeta \in T : |\varphi(\zeta)| = 1\}$ is such that $m(E_\varphi) > 0$, then $C_\varphi$ is an isolated element of the space of all composition operators in $L(H^2)$ endowed with the uniform operator topology. Hence, we are interested mainly in symbols $\varphi \in \mathcal{S}$ with the property (4).

Howard Schwartz was the first to consider the problem of relating convergence of symbols and convergence of the induced composition operators. In [6] he settled the cases of weak and strong operator–convergence, proving that $C_{\varphi_n} \to C_\varphi$ in the weak operator topology if and only if $\varphi_n \to \varphi$ weakly in $H^2$, respectively that $C_{\varphi_n} \to C_\varphi$ strongly if and only if $\|\varphi_n - \varphi\|_2 \to 0$. He obtained partial results on uniform convergence too. According to [6, Theorem 4.5], $\|C_{\varphi_n} - C_\varphi\| \to 0$ if $\varphi_n \to \varphi$ a.e.,

\[ \int_T \frac{dm}{1 - |\varphi_n|^2} < \infty \quad n = 1, 2, \ldots, \]

and

\[ \int_T \frac{dm}{1 - |\varphi_n|^2} \to \int_T \frac{dm}{1 - |\varphi|^2} < \infty. \]

The operators involved in the theorem above are Hilbert–Schmidt composition operators. It is natural to ask if this theorem holds in the stronger sense of Hilbert–Schmidt norm–convergence.

In Section 2 we show that this is the case and, besides improving Schwartz’s result, obtain a necessary and sufficient characterization of the situation when a sequence of Hilbert-Schmidt composition operators $\{C_{\varphi_n}\}$ converges in the Hilbert Schmidt norm to some composition operator $C_\varphi$. More exactly we prove that this happens if and only if $\varphi_n \to \varphi$ a.e.,

\[ \int_T \frac{dm}{1 - |\varphi|^2} < \infty, \quad \|\varphi_n - \varphi\|_2 \to 0, \quad \text{and} \quad \int_T \frac{dm}{1 - |\varphi_n|^2} \to \int_T \frac{dm}{1 - |\varphi|^2}. \]

Recent results of [5] are also extended in Section 2. In Section 3 we consider the sequence $\{C^n_w\}$ for some non–inner $\varphi \in \mathcal{S}$ having a fixed point $w \in \mathcal{U}$. We prove that $\|C^n_w - C_w\| \to 0$. Since $C_w$ is a rank–one idempotent, hence, in particular, a Hilbert–Schmidt operator, it is normal to ask if the previous result can be improved to Hilbert–Schmidt norm convergence, in select cases. We show that this happens if and only if $C^n_w$ is Hilbert–Schmidt for some $k$.

In section 4 we prove the norm estimate

\[ \|C_\varphi - C_\psi\| \leq 2 \sqrt{\int_T \frac{|\varphi(u) - \psi(u)|}{(1 - |\varphi(u)|)(1 - |\psi(u)|)} dm(u)}. \]
and deduce that \( \| C_{\varphi_n} - C_{\varphi} \| \to 0 \) if
\[
\int_{\mathbb{T}} \frac{|\varphi_n - \varphi|}{(1 - |\varphi|)(1 - |\varphi_n|)} \, dm \to 0,
\]
which relates to a result in Section 2, where it is proved that, if the condition above holds and \( C_{\varphi} \) is Hilbert–Schmidt, then \( C_{\varphi_n} \) tends to \( C_{\varphi} \) in the Hilbert–Schmidt norm.

As a final remark in this introductory section, we would like to observe that, if a sequence of composition operators tends weakly to an operator, then that operator too must be a composition operator. Indeed:

**Remark 1 ( [6])**. The set of all composition operators is weakly sequentially compact.

### 2. Hilbert-Schmidt Norm Convergence

On any Hilbert space \( \mathcal{H} \), the Hilbert-Schmidt norm \( \| T \|_{\text{HS}} \) of an operator \( T \) is defined as follows,

\[
\| T \|_{\text{HS}} = \sqrt{\sum_{n=0}^{\infty} \| T e_n \|^2},
\]

where \( \{ e_n \} \) is an orthonormal basis of \( \mathcal{H} \). The quantity in (5) does not depend on the orthonormal basis chosen [4], thus it is larger than or equal the operator norm \( \| T \| \) of \( T \). Therefore, if we can prove that, under certain assumptions on \( \{ \varphi_n \} \), one has that \( \| C_{\varphi_n} - C_{\varphi} \|_{\text{HS}} \to 0 \), then we can deduce \( \| C_{\varphi_n} - C_{\varphi} \| \to 0 \).

Recall that a Hilbert-space operator is called a Hilbert-Schmidt operator if it has finite Hilbert-Schmidt norm. It is well-known [7], that a composition operator \( C_{\varphi} \) on \( H^2 \) has Hilbert-Schmidt norm given by

\[
\| C_{\varphi} \|_{\text{HS}} = \sqrt{\int_{\mathbb{T}} \frac{dm}{1 - |\varphi|^2}}.
\]

**Lemma 1.** The sequence \( \{ C_{\varphi_n} \} \) of Hilbert-Schmidt composition operators tends toward the composition operator \( C_{\varphi} \) in the Hilbert-Schmidt norm if \( \varphi_n \to \varphi \), a.e. on \( \mathbb{T} \),

\[
\int_{\mathbb{T}} \frac{dm}{1 - |\varphi|^2} < \infty,
\]

and

\[
\int_{\mathbb{T}} \frac{dm}{1 - |\varphi_n|^2} \to \int_{\mathbb{T}} \frac{dm}{1 - |\varphi|^2}.
\]

**Proof.** Using formula (5) and the standard orthonormal basis \( \{ 1, z, z^2, z^3, \ldots \} \) of \( H^2 \), one gets

\[
\| C_{\varphi_n} - C_{\varphi} \|_{\text{HS}}^2 = \int_{\mathbb{T}} \left( \frac{1}{1 - |\varphi_n|^2} + \frac{1}{1 - |\varphi|^2} - 2 \Re \frac{1}{1 - \overline{\varphi_n} \varphi} \right) \, dm = \int_{\mathbb{T}} \frac{1}{1 - |\varphi_n|^2} \, dm + \int_{\mathbb{T}} \frac{1}{1 - |\varphi|^2} \, dm - 2 \Re \int_{\mathbb{T}} \frac{1}{1 - \overline{\varphi_n} \varphi} \, dm.
\]
By hypothesis, 
\[ \int_T \frac{dm}{1 - |\varphi_n|^2} \to \int_T \frac{dm}{1 - |\varphi|^2}. \]

Since 
\[ \left| \frac{1}{1 - \varphi_n \varphi} \right| \leq \frac{1}{1 - |\varphi|^2} \quad m - a.e., \]
the a.e. convergence hypothesis and the dominated convergence theorem combine to show that 
\[ \int_T \frac{1}{1 - \varphi_n \varphi} dm \to \int_T \frac{1}{1 - |\varphi|^2} dm. \]

By (9), it follows that 
\[ \| C_{\varphi_n} - C_{\varphi} \|_{HS} \to 0. \]

The lemma above is an improvement in the framework of $H^2$ of [6, Theorem 4.5] and allows us to prove the main result of this section.

**Theorem 1.** If $C_\varphi$ is a Hilbert–Schmidt operator, then $\| C_{\varphi_n} - C_\varphi \|_{HS} \to 0$ if and only if $\| C_{\varphi_n} \|_{HS} \to \| C_\varphi \|_{HS}$ and $\| \varphi_n - \varphi \|_2 \to 0$.

**Proof.** First note that if $\| C_{\varphi_n} - C_\varphi \|_{HS} \to 0$ then clearly $\| C_{\varphi_n} \|_{HS} \to \| C_\varphi \|_{HS}$ and $\| \varphi_n - \varphi \|_2 = \| C_{\varphi_n}(z) - C_\varphi(z) \|_2 \leq \| C_{\varphi_n} - C_\varphi \|_{HS} \to 0$.

Conversely, assume by way of contradiction that $\| C_{\varphi_n} \|_{HS} \to \| C_\varphi \|_{HS} < \infty$, $\| \varphi_n - \varphi \|_2 \to 0$, but $\| C_{\varphi_n} - C_\varphi \|_{HS} \not\to 0$. Then one can find some $\epsilon_0 > 0$ and a subsequence \{C_{\varphi_{n_k}}\} such that
\[ \| C_{\varphi_{n_k}} - C_\varphi \|_{HS} \geq \epsilon_0. \]

Since, $\| \varphi_{n_k} - \varphi \|_2 \to 0$, there is a subsequence of \{\varphi_{n_k}\} that converges a.e. to $\varphi$. Applying Lemma 1 to that subsequence one gets a contradiction to (10).

**Corollary 1.** Let $C_\varphi$ be a Hilbert–Schmidt composition operator and \{\varphi_n\} a sequence in $S$. If
\[ \int_T \frac{|\varphi_n - \varphi|}{(1 - |\varphi|)(1 - |\varphi_n|)} dm \to 0 \]
then $\| C_{\varphi_n} - C_\varphi \|_{HS} \to 0$.

**Proof.** Clearly
\[ \| \varphi_n - \varphi \|_1 \leq \int_T \frac{|\varphi_n - \varphi|}{(1 - |\varphi|)(1 - |\varphi_n|)} dm \quad n = 1, 2, 3, \ldots \]
so $\| \varphi_n - \varphi \|_2 \to 0$, since $\| \varphi_n - \varphi \|_2^2 \leq 2\| \varphi_n - \varphi \|_1$. Also
\[ \| C_{\varphi_n} \|_{HS}^2 - \| C_{\varphi} \|_{HS}^2 = \left| \int_T \frac{|\varphi_n|^2 - |\varphi|^2}{(1 - |\varphi|^2)(1 - |\varphi_n|^2)} dm \right| \leq 2 \int_T \frac{|\varphi_n - \varphi|^2}{(1 - |\varphi|^2)(1 - |\varphi_n|^2)} dm \to 0. \]

As another application, we prove the following "dominated convergence principle" for Hilbert-Schmidt norm-convergence of composition operators, which improves a result in [6], where uniform convergence is proved under assumptions slightly more restrictive that the ones below.
Theorem 2. Let $\varphi, \varphi_n \in \mathcal{S}$, $n = 1, 2, \ldots$. If there exists a measurable function $\chi : \mathbb{T} \to [0, \infty]$ so that, for each $n$

$$|\varphi_n| \leq \chi \leq 1 \quad m - a.e.,$$

$$\int_{\mathbb{T}} \frac{dm}{1 - |\chi|^2} < \infty,$$

and $\|\varphi_n - \varphi\|_2 \to 0$, then $\|C_{\varphi_n} - C_{\varphi}\|_{HS} \to 0$.

Proof. Observe that

$$\frac{1}{1 - |\varphi_n|^2} \leq \frac{1}{1 - |\chi|^2}, \quad m - a.e. \quad n = 1, 2, \ldots$$

Since $\{\varphi_n\}$ has a subsequence $\{\varphi_{n_k}\}$ that converges a.e. to $\varphi$, Lebesgue’s dominated convergence theorem leads to

$$\|C_{\varphi_{n_k}}\|_{HS} \to \|C_{\varphi}\|_{HS} \leq \int_{\mathbb{T}} \frac{dm}{1 - |\chi|^2} < \infty.$$

Hence, by Theorem 1, $\|C_{\varphi_{n_k}} - C_{\varphi}\|_{HS} \to 0$. Based on that, one can prove by contradiction that $\|C_{\varphi_n} - C_{\varphi}\|_{HS} \to 0$ exactly as in the proof of that theorem.

As a corollary, we obtain Theorem 1 of [5]:

Corollary 2. The map $\varphi \to C_{\varphi}$ is continuous from the open unit ball of $H^\infty$ endowed with $\|\|_\infty$ into the set of Hilbert–Schmidt composition operators.

Proof. Indeed, one may choose a positive constant $r$ so that $\|\varphi\|_\infty + r < 1$, set $\chi := \|\varphi\|_\infty + r$, and apply the previous theorem, (which is possible, since $\|\varphi_n - \varphi\|_\infty \to 0$ implies that $\|\varphi_n\|_\infty \leq \|\varphi\|_\infty + r$ for all values of $n$ large enough).

Actually, in [5], the statement above is deduced as a consequence of the fact that the map $\varphi \to C_{\varphi}$ is Lipschitz continuous from each ball of $H^\infty$ of radius $r$, $0 < r < 1$, endowed with $\|\|_\infty$ into the set of Hilbert–Schmidt composition operators, (a fact the authors of [5] establish). As a last remark in this section, we would like to note that a simple upper norm estimate for the Hilbert–Schmidt norm of a difference of two composition operators proves that the map $\varphi \to C_{\varphi}$ is Lipschitz continuous on subsets of the unit ball of $H^\infty$ larger than the balls above.

Remark 2. For any pair of distinct symbols $\varphi, \psi \in \mathcal{S}$, let $\chi := \max\{|\varphi|, |\psi|\}$. The following upper estimate of $\|C_{\varphi} - C_{\psi}\|_{HS}$ holds

$$\|C_{\varphi} - C_{\psi}\|_{HS} \leq \sqrt{\int_{\mathbb{T}} \frac{1 + \chi^2}{(1 - \chi^2)^3} dm} \|\varphi - \psi\|_\infty. \quad (12)$$

Hence for each $R > 0$, the map $\varphi \to C_{\varphi}$ is Lipschitz continuous on $\mathcal{S}_R := \{\varphi \in \mathcal{S} : \int_{\mathbb{T}} dm/(1 - |\varphi|)^3 \leq R\}$, that is there is some $M > 0$ such that

$$\|C_{\varphi} - C_{\psi}\|_{HS} \leq M \|\varphi - \psi\|_\infty \quad \varphi, \psi \in \mathcal{S}_R.$$
Proof. By [3, pp. 339], one can write
\[
\|C_\psi - C_\varphi\|_{HS}^2 = \int_T \left| \frac{\psi - \varphi}{1 - \overline{\varphi}\psi} \right|^2 \left( \frac{1}{1 - |\varphi|^2} + \frac{1}{1 - |\psi|^2} - 1 \right) dm \leq 
\int_T |\varphi - \psi|^2 \frac{1 + |\chi|^2}{(1 - |\chi|^2)^3} dm,
\]
hence (12) holds.

\[\square\]

3. The Powers of a Composition Operator

In this section we treat the norm convergence of the operator sequence \(\{C^m_\varphi\}\), where \(\varphi \in S\) has a fixed point \(w \in U\) and is not an inner function. Recall that an inner function is an analytic selfmap of \(U\) whose radial limit-function is unimodular \(m\text{-a.e.}\) on \(T\).

For each \(w \in U\), \(C_w\) denotes the composition operator of constant symbol \(w\). Denote \(\varphi^{[n]} = \varphi \circ \cdots \circ \varphi\), \(n\) times for each \(n = 1, 2, \ldots\). Clearly \(C_{\varphi^{[n]}} = C^n_\varphi\).

Theorem 3. Let \(\varphi \in S\) be a non-inner symbol. If for some \(w \in U\), \(\varphi(w) = w\), then \(\|C^n_\varphi - C_nw\| \to 0\).

Proof. Assume first that \(w = 0\). Let \(H^2_0 = \{f \in H^2 : f(0) = 0\}\). Recall that \(\|C_\varphi|_{H^2_0}\| = \delta < 1, [8]\). Consider any \(f \in H^2\), \(\|f\|_2 = 1\), and note that \(\|C_\varphi f - C_0 f\|_2 = \|C_\varphi(f - f(0))\|_2 \leq \delta \|f - f(0)\|_2\). Hence \(\|C^n_\varphi f - C_0 f\|_2 = \|C^n_\varphi(f \circ \varphi^{[n-1]} - f(0))\|_2 \leq \delta \|f \circ \varphi^{[n-1]} - f(0)\|_2 = \delta \|C^n_\varphi f - C_0 f\|_2\). Iterating, one gets \(\|C^n_\varphi f - C_0 f\| \leq \delta^n \to 0\). A conformal conjugation argument takes care of the case \(w \neq 0\). Indeed, consider the selfinverse conformal automorphism \(\omega_w(z) = (w - z)/(1 - \overline{w}z)\), and set \(\psi = \alpha_w \circ \varphi \circ \alpha_w\). Note that \(\psi(0) = 0\), hence \(\|C^n_\varphi - C^n_\psi\| \to 0\) if \(m, n \to \infty\). This fact implies \(\|C^n_\varphi - C^n_\psi\| \to 0\) if \(m, n \to \infty\). Indeed, for each \(k\) one has \(C^n_{\varphi^k} = C_\varphi^k C^n_\psi C_\varphi^k\) and hence \(\|C^n_\varphi - C^n_\psi\| \leq \|C_\varphi^k\| \|C^n_\psi\|, m = 1, 2, \ldots\). We established that the sequence \(\{C^n_\varphi\}\) is norm-convergent. Let \(T\) denote its limit. It is well-known that \(\varphi^{[n]} \to w\) uniformly on compact subsets of \(U\), hence also weakly in \(H^2\), (see [7, the Denjoy–Wolff Theorem]). Since \(C^n_\varphi(z) = \varphi^{[n]}\), it follows by Remark 1 that \(T = C_w\).

\[\square\]

The argument used to prove Theorem 3, occurs, with minor changes, in [2]. There it is used to show that the iterates of \(\varphi\) converge to \(w\) in the \(H^2\) norm. We included the proof of Theorem 3 for the sake of completeness.

If \(\varphi\) is inner, then for any \(n\): \(\|C^n_\varphi - C_w\| \geq \|(C^n_\varphi - C_w)(z)\| \geq 1 - |w|\), so that Theorem 3 cannot be extended to this case. In fact, by a result of Berkson ([1], see also [9]) \(\|C^n_\varphi - C_w\| \geq 1\) in this case.

The situation when \(\|C^n_\varphi - C_w\|_{HS} \to 0\) is characterized in the following.

Theorem 4. Let \(\varphi\) be a non-inner function with a fixed point \(w\) in \(U\). Then \(\|C^n_\varphi - C_w\|_{HS} \to 0\) if and only if there is some positive integer \(k\) such that \(C^k_\varphi\) is a Hilbert-Schmidt operator.

Proof. The necessity is evident, given that obviously \(C_w\) is Hilbert-Schmidt. To prove the sufficiency, assume first that \(w = 0\). Note that \(\varphi^{[n]} \to 0, m\text{-a.e.}\) Indeed,
using the notation in the proof of Theorem 3, observe that, in that proof, we obtained that
\[ \| C_n^\varphi - C_0 \| \leq \delta^n \quad n = 1, 2, 3, \ldots \]
hence
\[ \sum_{n=1}^{\infty} \| (C_n^\varphi - C_0)(z) \|^2 < \infty, \]
that is
\[ \sum_{n=1}^{\infty} \| \varphi^{[n]} \|^2 < \infty, \]
so, by Lebesgue’s monotone convergence theorem,
\[ \int_T \left( \sum_{n=1}^{\infty} |\varphi^{[n]}|^2 \right) dm < \infty, \]
which implies \( \varphi^{[n]} \to 0, \ m \)-a.e.

Now, by the Schwarz lemma in classical complex analysis,
\[ |\varphi^{[n]}| \leq |\varphi^{[k]}| \quad m - a.e. \quad n \geq k \]
so, setting \( \chi := |\varphi^{[k]}| \) in Theorem 2 leads to the desired conclusion when \( w = 0 \). A standard conformal conjugation argument takes care of the general case like in the proof of Theorem 3. Indeed, for \( w \) arbitrary, one can associate to \( \varphi \) the conformal conjugate \( \psi \) as in that proof and note that \( \| C_n^\varphi - C_0 \|_{\text{HS}} \leq \| C_n^\psi - C_0 \|_{\text{HS}}, \)
\( m, n = 1, 2, \ldots, \) by [4, pp. 1012, Corollary 5]. Thus, by the first part of this proof, the sequence \( \{ C_n^\varphi \} \) tends to an operator \( T \) in the Hilbert–Schmidt norm. One shows that \( T = C_w \) exactly as in the proof of Theorem 3. \( \square \)

In the argument above we needed the fact that, if \( \varphi \) fixes a point \( w \) in \( U \) and is not an inner function, then its iterates tend \( \text{a.e.} \) to \( w \). This was first established in [2]. For the sake of the self-sufficiency of the current paper, we decided to include the proof, rather than just refer the reader to [2].

The situation when the assumptions in Theorem 3 hold but those in Theorem 4 don’t, may occur, as we show in the following.

**Example 1.** Let \( \varphi(z) = (z^3 + 1)/2 \). This symbol satisfies the assumptions in Theorem 3, hence there is \( w \in U \) so that \( \| C_n^\varphi - C_w \| \to 0, \) but \( \| C_n^\varphi - C_w \|_{\text{HS}} \to 0. \)

**Proof.** Clearly \( \varphi \) is not inner. Indeed, by the triangle inequality, \( |(z^3 + 1)/2| \leq 1, \)
for all \( z \) in the closed unit disk, and equality occurs only if \( z \) is a cube root of 1.
The fixed points of \( \varphi \) are the zeros of \( z^3 - 2z + 1, \) a polynomial that is real on the real line, positive at .5 and negative at .7. Therefore, \( \varphi \) has a fixed point \( w \in U \) and hence satisfies the assumptions in Theorem 3. On the other hand, all the iterates of \( \varphi \) have finite angular derivatives at 1. Thus, \( C_n^\varphi \) is not compact, \( n = 1, 2, \ldots, \)
(see [3] or [7]), and hence, \( C_n^\varphi \) cannot be Hilbert-Schmidt. \( \square \)

By Remark 1, the power-sequence \( \{ C_n^\varphi \} \) of a composition operator with symbol without fixed points in \( U \) is weakly divergent, since in that case, there is a unimodular constant function \( \omega \) toward which \( \{ \varphi^{[n]} \} \) tends weakly, (by the Denjoy–Wolff theorem, [3], [7]).
4. Uniform Convergence

In this section we establish an upper norm estimate for the norm of a difference of two composition operators and show that if condition (11) holds, but one drops the requirement that $C_{\varphi}$ be Hilbert–Schmidt, one can still prove that $\|C_{\varphi_n} - C_{\varphi}\| \to 0$. We begin with the norm estimate.

**Theorem 5.** For any $\varphi, \psi \in S$ the following inequality holds

\[
\|C_{\varphi} - C_{\psi}\| \leq 2\sqrt{\int_{\mathbb{T}} \frac{|\varphi(u) - \psi(u)|}{(1 - |\varphi(u)|)(1 - |\psi(u)|)} \, dm(u)}.
\]

**Proof.** First we prove a simple inequality involving the usual Poisson kernel $P(z, \xi)$, $z \in \mathbb{U}$, $\xi \in \mathbb{T}$, namely

\[
|P(z, \xi) - P(w, \xi)| \leq 2 \frac{|z - w|}{|\xi - z||\xi - w|}, \quad z, w \in \mathbb{U}, \xi \in \mathbb{T}.
\]

Indeed,

\[
|P(z, \xi) - P(w, \xi)| = \left| \text{Re} \left( \frac{\frac{z + \xi}{\xi - z} - \frac{w + \xi}{\xi - w}}{\frac{z - w}{\xi - z}} \right) \right| \leq 2 \frac{|z - w|}{|\xi - z||\xi - w|}.
\]

Next, note that the above inequality can be used to show that

\[
|f(z) - f(w)|^2 \leq 4|z - w| \sup_{\xi \in \mathbb{T}} \left( \frac{1}{|\xi - z||\xi - w|} \right) \|f\|_2^2
\]

\[
\leq 4|z - w| \|f\|_2^2 \left( \frac{1}{(1 - |z|)(1 - |w|)} \right), \quad z, w \in \mathbb{U}, f \in H^2.
\]

Indeed, using the Cauchy-Schwartz inequality,

\[
|f(z) - f(w)|^2 \leq \left( \int_{\mathbb{T}} |P(z, \xi) - P(w, \xi)| \|f(\xi)\| \, dm(\xi) \right)^2
\]

\[
\leq \int_{\mathbb{T}} |\xi| (P(z, \xi) - P(w, \xi)|^2 \, dm(\xi) \|f\|_2^2
\]

\[
\leq \sup_{\xi \in \mathbb{T}} |P(z, \xi) - P(w, \xi)| \int_{\mathbb{T}} |P(z, \xi) - P(w, \xi)| \, dm(\xi) \|f\|_2^2
\]

\[
\leq 4|z - w| \sup_{\xi \in \mathbb{T}} \left( \frac{1}{|\xi - z||\xi - w|} \right) \|f\|_2^2
\]

\[
\leq 4|z - w| \left( \frac{1}{(1 - |z|)(1 - |w|)} \right) \quad z, w \in \mathbb{U}, f \in H^2.
\]

Substitute $z$ by $\varphi(u)$, $w$ by $\psi(u)$, and integrate $dm(u)$ to obtain (13).

**Corollary 3.** If condition (11) holds then $\|C_{\varphi_n} - C_{\varphi}\| \to 0$.

Clearly, inequality (13) is interesting only if $\varphi \neq \psi$, $|\varphi| < 1$, and $|\psi| < 1$, m-a.e. Indeed, the integral involved in it is infinite if $\varphi \neq \psi$ and any of these functions has unimodular radial function on a measurable subset of $\mathbb{T}$ having positive measure.

The paper [9] contains an upper norm–estimate for the difference of two composition operators. The methods used in [9, Theorem 3.2] can be adapted to show that, if the integral in estimate (13) is finite, then the operator $C_{\varphi} - C_{\psi}$ must be compact.
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