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COMPOSITION OPERATORS ON A CLASS OF ANALYTIC FUNCTION SPACES RELATED TO BRENNAN'S CONJECTURE

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Abstract. Brennan's conjecture in univalent function theory states that if τ is any analytic univalent transform of the open unit disk $\mathbb{D}$ onto a simply connected domain G and $-1/3 < p < 1$, then $1/(\tau')^p$ belongs to the Hilbert Bergman space of all analytic square integrable functions with respect to the area measure. We introduce a class of analytic function spaces $L^2_a(\mu_p)$ on G and prove that Brennan's conjecture is equivalent to the existence of compact composition operators on these spaces for every simply connected domain G and all $p \in (-1/3, 1)$. Motivated by this result, we study the boundedness and compactness of composition operators in this setting.

1. Introduction

Given a selfmap φ of some set E and a space S consisting of complex functions on E, we denote by C_φ and call the composition operator of symbol φ, (or induced by φ) the transform

$$C_\varphi f = f \circ \varphi \qquad f \in S.$$

A weighted composition operator is a composition operator followed by a multiplication operator. More exactly, if ψ is a complex function on E, then the transform

$$T_{\psi,\varphi} f = M_\psi C_\varphi f = \psi f \circ \varphi \qquad f \in S$$

is called the weighted composition operator of symbols ψ and φ. Note that the first symbol is that of the multiplication operator M_ψ and the second that of the composition operator. We will use this notation throughout this paper. Recently, weighted composition operators have been tied to Brennan's conjecture in univalent function theory [13].

Throughout this paper τ will denote an analytic, univalent transform of the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto some simply connected domain $G \subseteq \mathbb{C}$. We denote $g = \tau^{-1}$ and refer to g as a *Riemann transform of G onto $\mathbb{D}$*, given Riemann's well known conformal equivalence theorem. Using this terminology, we recall the following important conjecture in univalent function theory:

> **Brennan's conjecture:** *If g is a Riemann transform of a simply connected domain $G \subsetneqq \mathbb{C}$ onto $\mathbb{D}$ and $4/3 < p < 4$, then*
>
> $$\int_G |g'|^p \, dA < +\infty. \tag{1.1}$$

Of course, dA denotes the area measure. That (1.1) holds when $4/3 < p < 3$ is an easy consequence of the Koebe distortion theorem. Brennan [1] extended this to $4/3 < p < 3+\delta$ for some small $\delta > 0$, and conjectured it to hold for $4/3 < p < 4$.

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This range of p can not be extended, as shown by the example $G = \mathbb{C} \setminus (-\infty, -1]$. The upper bound of those p for which (1.1) is known to hold has been increased by several authors, in particular to approximately 3.78 by S. Shimorin in [12].

Let us consider the Hilbert space $L^2_a(\mathbb{D})$, the space of all analytic functions on $\mathbb{D}$ that are square integrable dA. Brennan's conjecture can be easily reformulated in terms of $\tau = g^{-1}$. Indeed, elementary computations lead to the following equivalent formulation of Brennan's conjecture:

If τ is a Riemann transform of $\mathbb{D}$ onto a simply connected domain $G \subsetneqq \mathbb{C}$ and $-1/3 < p < 1$, then $1/(\tau')^p \in L^2_a(\mathbb{D})$.

Brennan's conjecture can also be formulated in terms of the compactness of some special weighted composition operators. Denote by $A_{\varphi,p}$ the weighted composition operator

$$A_{\varphi,p} = T_{(\tau'\circ\varphi/\tau')^p,\varphi}.$$

The main result in [13] is:

Theorem 1 ([13, Theorem 1.1])**.** *$1/(\tau')^p \in L^2_a(\mathbb{D})$ if and only if there is some analytic selfmap φ of $\mathbb{D}$ so that $A_{\varphi,p}$ is a compact operator on $L^2_a(\mathbb{D})$.*

Our work on this paper began with the question: *How does one formulate Brennan's conjecture in terms of ("unweighed") composition operators?* The answer is in the next section and it involves the introduction of a new class of spaces of analytic functions. Section 2 contains a brief investigation of the properties of those spaces followed by a first approach to the study of their composition operators. Several necessary conditions for boundedness and compactness are obtained. As an application, bounded automorphic composition operators are characterized.

Section 3 contains our main results. We give general necessary and sufficient criteria for boundedness and compactness of composition operators in terms of pull–back Carleson measures induced by their symbols (Theorems 3, 4, and 5). As an application of these criteria we prove integral–transform criteria for both boundedness and compactness (Theorem 6). In section 4 we demonstrate the utility of the results proved in the previous sections by applying them to the study of composition operators on the spaces under consideration, constructed over two particular domains.

2. A new class of spaces

Denote by $\mathcal{H}(G)$ the space of holomorphic functions on G. We introduce the function spaces

$$L^2_a(\mu_p) := \left\{ F \in \mathcal{H}(G) : \int_G |F|^2 \, d\mu_p < +\infty \right\}$$

where p is any fixed real number and $d\mu_p = |g'|^{2p+2} dA$. It should be noted that the space $L^2_a(\mu_p)$ does not depend on the Riemann map chosen from G onto $\mathbb{D}$. Indeed, if g and g_1 are two such maps, then $g \circ g_1^{-1}$ is a disk automorphism and hence has the form $\lambda\alpha_a$ where λ is a unimodular constant, $a \in \mathbb{D}$, and $\alpha_a(z) = (a-z)/(1-\bar{a}z)$. Thus

$$|g'| = \frac{1-|a|^2}{|1-\bar{a}g_1|^2}|g_1'|,$$

which implies

$$\frac{1-|a|}{1+|a|}|g_1'| \le |g'| \le \frac{1+|a|}{1-|a|}|g_1'|.$$

It follows that the two function spaces are the same and the norms are equivalent.

Throughout this paper an analytic selfmap of G will be denoted by ϕ and $\varphi = g \circ \phi \circ g^{-1}$ will denote the analytic selfmap of $\mathbb{D}$ that is conformally conjugate to ϕ. With this notation we prove:

Proposition 1. *The composition operator C_ϕ on $L_a^2(\mu_p)$ is unitarily equivalent to $A_{\varphi,p}$ on $L_a^2(\mathbb{D})$ and hence, Brennan's conjecture is equivalent to the statement that the spaces $L_a^2(\mu_p)$ endowed with the norm*

$$\|F\| = \sqrt{\frac{1}{\pi}\int_G |F|^2 |g'|^{2p+2} dA}$$

support compact composition operators if $-1/3 < p < 1$.

Proof. One easily checks that the weighted composition operators

$$T_{\frac{1}{(\tau')^p},\tau} F = \frac{1}{(\tau')^p} F \circ \tau \qquad\qquad F \in L_a^2(\mu_p)$$

and

$$T_{\frac{1}{(g')^p},g} f = \frac{1}{(g')^p} f \circ g \qquad\qquad f \in L_a^2(\mathbb{D})$$

are onto isometries inverse to each other. Note that, by a straightforward computation,

$$C_\phi = T_{\frac{1}{(g')^p},g} A_{\varphi,p} T_{\frac{1}{(\tau')^p},\tau}.$$

By [13, Theorem 1.1], this ends the proof. □

In the following we examine the spaces $L_a^2(\mu_p)$. We used the coefficient $1/\pi$ in the definition of their norm above, since the Bergman space $L_a^2(\mathbb{D})$ is usually constructed by using the normalized area measure. Among other things, that makes the reproducing kernel–functions of $L_a^2(\mu_p)$ simpler. Recall that a reproducing kernel Hilbert space (RKHS) is a Hilbert space consisting of functions on some set S with the property that point–evaluations are continuous functionals. Therefore one can identify the special functions K_z, $z \in S$, called the kernel–functions, (or the evaluation kernels) of the space, which have the "reproducing property":

$$f(z) = \langle f, K_z \rangle \qquad z \in S$$

for all functions f in the space.

The spaces $L_a^2(\mu_p)$ inherit the Hilbert structure from $L_a^2(\mathbb{D})$ via the isometries used in the proof of Proposition 1. Furthermore, they are RKHS. Recall that the evaluation kernels for $L_a^2(\mathbb{D})$ are the functions

$$k_a(z) = \frac{1}{(1-\bar{a}z)^2} \qquad z, a \in \mathbb{D}.$$

Based on that, one obtains:

Proposition 2. *The evaluation kernels of $L_a^2(\mu_p)$ are*

$$K_b(w) = \frac{1}{(\overline{g'(b)}g'(w))^p(1-\overline{g(b)}g(w))^2} \qquad b, w \in G$$

with norm given by

$$\|K_b\| = \frac{1}{|g'(b)|^p(1-|g(b)|^2)} \qquad b \in G.$$

Proof. Denote $a = g(b)$, so $\tau(a) = b$. Let $F \in L^2_a(\mu_p)$ be arbitrary and fixed and let $f := T_{\frac{1}{(\tau')^p},\tau}F$. One can write

$$\langle F, T_{\frac{1}{(g')^p},g}k_a\rangle = f(a).$$

So

$$\langle F, T_{\frac{1}{(g')^p},g}k_a\rangle = \frac{1}{(\tau'(a))^p}F(\tau(a))$$

that is

$$F(b) = \langle F, \frac{\overline{(\tau'(a))^p}}{(g')^p}k_a \circ g\rangle = \langle F, K_b\rangle.$$

The norm of K_b is computed with the formula $\|K_b\|^2 = K_b(b)$, a consequence of the reproducing property. □

An important observation is that these spaces have kernels "norm–bounded on compacts", that is, if $C \subseteq G$ is any nonempty compact then $\sup\{\|K_b\| : b \in C\} < +\infty$, (since the map $b \in G \to \|K_b\|$ is continuous). As a consequence of this property:

Remark 1. *In the spaces $L^2_a(\mu_p)$, weak convergence is equivalent to norm boundedness plus uniform convergence on compacts.*

Indeed, in any RKHS a sequence is weakly convergent if and only it is norm–bounded and pointwise convergent to the weak limit-function. If the space consists of analytic functions and has kernels "norm–bounded on compacts", then a weakly convergent sequence necessarily tends uniformly on compacts to its weak limit–function because the sequence tends pointwise to that function and, on each compact C the estimate

$$|f_n(z)| \le \|f_n\|\|K_z\| \le \sup\{\|K_b\| : b \in C\}\sup\{\|f_n\| : n = 1,2,3,\dots\} \qquad z \in C.$$

shows that the weakly convergent sequence $\{f_n\}$ is a normal family of analytic functions. Thus, the sequence must converge uniformly on compacts.

The facts above were the few things we wanted to establish on the class of spaces under consideration. We turn now to the problem of understanding when composition operators on these spaces are bounded, respectively compact. First let us note the cases when the study can be reduced to that of composition operators on $L^2_a(\mathbb{D})$.

Remark 2. *Suppose that either $p = 0$, or $p \neq 0$ and g' (or, equivalently τ') is both bounded and bounded away from 0. Then all composition operators on $L^2_a(\mu_p)$ are bounded and the compact composition operators C_ϕ are exactly those whose conformally conjugate symbols φ satisfy the relation*

$$\lim_{|z|\to 1^-} \frac{1-|z|}{1-|\varphi(z)|} = 0. \tag{2.1}$$

Indeed, in both cases, the multiplication operator $M_{(\tau'\circ\varphi)^p/(\tau')^p}$ that appears in the expression of $A_{\varphi,p}$ is a bounded invertible operator and so the boundedness and compactness of C_ϕ on $L^2_a(\mu_p)$ are respectively equivalent to the corresponding properties of C_φ on $L^2_a(\mathbb{D})$.

Recall the elementary but very useful formula for composition operators on RKHS that

$$C_\phi^* K_b = K_{\phi(b)} \qquad b \in G.$$

When C_ϕ is bounded, this immediately gives the estimate that

$$\sup\{\|K_{\phi(b)}\|/\|K_b\| : b \in G\} \le \|C_\phi^*\| = \|C_\phi\|.$$

Combined with Proposition 2, this tells us that

$$\sup\left\{\frac{|g'(b)|^p(1-|g(b)|^2)}{|g'(\phi(b))|^p(1-|g(\phi(b))|^2)} : b \in G\right\} =$$

$$\sup\left\{\frac{|\tau'(\varphi(a))|^p(1-|a|^2)}{|\tau'(a)|^p(1-|\varphi(a)|^2)} : a \in \mathbb{D}\right\} \le \|C_\phi\|. \tag{2.2}$$

While the supremum in (2.2) being finite is a useful necessary condition for C_ϕ to be bounded, an example will be presented in section 4 that shows it is not sufficient; see Example 4. A condition necessary and sufficient for C_ϕ to be bounded will be given in the next section. Our first application of (2.2) is to find which automorphic composition operators are bounded.

Proposition 3. *Suppose $p \neq 0$ and the analytic selfmap ϕ of G is conformally conjugate to a finite Blaschke product φ. Then C_ϕ is bounded if and only if $(g'/g'\circ\phi)^p$, or equivalently $(\tau'\circ\varphi/\tau')^p$, is bounded .*

Proof. If φ is a finite Blaschke product, then the quantity $(1-|a|^2)/(1-|\varphi(a)|^2)$ is both bounded and bounded away from 0, as $a \in \mathbb{D}$. Thus (2.2) is equivalent to the fact that $(\tau'\circ\varphi/\tau')^p$ is a bounded analytic function. On the other hand, if $(\tau'\circ\varphi/\tau')^p$ is a bounded analytic function, then $A_{\varphi,p}$ is bounded and hence, so is C_ϕ. □

A second application of (2.2) is a characterization of when all composition operators on $L^2_a(\mu_p)$ are bounded. The method of proof comes from [11, Theorem 6.1].

Proposition 4. *If $p \neq 0$, all composition operators on $L^2_a(\mu_p)$ are bounded if and only if g' is both bounded and bounded away from zero.*

Proof. We already observed that the if part of the equivalence is true. Assume now that all composition operators are bounded. Then all automorphic composition operators are bounded and from the preceding proposition, $\sup\{\tau'(\lambda z)/\tau'(z) : z \in \mathbb{D}\} = \|\tau'(\lambda z)/\tau'(z)\|_\infty$ is a real-valued function of $\lambda \in \partial\mathbb{D}$. We will show that in fact $\sup\{\|\tau'(\lambda z)/\tau'(z)\|_\infty : \lambda \in \partial\mathbb{D}\} = S < \infty$. But before proving this, observe that the fact that τ' is both bounded and bounded away from 0, which will complete the proof of the proposition, is a consequence of $S < \infty$ and the following principle which appears in [11].

If a zero–free analytic function f on $\mathbb{D}$ has the property

$$|f(\lambda z)|/|f(z)| \leq S \qquad z \in \mathbb{D},\, \lambda \in \partial\mathbb{D}$$

for some constant S, then f must be both bounded and bounded away from 0.

Indeed, pick an arbitrary $a \in \mathbb{D}$ and consider the function $f(az)$, whose maximum modulus on $\overline{\mathbb{D}}$ must be attained on $\partial\mathbb{D}$. This produces a $\lambda \in \partial\mathbb{D}$ with the property $|f(\lambda a)| \geq |f(0)|$, hence $S \geq |f(\lambda a)|/|f(a)| \geq |f(0)|/|f(a)|$, that is f is bounded away from 0. If f has the property above, then so does $1/f$, so f must also be bounded.

To complete the proof we only need to show that $S < \infty$. First note that, for all fixed $z \in \mathbb{D}$, the map $\lambda \to |\tau'(\lambda z)/\tau'(z)|$ is continuous, and hence $\lambda \to \|\tau'(\lambda z)/\tau'(z)\|_\infty$ is measurable, which implies that $\lambda \to \|\tau'(\lambda z)/\tau'(z)\|_\infty$ is uniformly bounded on some measurable subset $E \subseteq \partial\mathbb{D}$ having positive measure. That is $\|\tau'(\lambda z)/\tau'(z)\|_\infty \leq M$, $\lambda \in E$. This is a consequence of the fact that $\lambda \to \|\tau'(\lambda z)/\tau'(z)\|_\infty$ is finite-valued and measurable, hence the measurable sets $\{\lambda \in \partial\mathbb{D} : \|\tau'(\lambda z)/\tau'(z)\|_\infty \leq n\}$, $n = 1, 2, 3, \ldots$ cannot be all negligible. Since E has positive measure, $E \cdot E$ contains a non–degenerate arc J of $\partial\mathbb{D}$ [8, Ch. 7, Problem 5, pp. 158], and it follows that $E^{2n} = \partial\mathbb{D}$ for some integer n. On the other hand

$$\|\tau'(\lambda_1\lambda_2 z)/\tau'(z)\|_\infty \leq \|\tau'(\lambda_1\lambda_2 z)/\tau'(\lambda_1 z)\|_\infty \|\tau'(\lambda_1 z)/\tau'(z)\|_\infty \leq M^2 \quad \lambda_1, \lambda_2 \in E.$$

Since $E^{2n} = \partial\mathbb{D}$, similar reasoning shows $S \leq M^{2n} < \infty$, as required. □

The following theorem records the facts we have proved.

Theorem 2. *In the interesting case $p \neq 0$, the following are equivalent.*

(i) *All composition operators on $L^2_a(\mu_p)$ are bounded.*

(ii) *All automorphic composition operators on $L^2_a(\mu_p)$ are bounded.*

(iii) *All automorphic composition operators on $L^2_a(\mu_p)$ with symbols conjugated to rotations are bounded.*

(iv) *The ratio of the derivatives of any two Riemann maps of $\mathbb{D}$ onto G is a bounded analytic function.*

(v) *The map g' is both bounded and bounded away from zero.*

In all cases that we have examined, the composition operator C_ϕ induced by an automorphism $\phi : G \to G$ is bounded on $L^2_a(\mu_p)$ exactly when the composition operator induced by ϕ^{-1} is also bounded. A natural question we are not able to answer is whether this holds in general: *Do the conformal automorphisms of G inducing bounded composition operators on $L^2_a(\mu_p)$ form a subgroup of the group of all conformal automorphisms?* This is equivalent to the following question in univalent function theory:

If τ_1 and τ_2 are Riemann maps of $\mathbb{D}$ onto G and $\tau_1'/\tau_2' \in H^\infty$, then does it follow that $\tau_2'/\tau_1' \in H^\infty$?

Regarding the group of conformal automorphisms of G inducing bounded composition operators on $L^2_a(\mu_p)$, it should be noted that it can be as poor as the trivial group:

Example 1. *Let $G = \mathcal{P}$ be the interior of a convex polygon with angles $\{\pi\alpha_j\}$ at vertices $\{w_j\}_{j=1}^n$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$. Then the only automorphism ϕ of $\mathcal{P}$ that induces a bounded composition operator is the identity.*

Proof. Let ϕ be an automorphism of $\mathcal{P}$ and assume that C_ϕ is bounded. The associated selfmap φ of $\mathbb{D}$ is also an automorphism, and so the quantities $(1-|a|^2)/(1-|\varphi(a)|^2)$ and $|\varphi'(a)|$ are both bounded and bounded away from 0, $a \in \mathbb{D}$. Thus we see from (2.2) that $(\tau\circ\varphi)'/\tau' \in H^\infty$. Note that $\tau\circ\varphi$ maps $z_j = \varphi^{-1}\circ g(w_j)$ to w_j, a vertex of $\mathcal{P}$ with angle $\pi\alpha_j$, and so $|(\tau\circ\varphi)'(z)| \approx |z-z_j|^{\alpha_j-1}$ for z near z_j; see for example [6, Theorem 3.9].

If ϕ is not the identity, then φ is also not the identity and so can not fix all of the points z_j. Let j_0 be the smallest index such that $\varphi(z_{j_0}) \neq z_{j_0}$. Then τ maps z_{j_0} either to an edge of $\mathcal{P}$ or to a vertex with angle $\pi\alpha_{j_1} > \pi\alpha_{j_0}$. In the first case $|\tau'(z)| \approx 1$ for z near z_{j_0}, while $|\tau'(z)| \approx |z-z_{j_0}|^{\alpha_{j_1}-1}$ in the second case. Since $|(\tau\circ\varphi)'(z)| \approx |z-z_{j_0}|^{\alpha_{j_0}-1}$ for z near z_{j_0}, where $\alpha_{j_0} < \alpha_{j_1}$, in either case it follows that $(\tau\circ\varphi)'/\tau'$ is unbounded near z_{j_0}. This contradicts $(\tau\circ\varphi)'/\tau' \in H^\infty$, and hence ϕ must be the identity. □

Another condition necessary that ϕ induces a bounded composition operator is given in the following proposition. In it and throughout this paper we write $w \to \partial G$ if w approaches the boundary of G on the Riemann sphere, i.e. w approaches the boundary of G or the point at infinity.

Proposition 5. *If C_ϕ is bounded on $L^2_a(\mu_p)$ then*

$$\lim_{w\to\partial G} \frac{|g'(w)|^p(1-|g(w)|^2)}{|g'(\phi(w))|^p} = 0 \tag{2.3}$$

or equivalently

$$\lim_{|z|\to 1^-} \frac{|\tau'(\varphi(z))|^p(1-|z|^2)}{|\tau'(z)|^p} = 0. \tag{2.4}$$

Proof. If C_ϕ is bounded, then $A_{\varphi,p}(1) = (\tau'\circ\varphi)^p/(\tau')^p$ is a function in $L^2_a(\mathbb{D})$. Any $f \in L^2_a(\mathbb{D})$ has the property $|f(a)|(1-|a|^2) \to 0$ if $|a| \to 1^-$, which is a direct consequence of the fact that the normalized kernels $k_a/\|k_a\|$ tend weakly to 0 if $|a| \to 1^-$. □

A basic principle in operator theory is that *if a big–O condition relates to boundedness, then the corresponding little–O condition relates to compactness.* Indeed the little–O condition associated to (2.2) must hold if C_ϕ is compact.

Proposition 6. *If C_ϕ is compact, then*

$$\lim_{w\to\partial G} \frac{|g'(w)|^p(1-|g(w)|^2)}{|g'(\phi(w))|^p(1-|g(\phi(w))|^2)} = \lim_{|a|\to 1^-} \frac{|\tau'(\varphi(a))|^p(1-|a|^2)}{|\tau'(a)|^p(1-|\varphi(a)|^2)} = 0. \tag{2.5}$$

Proof. This is a consequence of the fact that the normalized reproducing kernels $K_b/\|K_b\|$ tend weakly to 0 as $b \to \partial G$ and the identities (where $a = g(b)$)

$$\|C^*_\phi(K_b/\|K_b\|)\| = \frac{\|K_{\phi(b)}\|}{\|K_b\|} = \frac{|g'(b)|^p(1-|g(b)|^2)}{|g(\phi(b))|^p(1-|g(\phi(b))|^2)} = \frac{|\tau'(\varphi(a))|^p(1-|a|^2)}{|\tau'(a)|^p(1-|\varphi(a)|^2)}.$$

□

We noted earlier that if the map g' is both bounded and bounded away from zero, then for all p, the compact composition operators on $L^2_a(\mu_p)$ are exactly those whose conjugate symbol φ satisfies the condition (2.1).

If g' is unbounded or unbounded away from zero, then condition (2.1) might not characterize compactness any more.

To see that, let us first denote by $B_p(g)$ *the Brennan integral of index p of g*, that is

$$B_p(g) = \sqrt{\frac{1}{\pi}\int_G |g'|^{2p+2} dA}.$$

Example 2. *Let $p > 0$ such that $B_p(g) < +\infty$. If there exist $\omega, \eta \in \partial\mathbb{D}$ such that $\lim_{z\to\omega} |\tau'(z)| = +\infty$ and τ' is bounded near η, then there are compact composition operators on $L^2_a(\mu_p)$ whose conjugate symbols do not satisfy (2.1). If $p < 0$ and $B_p(g) < +\infty$, then there are compact composition operators on $L^2_a(\mu_p)$ whose conjugate symbols do not satisfy (2.1) whenever there exist $\omega, \eta \in \partial\mathbb{D}$ such that $\lim_{z\to\omega} |\tau'(z)| = 0$ and $1/\tau'$ is bounded near η.*

Proof. Assume we are in the case $p > 0$. Consider the symbol ϕ conjugated to $\varphi(z) = \eta\overline{\omega}(z + \omega)/2$. Visibly, φ has a finite angular derivative at ω and for that reason, does not satisfy condition (2.1). Also $\lim_{z\to\omega}(\tau' \circ \varphi(z))^p/(\tau'(z))^p = 0$, since $\varphi(\omega) = \eta$. This enables us to prove that $A_{\varphi,p}$ is compact, and hence C_ϕ is also compact. Indeed, for arbitrary fixed $\epsilon > 0$, one can choose $\delta > 0$ small enough so that

$$|\tau'(\varphi(z))^{2p}/(\tau'(z))^{2p}| < \epsilon \qquad \text{if} \qquad |\omega - z| < \delta.$$

Then, for each weakly null sequence $\{f_n\}$ in $L^2_a(\mathbb{D})$, that is, for each norm–bounded sequence tending to 0 uniformly on compacts, one can write

$$\int_{\{z\in\mathbb{D}:|\omega-z|<\delta.\}} |\tau'(\varphi(z))^{2p}/(\tau'(z))^{2p}|\, |f_n \circ \varphi|^2\, dA \le \epsilon\|C_\varphi\|^2 M, \qquad n = 1, 2, \ldots$$

where $M > 0$ is a constant bounding above the norms squared of the functions in $\{f_n\}$. On the other hand

$$\lim_{n\to+\infty} \int_{\{z\in\mathbb{D}:|\omega-z|\ge\delta.\}} |\tau'(\varphi(z))^{2p}/(\tau'(z))^{2p}||f_n \circ \varphi|^2\, dA = 0$$

due to the uniform convergence on compacts of $\{f_n\}$ to 0 and the assumption that $dA/|\tau'|^{2p}$ is a finite measure. The proof in the case that $p < 0$ is similar. □

Concrete examples satisfying the assumptions in Example 2 can be found in the last section of this paper; see Corollary 1.

It is interesting to note that:

Proposition 7. *If $B_p(g) < +\infty$ then*

$$|g'(b)|^p \le \frac{B_p(g)}{1 - |g(b)|^2} \qquad b \in G \tag{2.6}$$

and

$$\lim_{b\to\partial G} |g'(b)|^p(1 - |g(b)|^2) = 0. \tag{2.7}$$

Proof. The fact that $B_p(g) < +\infty$ implies that $L^2_a(\mu_p)$ contains the constant functions and hence the composition operators of constant symbols are bounded on $L^2_a(\mu_p)$. In such a case, let us consider $w \in G$ and $\phi \equiv w$. One has that

$$\|C_w\| = B_p(g)\|K_w\| \tag{2.8}$$

since

$$\|C_w F\| = \sqrt{\frac{1}{\pi}\int_G |F(w)|^2 |g'|^{2p+2} dA} = |\langle F, K_w\rangle|\, B_p(g).$$

Relation (2.6) is the direct consequence of (2.2), (2.8), and the formula for $\|K_b\|$. Relation (2.7) is the consequence of taking $\phi \equiv w$ in (2.5), which is possible, since, if composition operators of constant symbols are bounded on $L^2_a(\mu_p)$, then it is easy to see they are actually compact. □

If Brennan's conjecture holds, then condition (2.7) holds for each $-1/3 < p < 1$. In fact, a straightforward application of Koebe's distortion theorem establishes that $\sup\{(1-|g(b)|^2)|g'(b)|^p : b \in G\} < \infty$ whenever $-1/3 \le p \le 1$.

As was noted before, the spaces $L^2_a(\mu_p)$ support compact composition operators if and only if $B_p(g) < +\infty$. In case this happens, an easy source of compact composition operators is characterizing the Hilbert-Schmidt composition operators.

Proposition 8. *A composition operator C_ϕ on $L^2_a(\mu_p)$ is Hilbert–Schmidt if and only if*

$$\int_G \frac{|g'|^{2p+2}}{|g'\circ\phi|^{2p}(1-|g\circ\phi|^2)^2}\, dA < +\infty. \tag{2.9}$$

Proof. The unitary operator $T_{\frac{1}{(g')^p},g}$ transforms the standard orthonormal basis $\{\sqrt{n+1}z^n : n = 0,1,2,\dots\}$ of $L^2_a(\mathbb{D})$ into the following complete orthonormal basis of $L^2_a(\mu_p)$

$$\left\{ f_n = \sqrt{n+1}\frac{g^n}{(g')^p} : n = 0,1,2,\dots \right\}.$$

One immediately obtains that the condition

$$\sum_{n=0}^{+\infty} \|C_\phi f_n\|^2 < +\infty$$

is equivalent to (2.9). □

As an application we analyze the status of operators C_ϕ induced by an analytic selfmap ϕ of G that transforms G into a relatively compact set whose closure is contained in G.

Proposition 9. *If ϕ is an analytic selfmap of G with the properties that $\overline{\phi(G)}$ is compact and $\overline{\phi(G)} \subseteq G$, then C_ϕ is Hilbert–Schmidt if $B_p(g) < +\infty$, respectively unbounded if $B_p(g) = +\infty$.*

Proof. The fact that C_ϕ is Hilbert–Schmidt if $B_p(g) < +\infty$ is a direct consequence of (2.9). If $B_p(g) = +\infty$, note that φ, the conjugate symbol, has the property $\|\varphi\|_\infty < 1$ so there is some $c > 0$ with the property

$$\|A_{\varphi,p}(1)\| = \left\| \left(\frac{\tau'\circ\varphi}{\tau'}\right)^{2p} \right\| \ge cB_p(g).$$

$\square$

3. Carleson Measures

Let $t > 0$. We need to characterize those positive Borel measures ν on G such that

$$\int_G |f|^t \, d\nu \leq C \int_G |f|^t \, d\mu_p \tag{3.1}$$

for some constant C and for all $f \in \mathcal{H}(G)$. We remark that by the Closed Graph Theorem this is equivalent to the inclusion

$$L^t_a(\mu_p) \subset L^t_a(\nu).$$

In the case that G is the unit disk (so $d\mu_p = dA$), such measures have been extensively studied and are known as Carleson measures. Their characterization is well known (see for example [4] or [2]), and is independent of the exponent t . We will see that this is the case in our more general setting as well, so we can say ν *is a* μ_p*-Carleson measure on* G if (3.1) holds for some (and hence all) $t > 0$.

Our work requires some background on the hyperbolic metric on G. For the following facts see, for example, [10, §9.5] or [6, §4.6]. The density function h_G for the hyperbolic metric on G is given by

$$h_G(w) = \frac{2|g'(w)|}{1 - |g(w)|^2}. \tag{3.2}$$

For each $b \in G$, let $\delta_G(b)$ denote the Euclidean distance from b to the boundary of G. A basic estimate is that

$$\frac{1}{2\delta_G(b)} \leq h_G(b) \leq \frac{2}{\delta_G(b)}. \tag{3.3}$$

The hyperbolic metric λ_G of G is defined by

$$\lambda_G(w_1, w_2) = \inf_\gamma \int_\gamma h_G(w) \, |dw|, \tag{3.4}$$

where the infimum is over all smooth curves in G connecting w_1 to w_2.

For $b \in G$ and $r > 0$, let $\Delta_{G,r}(b)$ denote the hyperbolic disk in G with center b and radius r:

$$\Delta_{G,r}(b) = \{w \in G \,:\, \lambda_G(w, b) \leq r\}.$$

The exact value of $r > 0$ will not be important below and can be considered as a fixed constant. So, to simplify the notation, we will often write Δ_G in place of $\Delta_{G,r}$. It is easily seen from (3.3) and (3.4) that δ_G is approximately constant in each $\Delta_G(b)$:

$$\delta_G(w) \approx \delta_G(b), \qquad w \in \Delta_{G,r}(b); \tag{3.5}$$

see for example [10, p. 157]. It easily follows from this and (3.3) that $\Delta_{G,r}(b)$ is roughly a Euclidean disk with radius comparable to $\delta_G(b)$:

$$B(b, \delta_G(b)(r \wedge 1)/4) \subset \Delta_{G,r}(b) \subset B(b, \delta_G(b)e^{2r}), \tag{3.6}$$

where $B(b, t)$ denotes the Euclidean disk with center b and radius t, and $r \wedge 1$ is the minimum of r and 1. We will also need the estimates

$$e^{-6r}|g'(b)| \leq |g'(w)| \leq e^{6r}|g'(b)|, \qquad w \in \Delta_{G,r}(b), \tag{3.7}$$

which come from an invariant form of the Koebe distortion theorem; see [6, Corollary 1.5].

Our characterization of μ_p-Carleson measures involves the averaging function defined by

$$\widehat{\nu}_p(w) = \frac{\nu(\Delta_G(w))}{\mu_p(\Delta_G(w))}.$$

Note that it follows from (3.6) and (3.7) that

$$\mu_p(\Delta_G(a)) \approx \delta_G^2(a)|g'(a)|^{2p+2}. \tag{3.8}$$

We will often write $X \lesssim Y$ or $Y \gtrsim X$ if $X \leq CY$ for some positive constant C dependent only on allowed parameters, and $X \approx Y$ if $X \lesssim Y \lesssim X$.

Theorem 3. *Let ν be a positive Borel measure on G and let $t > 0$. The following are equivalent:*

(a) $\widehat{\nu}_p \in L^\infty(G)$;

(b) *There is a constant C such that $\int_G |f|^t \, d\nu \leq C \int_G |f|^t \, d\mu_p$ for all $f \in \mathcal{H}(G)$.*

Moreover, $\|\widehat{\nu}_p\|_\infty \approx C_1$, where C_1 is the norm of the embedding $L_a^t(\mu_p) \subset L_a^t(\nu)$.

Remark. It is worth noting that Theorem 3(a) is independent of the parameter t, and hence if Theorem 3(b) holds for one t, it holds for all $t > 0$.

Proof. Let $h \geq 0$ be subharmonic on G. For $a \in G$, the subharmonic mean value inequality on the disk $B(a, \delta_G(a)(r \wedge 1)/4) \subset \Delta_{G,r}(a)$ and (3.5) show that

$$h(a) \lesssim \int_{\Delta_G(a)} \frac{h(w)}{\delta_G^2(a)} dA(w) \approx \int_G \chi_{\Delta_G(a)}(w) \frac{h(w)}{\delta_G^2(w)|g'(w)|^{2p+2}} d\mu_p(w).$$

Since $\chi_{\Delta_G(a)}(w) = \chi_{\Delta_G(w)}(a)$, integrating both sides of this inequality against the measure $d\nu(a)$, changing the order of integration, and using (3.8) gives

$$\int_G h(a) \, d\nu(a) \lesssim \int_G \widehat{\nu}_p(w) h(w) d\mu_p(w).$$

Thus letting $h = |f|^t$, we see that (a) implies (b).

For the converse, assume (b) and let $a \in G$. Application of (b) to the test function $(K_a)^{2/t}$ yields

$$\int_G |K_a|^2 \, d\nu \leq C \int_G |K_a|^2 \, d\mu_p = C \frac{1}{|g'(a)|^{2p}(1 - |g(a)|^2)^2}. \tag{3.9}$$

Also, from (3.7) we see that

$$|K_a(w)|^2 |g'(a)|^{4p}(1 - |g(a)|^2)^4 \approx 1, \qquad w \in \Delta_G(a).$$

Hence

$$\frac{\nu(\Delta_G(a))}{|g'(a)|^{4p}(1 - |g(a)|^2)^4} \approx \int_{\Delta_G(a)} |K_a|^2 \, d\nu \leq \int_G |K_a|^2 \, d\nu. \tag{3.10}$$

Combining (3.9) and (3.10) we get that

$$\nu(\Delta_G(a)) \lesssim |g'(a)|^{2p}(1 - |g(a)|^2)^2.$$

Next, observe from (3.2) and (3.3) that

$$1 - |g(a)|^2 \approx |g'(a)|\delta_G(a).$$

Hence, using (3.8), we get

$$\nu(\Delta_G(a)) \lesssim \mu_p(\Delta_G(a)),$$

and so $\widehat{\nu}_p \in L^\infty(G)$.

Examination of the proof shows that $\|\widehat{\nu}_p\|_\infty$ is comparable to the norm of the embedding $L^t_a(\mu_p) \subset L^t_a(\nu)$. This completes the proof. $\square$

We say ν *is a compact μ_p-Carleson measure on G* if the embedding

$$L^t_a(\mu_p) \subset L^t_a(\nu)$$

is compact. As expected, these measures can be characterized by a little-oh version of the μ_p-Carleson criteria.

Theorem 4. *Let ν be a positive Borel measure on G which is finite on the compact subsets, let $t > 0$, and let $w_0 \in G$. The following are equivalent:*

(a) $\displaystyle\lim_{\lambda_G(w_0,w)\to\infty} \widehat{\nu}_p(w) = 0$*;*

(b) *The embedding $L^t_a(\mu_p) \subset L^t_a(\nu)$ is compact.*

Remark. The proof of this kind of result is now routine in the setting of the unit disk. For completeness, we show the same approach works in the present setting.

Proof. Assume first that (a) holds. Let $\{f_n\}$ be a bounded sequence in $L^t_a(\mu_p)$. It must be shown that there is a subsequence that converges in $L^t_a(\nu)$. A normal families argument produces a subsequence that converges locally uniformly to a function f which by Fatou's Lemma must belong to $L^t_a(\mu_p)$. By re-indexing, subtracting f, and scaling, we may assume that the original sequence $f_n \to 0$ locally uniformly and $\|f_n\|_{L^t_a(\mu_p)} \le 1$, and we must show that $f_n \to 0$ in $L^t_a(\nu)$. Let $\varepsilon > 0$ and put $K = \{w \in G : \widehat{\nu}_p(w) \ge \varepsilon\}$. Then K is compact and since $f_n \to 0$ locally uniformly,

$$\int_K |f_n|^t d\nu \le \varepsilon$$

for all n sufficiently large. It is easily checked that the measure $\eta = \chi_{G\setminus K} d\nu$ satisfies $\widehat{\eta}_p(w) \le C\varepsilon$. Thus Theorem 3 shows that

$$\int_{G\setminus K} |f_n|^t d\nu \le C\varepsilon \int_G |f_n|^t d\mu_p \le C\varepsilon.$$

Combined with the previous display, this shows that $f_n \to 0$ in $L^t_a(\nu)$ as required, and completes the proof that (a) implies (b).

For the proof that (b) implies (a), note that the normalized test functions $(K_w)^{2/t}/\|(K_w)^{2/t}\|_{L^t_a(\mu_p)} \to 0$ weakly in $L^t_a(\mu_p)$ as $\lambda_G(w_0, w) \to \infty$.

Hence compactness of the embedding $L^t_a(\mu_p) \subset L^t_a(\nu)$ implies $\|(K_w)^{2/t}\|_{L^t_a(\nu)}/\|(K_w)^{2/t}\|_{L^t_a(\mu_p)} \to 0$ as $\lambda_G(w_0, w) \to \infty$. This in place of (3.9) and using (3.10) as in the proof of Theorem 3 shows that $\widehat{\nu}_p(w) \to 0$. The proof is complete. $\square$

Standard methods now give Carleson measure criteria of when C_ϕ is bounded or compact. A change of variables formula from measure theory involving the pullback measure defined by $\mu_p \circ \phi^{-1}(E) = \mu_p(\phi^{-1}(E))$ shows that

$$\|f \circ \phi\|_{L^t_a(\mu_p)} = \|f\|_{L^t_a(\mu_p\circ\phi^{-1})}.$$

This gives the following Carleson measure criteria:

Theorem 5. *C_ϕ is bounded if and only if $\mu_p \circ \phi^{-1}$ is a μ_p-Carleson measure. C_ϕ is compact if and only if $\mu_p \circ \phi^{-1}$ is a compact μ_p-Carleson measure.*

Here's an example showing how the above criteria work.

Example 3. *Let $G = \{w : |\mathrm{Im}\, w| < \pi/2\}$, $\tau(z) = \log[(1+z)/(1-z)]$, and $g(w) = (1-e^{-w})/(1+e^{-w})$. Consider $\phi : G \to G$ defined by $\phi(w) = w/2$. Then C_ϕ is bounded if and only if $p \geq -1$ and C_ϕ is compact if and only if $p > -1$.*

Proof. A computation shows that $|g'(w)| \approx e^{-|\mathrm{Re}\, w|}$, $w \in G$. Hence $\mu_p(\Delta(w)) \approx e^{-|\mathrm{Re}\, w|(2p+2)} A(\Delta(w))$, while $\mu_p(\phi^{-1}\Delta(w)) \approx e^{-2|\mathrm{Re}\, w|(2p+2)} A(\phi^{-1}\Delta(w))$. Since $A(\phi^{-1}\Delta(w)) \leq 4A(\Delta(w))$, it follows that $\mu_p \circ \phi^{-1}$ is a μ_p-Carleson measure for all $p \geq -1$. Also, for $0 < x < \infty$,

$$\widehat{\mu_p \circ \phi^{-1}}(x) = \frac{\mu_p(\Delta(2x))}{\mu_p(\Delta(x))} \approx e^{-x(2p+2)}.$$

If $\mu_p \circ \phi^{-1}$ is a μ_p-Carleson measure, then this remains bounded as $x \to \infty$ and so $p \geq -1$. On the other hand, if $\mu_p \circ \phi^{-1}$ is a compact μ_p-Carleson measure, then this approaches 0 as $x \to \infty$ and so $p > -1$. Conversely, suppose that $p > -1$. To show $\mu_p \circ \phi^{-1}$ is a compact μ_p-Carleson measure, we must show $\widehat{\mu_p \circ \phi^{-1}}(w)$ is small when $\lambda_G(0, w)$ is large. If $\Delta(w) \cap \phi(G) = \emptyset$, then $\widehat{\mu_p \circ \phi^{-1}}(w) = 0$. Otherwise $\Delta(w) \cap \phi(G) \neq \emptyset$, and then $|\mathrm{Re}\, w|$ must be large when $\lambda_G(0, w)$ is large. Hence $\widehat{\mu_p \circ \phi^{-1}}(w) \lesssim e^{-|\mathrm{Re}\, w|(2p+2)}$ is small, as required. The proof is complete. □

One of the benefits of Theorem 5 is that it shows that an analytic selfmap ϕ of G simultaneously induces a bounded composition operator, (respectively a compact composition operator) on all spaces $L^t_a(\mu_p)$. A deeper application of the same theorem is proving, (based on it), what we call integral–transform criteria for the boundedness and compactness of composition operators. Such criteria were first produced by the authors of [3] for weighted composition operators on the space $L^2_a(\mathbb{D})$. Their technical tools for proving such criteria were not Carleson measures, as in our case.

For each analytic $\phi : G \to G$, consider the integral transform

$$T_\phi(z) := (1-|g(z)|^2)^2 \int_G \frac{|g'|^{2p+2}\, dA}{|g' \circ \phi|^{2p} |1 - \overline{g(z)} g \circ \phi|^4} \in [0, +\infty] \qquad z \in G. \tag{3.11}$$

Theorem 6. *Under the assumptions and notations above, the following hold.*

$$\|C_\phi\| < +\infty \iff \|T_\phi\|_\infty < +\infty. \tag{3.12}$$

$$C_\phi \text{ is compact} \iff \lim_{z \to \partial G} T_\phi(z) = 0. \tag{3.13}$$

Proof. First note that T_ϕ has the alternative representation

$$T_\phi(z) = \frac{1}{\|K_z\|^2} \int_G |K_z \circ \phi|^2 \, d\mu_p = \frac{1}{\|K_z\|^2} \int_G |K_z|^2 \, d\mu_p \phi^{-1} \qquad z \in G. \tag{3.14}$$

If $\|C_\phi\| < +\infty$, then

$$\|T_\phi\|_\infty = \sup\{\|C_\phi K_z\|^2 / \|K_z\|^2 : z \in G\} \leq \|C_\phi\|^2 < +\infty.$$

Thus, T_ϕ is bounded.

Conversely, if T_ϕ is bounded, then note that one can write (3.10) for $\mu_p\phi^{-1}$, that is, (given that $\mu_p(\Delta_G(z)) \approx \delta_G^2(z)|g'(z)|^{2p+2} \approx |g'(z)|^{2p}(1-|g(z)|^2)^2$),

$$\frac{\mu_p\phi^{-1}(\Delta_G(z))}{\mu_p(\Delta_G(z))} \leq \frac{1}{\|K_z\|^2}\int_G |K_z|^2\, d\mu_p\phi^{-1} \qquad z \in G. \tag{3.15}$$

By the boundedness of T_ϕ and Theorem 3, it follows that $\mu_p\phi^{-1}$ is a μ_p–Carleson measure and hence C_ϕ is bounded.

The second equivalence in this theorem follows by Theorem 4 and the little-oh version of the proof above. For the sake of completeness, here are the details. As was noted before, $K_z/\|K_z\| \to 0$ weakly as $z \to \partial G$. Therefore, if C_ϕ is compact, then $\|C_\phi K_z\|^2/\|K_z\|^2 \to 0$ as $z \to \partial G$. By (3.14) this means that $\lim_{z\to\partial G} T_\phi(z) = 0$.

Conversely, assume that $\lim_{z\to\partial G} T_\phi(z) = 0$. By (3.15), this implies that $\mu\phi^{-1}$ is a compact μ_p–Carleson measure and hence C_ϕ is compact. □

4. Spaces over two particular domains

We already noted that, if τ' is both bounded and bounded away from zero then the boundedness or compactness of C_ϕ on $L_a^2(\mu_p)$ is equivalent to the boundedness, respectively compactness of C_φ on $L_a^2(\mathbb{D})$. We consider two of the simplest domains where τ' does not have one of the properties above, namely the right half–plane $\prod^+ = \left\{\frac{1+z}{1-z} : z \in \mathbb{D}\right\}$ and the cardioid $\mathcal{C} = \{z - z^2/2 : z \in \mathbb{D}\}$. In the case of the half–plane $\tau'(z) = 2/(1-z)^2$ fails to be bounded, whereas $\tau'(z)$ equals $1-z$, a map that is not bounded away from zero, in the case of the cardioid.

The purpose of this section is to illustrate the utility of our criteria for boundedness and compactness in the case of the function spaces we consider in this paper, constructed over the domains above. As usual ϕ is an arbitrary analytic selfmap of G and φ its conjugate. Since our Carleson criteria involve hyperbolic disks, it is useful to transfer everything in $\mathbb{D}$ where hyperbolic disks are round. More exactly, let $\Delta_r(a)$ be the hyperbolic disc in $\mathbb{D}$ having hyperbolic center $a \in \mathbb{D}$ and hyperbolic radius r. Recall that this is the circular Euclidean disk of center and radius $C(a,r)$ and $R(a,r)$ respectively, where

$$C(a,r) = \frac{1-\tanh^2 r}{1-|a|^2\tanh^2 r}a \qquad R(a,r) = \frac{1-|a|^2}{1-|a|^2\tanh^2 r}\tanh r.$$

Our Carleson–measure criteria for boundedness and compactness lead then to the following.

Theorem 7. *The operator C_ϕ is bounded on $L_a^2(\mu_p)$, if and only if*

$$\sup_{a\in\mathbb{D}} \frac{|1-a|^{2\alpha}}{(1-|a|)^2}\int_{\varphi^{-1}(\Delta_r(a))} |1-z|^{-2\alpha}\, dA(z) < +\infty \tag{4.1}$$

with α related to p as follows: $\alpha = -2p$ in the case of the half–plane and $\alpha = p$ in the case of the cardioid. With the same notations, C_ϕ is compact if and only if

$$\lim_{a\to\partial\mathbb{D}} \frac{|1-a|^{2\alpha}}{(1-|a|)^2}\int_{\varphi^{-1}(\Delta_r(a))} |1-z|^{-2\alpha}\, dA(z) = 0. \tag{4.2}$$

Proof. Based on relation (3.7), one gets

$$\int_{\Delta_r(a))} \frac{1}{|\tau'|^{2p}}\, dA \approx \frac{(1-|a|)^2}{|\tau'(a)|^{2p}} \qquad a \in \mathbb{D}. \tag{4.3}$$

That fact and straightforward computations involving the change of variable formula dA, show that Condition (a) in Theorem 3, with $\nu = \mu_p \phi^{-1}$ is equivalent to (4.1), whereas condition (a) in Theorem 4 is equivalent to (4.2). □

Straightforward computations can be used to transfer the results in Theorem 6 to $\mathbb{D}$ and treat spaces over the cardioid and the half–plane simultaneously. One obtains:

Theorem 8. *Let $\alpha = -2p$ in the case of the half–plane and $\alpha = p$ in the case of the cardioid. Denoting by $\|\quad\|_e$ the essential norm, the following equivalences are valid.*

$$\|C_\phi\| < +\infty \Longleftrightarrow \sup_{a\in\mathbb{D}} (1-|a|^2)^2 \int_{\mathbb{D}} \left|\frac{1-\varphi(z)}{1-z}\right|^{2\alpha} \frac{dA(z)}{|1-\overline{a}\varphi(z)|^4} < +\infty. \tag{4.4}$$

$$\|C_\phi\|_e = 0 \Longleftrightarrow \limsup_{|a|\to 1^-} (1-|a|^2)^2 \int_{\mathbb{D}} \left|\frac{1-\varphi(z)}{1-z}\right|^{2\alpha} \frac{dA(z)}{|1-\overline{a}\varphi(z)|^4} = 0. \tag{4.5}$$

Recall that necessary conditions for boundedness and compactness of C_ϕ were given in section 2. As an application of Theorem 8 we can now give the example promised in that section, showing that while the supremum in (2.2) being finite is necessary for C_ϕ to be bounded, it is not sufficient.

Example 4. *Let $p = 1$ and let C_ϕ be the composition operator induced by the constant self-map $\phi(w) \equiv 0$ of the cardiod $\mathcal{C} = \{z - z^2/2 : z \in \mathbb{D}\}$. Then the supremum in* (2.2) *is finite, but C_ϕ is unbounded on $L^2_a(\mu_1)$.*

Proof. We have $\tau(z) = z - z^2/2$ and $\varphi(z) \equiv 0$, so with $p = 1$ the supremum in (2.2) is $\sup\{(1-|a|^2)/|1-a| \,:\, a \in \mathbb{D}\} = 2 < \infty$. On the other hand, the integral in (4.4) with $a = 0$ is $\int_{\mathbb{D}} |1-z|^{-2} dA(z) = \infty$, and so C_ϕ is unbounded by Theorem 8. □

As a last application of the necessary conditions in section 2, we now identify the the conformal conjugates of symbols inducing bounded composition operators on the spaces considered in this section, as a select class of symbols having angular derivatives at 1 (the point transformed into the point at infinity in the case of the half–plane, respectively the inner cusp of the cardioid).

Several results in [5] are needed. Prior to stating them, we introduce the few concepts needed to understand them.

Let ω be a point on the unit circle $\partial\mathbb{D}$. The regions

$$\Gamma_M(\omega) = \left\{ z \in \mathbb{D} : \frac{|\omega - z|}{1-|z|} < M \right\} \qquad M > 1$$

are called *nontangential approach regions* with vertex at ω. If z tends to ω inside such a region, that fact is equivalent to convergence to ω inside an angle of aperture less than π having vertex at ω. The aperture can be anything between 0 and π as M ranges between 1 and $+\infty$.

An analytic selfmap φ of $\mathbb{D}$ has an angular derivative at a boundary point $\omega \in \partial\mathbb{D}$ if there is some $\eta \in \partial\mathbb{D}$ and some $c \in \mathbb{C}$, so that, for each $M > 1$,

$$\frac{\eta - \varphi(z)}{\omega - z} \to c \quad \text{as} \quad z \to \omega \quad \text{inside} \quad \Gamma_M(\omega).$$

In that case, the value c is called the angular derivative of φ at ω, and we denote $c = \varphi'(\omega)$. Clearly η is the angular limit of φ at ω, i.e. the limit of $\varphi(z)$ as $z \to \omega$ inside each region $\Gamma_M(\omega)$.

If the angular limit φ at ω exists and equals ω, we call ω a boundary fixed point of φ.

The following are established in [5].

• *For a fixed analytic selfmap φ of $\mathbb{D}$, fixed constants, $\omega, \eta \in \partial\mathbb{D}$, and $1 < \alpha < +\infty$ we denote*

$$\beta_\alpha = \sup\left\{\frac{|\eta - \varphi(z)|^\alpha (1 - |z|^2)}{|\omega - z|^\alpha (1 - |\varphi(z)|^2)} : z \in \mathbb{D}\right\} \in (0, +\infty].$$

• *If, for some $1 < \alpha < +\infty$, β_α is finite, then the angular derivative $\varphi'(\omega)$ of φ at ω exists and the angular limit $\varphi(\omega)$ of φ at ω equals η.*

• *If $\varphi'(\omega)$ exists and $\varphi(\omega) = \eta$, then β_α is finite for each $1 < \alpha \leq 2$, but may be infinite for all values $2 < \alpha < +\infty$.*

• *For each $1 < \alpha < +\infty$ one has*

$$\beta_\alpha < +\infty \Longleftrightarrow \limsup_{z\to\omega} \frac{|\eta - \varphi(z)|^\alpha (1 - |z|^2)}{|\omega - z|^\alpha (1 - |\varphi(z)|^2)} < +\infty. \tag{4.6}$$

• If $(\eta - \varphi(z))/(\omega - z)$ is bounded, then

$$\beta_\alpha < +\infty \qquad 1 < \alpha < +\infty.$$

Using the above one can prove:

Proposition 10. *Working with the substitutions $\alpha = -2p$, in the case of $\prod^+$, respectively $\alpha = p$ for $\mathcal{C}$, one has that $B_p(g) = +\infty \iff \alpha \geq 1$. If $\alpha > 1$ and $\|C_\phi\| < +\infty$, then 1 must be a boundary fixed point of φ where the angular derivative $\varphi'(1)$ exists. If $\alpha > 2$, then the class of symbols φ conjugated to the symbols inducing bounded composition operators on $L^2_a(\mu_p)$ is a strict subset of the set of all φ fixing 1 and having a finite angular derivative at that boundary fixed point. If $\alpha = 1$ and $\|C_\phi\| < +\infty$, then 1 must be a boundary fixed point of φ.*

Proof. The equivalence $B_p(g) = +\infty \iff \alpha \geq 1$ is established by a routine computation. Condition (2.2) looks as follows

$$\sup\left\{\frac{|1 - \varphi(z)|^\alpha (1 - |z|^2)}{|1 - z|^\alpha (1 - |\varphi(z)|^2)} : z \in \mathbb{D}\right\} \leq \|C_\phi\| < +\infty$$

that is, if $\alpha > 1$, one gets $\beta_\alpha < +\infty$, (where $\omega = \eta = 1$), which tells us $\varphi'(1)$ must exist and $\varphi(1) = 1$. On the other hand, if $\alpha > 2$, it is possible that $\varphi'(1)$ exist and $\varphi(1) = 1$, but $\beta_\alpha = +\infty$, hence condition (2.2) fails for such φ.

Finally, if $\alpha = 1$ and $\|C_\phi\| < +\infty$, condition (2.4) has the form

$$\lim_{|z|\to 1^-} (1 - |z|) \left|\frac{1 - \varphi(z)}{1 - z}\right| = 0.$$

Combining this with the estimate

$$|1-\varphi(z)| = \left|\frac{1-\varphi(z)}{1-z}\right| \frac{|1-z|}{1-|z|}(1-|z|) \leq M \left|\frac{1-\varphi(z)}{1-z}\right| (1-|z|)$$

which is valid for some $M > 1$ if $z \to 1$ inside a nontangential approach region with vertex at 1, one gets that $\varphi(z) \to 1$ if $z \to 1$ nontangentially.

□

If we consider symbols ϕ conjugated to linear fractional symbols φ, a complete characterization in terms of boundary fixed points and angular derivatives of those inducing bounded or compact composition operators is obtainable, as we prove in the following. Recall that, by Proposition 1, the weighted composition operator

$$A_{\varphi,\alpha} f(z) = \left(\frac{1-\varphi(z)}{1-z}\right)^{\alpha} f \circ \varphi(z) \qquad f \in L^2_a(\mathbb{D})$$

is unitarily equivalent to C_ϕ.

Proposition 11. *Let ϕ be conformally conjugated to a symbol φ that is extensible by analyticity at 1. If $\alpha \geq 1$, then C_ϕ is bounded if and only if $\varphi(1) = 1$. If $0 < \alpha < 1$ then C_ϕ is always bounded, except when $|\varphi(1)| = 1$ and $\varphi(1) \neq 1$. The operator C_ϕ is non–compact if $\varphi(1) = 1$.*

Proof. Assume $\alpha \geq 1$, then if C_ϕ is bounded, necessarily $\varphi(1) = 1$, by Proposition 10. For the sufficiency, note that if φ is extensible by analyticity at 1 and $\varphi(1) = 1$, then $A_{\varphi,\alpha}$ is bounded for all $\alpha > 0$ because $((1-\varphi(z))/(1-z))^{\alpha} \in H^\infty(\mathbb{D})$. Among other things this takes care of the sufficiency, (for the boundedness of C_ϕ), of condition $\varphi(1) = 1$, in the case $0 < \alpha < 1$. To establish the boundedness of C_ϕ when $0 < \alpha < 1$ and $|\varphi(1)| < 1$, it is enough to show that the weighted composition operator

$$f(z) \to (1/(1-z)^{\alpha}) f(\varphi(z))$$

acts on $L^2_a(\mathbb{D})$. Indeed, this is the consequence of the closed graph principle and the fact that $|1-\varphi(z)|^{\alpha}$ is bounded. The Brennan integral being finite under the assumption $\alpha < 1$, for each $f \in L^2_a(\mathbb{D})$, one has that $f \circ \varphi$ is bounded on some neighborhood N of 1, relative to $\mathbb{D}$, so

$$\int_N |(1/(1-z)^{\alpha}) f(\varphi(z))|^2 \, dA \lesssim \int_N |(1/(1-z)^{\alpha})|^2 \, dA < +\infty$$

and

$$\int_{\mathbb{D}\setminus N} |(1/(1-z)^{\alpha}) f(\varphi(z))|^2 \, dA \lesssim \int_{\mathbb{D}\setminus N} |f(\varphi(z))|^2 \, dA < +\infty.$$

If $|\varphi(1)| = 1$, $0 < \alpha < 1$, and $\varphi(1) \neq 1$, then let $\omega \in \partial\mathbb{D}$ be such that $\varphi(1) = \omega$. Note that C_ϕ is unbounded because condition (2.2) fails. Indeed:

$$\lim_{z\to 1} \frac{|1-\varphi(z)|^{\alpha}(1-|z|^2)}{|1-z|^{\alpha}(1-|\varphi(z)|^2)} = \frac{1}{|\varphi'(1)|} \frac{|1-\omega|^{\alpha}}{0^+} = +\infty.$$

If $\varphi(1) = 1$, condition (2.5) fails, since angular derivatives cannot be null and

$$\lim_{z\to 1} \frac{|1-\varphi(z)|^{\alpha}(1-|z|^2)}{|1-z|^{\alpha}(1-|\varphi(z)|^2)} = |\varphi'(1)|^{\alpha-1} \neq 0.$$

□

Corollary 1. *Suppose φ is a linear fractional map. If $\alpha > 0$, then the boundedness of C_ϕ is completely characterized by the considerations above. For $0 < \alpha < 1$, C_ϕ is compact if and only if either $\|\varphi\|_\infty < 1$, or φ is not an automorphism of $\mathbb{D}$ and $\varphi(\omega) = 1$ for some unimodular $\omega \neq 1$.*

Proof. By Proposition 9, C_ϕ is Hilbert–Schmidt and hence compact if $\|\varphi\|_\infty < 1$. The only other situation when C_ϕ might be compact is when $|\varphi(1)| < 1$ and $\|\varphi\|_\infty = 1$. In that case, there are unimodular numbers ω, η, $\omega \neq 1$ so that $\varphi(\omega) = \eta$. If $\eta \neq 1$, then condition (2.5) fails:

$$\lim_{z \to \omega} \frac{|1 - \varphi(z)|^\alpha (1 - |z|^2)}{|1 - z|^\alpha (1 - |\varphi(z)|^2)} = \frac{|1 - \eta|^\alpha}{|1 - \omega|^\alpha |\varphi'(\omega)|} \neq 0.$$

Hence C_ϕ is not compact in that case. If φ is not an automorphism and $\varphi(\omega) = 1$ one can repeat the proof in Example 2, to show C_ϕ is compact. □

To finish the characterization of boundedness and compactness when φ is linear fractional, we need to address the case $\alpha < 0$. We do this in the following:

Proposition 12. *If $\alpha < 0$, then C_ϕ is always bounded, except the case when $\varphi(\mathbb{D})$ is a disk tangent at 1 and $\varphi(1) \neq 1$. The situations when the operator C_ϕ is compact are when either $\|\varphi\|_\infty < 1$, or φ is not an automorphism of $\mathbb{D}$, $|\varphi(1)| = 1$, and $\varphi(1) \neq 1$.*

Proof. If $\alpha < 0$, note that $((1 - \varphi(z))/(1 - z))^\alpha \in H^\infty(\mathbb{D})$ and hence C_ϕ is bounded, unless $\varphi(\mathbb{D})$ is a disk tangent at 1 and $\varphi(1) \neq 1$. The only situation when the boundedness of $((1 - \varphi(z))/(1 - z))^\alpha$ is not evident is when $\varphi(\mathbb{D})$ is a disk tangent at 1 and $\varphi(1) = 1$. In that case, note that $\varphi'(1)$ cannot be null, since linear fractional maps are univalent.

Assume now that $\varphi(\mathbb{D})$ is a disk tangent at 1 and $\varphi(1) \neq 1$. In that case, there is $\omega \in \partial\mathbb{D}$ such that $\varphi(\omega) = 1$. The consequence is that C_ϕ is unbounded because, condition (2.2) fails. Indeed:

$$\lim_{z \to \omega} \frac{|1 - \varphi(z)|^\alpha (1 - |z|^2)}{|1 - z|^\alpha (1 - |\varphi(z)|^2)} = \frac{1}{|\varphi'(\omega)||1 - \omega|^\alpha} (0^+)^\alpha = +\infty.$$

Besides the situation $\|\varphi\|_\infty < 1$, the only other situation when C_ϕ might be compact is when $\varphi(1)$ is unimodular and $\varphi(1) \neq 1$. Indeed, if $|\varphi(1)| < 1$ one can repeat the argument in Corollary 1 to show C_ϕ is non–compact. In the situation that φ is not an automorphism, $\varphi(1) \neq 1$, and $|\varphi(1)| = 1$, take $\omega = 1$, $\eta = \varphi(\omega)$ and repeat the proof in Example 2 to show C_ϕ is compact. □

We conclude by introducing a class of function spaces that arise naturally in the process of discussing the boundedness of $A_{\varphi,\alpha}$, $\alpha \geq 1$. They seem to be "Bergman copies" of the already known local Dirichlet spaces.

The Closed Graph Theorem shows that $A_{\varphi,\alpha}$ is bounded if and only if

$$C_\varphi \left((1 - z)^\alpha L_a^2(\mathbb{D}) \right) \subseteq (1 - z)^\alpha L_a^2(\mathbb{D}).$$

This leads to the problem of finding the bounded composition operators on the spaces $\mathcal{S}_\alpha = \mathbb{C} + (1 - z)^\alpha L_a^2(\mathbb{D})$. It is easy to see that these spaces are the subspaces of $L_a^2(\mathbb{D})$ consisting of functions $f \in L_a^2(\mathbb{D})$ that have a nontangential limit $f(1)$ at 1 and the property that $(f(1) - f(z))/(1 - z)^\alpha \in L_a^2(\mathbb{D})$. Indeed, if $f \in \mathbb{C} + (1 - z)^\alpha L_a^2(\mathbb{D})$ has the representation $f(z) = c + (z - 1)^\alpha g(z)$ for some constant $c \in \mathbb{C}$

and some $g \in L^2_a(\mathbb{C})$, then, by an argument already used above, the nontangential limit of f at 1 must exist and equal c. Indeed

$$|f(z) - c| = \frac{|1-z|^\alpha}{1-|z|}|g(z)|(1-|z|) \leq M|g(z)|(1-|z|)$$

if $z \to 1$ in a nontangential approach region, (since, like in the proof of Proposition 5, $|g(z)|(1-|z|) \to 0$ because $g \in L^2_a(\mathbb{D})$).

Actually, for the case $\alpha > 1$, our argument shows that, for all $f \in \mathcal{S}_\alpha$ the limit $f(1)$ exists as $z \to 1$ inside any boundary approach region

$$R_{M,\alpha}(1) = \left\{ z \in \mathbb{U} : \frac{|1-z|^\alpha}{1-|z|^2} < M \right\} \qquad M > 0. \tag{4.7}$$

These regions, called *tangential approach regions* that make α – contact with the unit circle at 1, were introduced in [5].

We can endow S_α with the norm

$$|f| := \sqrt{\|f\|^2 + \left\| \frac{f(1) - f(z)}{(1-z)^\alpha} \right\|^2}.$$

The norm above satisfies the parallelogram law and a routine argument shows it is also complete. Thus S_α are Hilbert spaces where norm-convergence implies uniform convergence on compacts, (since the latter fact is valid for the smaller Bergman–norm). One can introduce similar spaces by replacing 1 with any unimodular number λ in the construction above.

The spaces $\mathbb{C} + (\lambda - z)H^2(\mathbb{D})$, $\lambda \in \partial\mathbb{D}$ are called local Dirichlet spaces and were introduced in [7]. Composition operators on local Dirichlet spaces are studied in [9]. The spaces $\mathbb{C} + (\lambda - z)^\alpha L^2_a(\mathbb{D}) \supsetneqq \mathbb{C} + (\lambda - z)^\alpha H^2(\mathbb{D})$, $\alpha \geq 1$, look like "Bergman versions" of local Dirichlet spaces.

It is easy to establish the following connection between composition operators on the spaces studied in this section and those acting on the spaces S_α.

Proposition 13. *For each $\alpha > 1$, the operator $A_{\varphi,\alpha}$ is bounded, if and only if 1 is a boundary fixed point of φ, $\beta_\alpha < +\infty$, and C_φ is a bounded operator on S_α.*

Proof. If $A_{\varphi,\alpha}$ is bounded, then 1 is a boundary fixed point of φ and $\beta_\alpha < +\infty$, by Proposition 10 and its proof. Also $C_\varphi((1-z)^\alpha L^2_a(\mathbb{D})) \subseteq (1-z)^\alpha L^2_a(\mathbb{D})$ and, since $C_\varphi 1 = 1$, one gets that $C_\varphi S_\alpha \subseteq S_\alpha$. Therefore, by the closed graph principle, C_φ is a bounded operator on S_α. Conversely, if C_φ is a bounded on S_α, $\varphi(1) = 1$, and $\beta_\alpha < +\infty$, then

$$\varphi(R_{M,\alpha}(1)) \subseteq R_{M\beta_\alpha,\alpha}(1) \qquad M > 0.$$

The consequence is that the composite $f \circ \varphi$ of each function $f \in \mathcal{S}_\alpha$ and φ has a nontangential limit at 1 and the equality $f \circ \varphi(1) = f(1)$ holds. Indeed, if $z_n \to 1$ nontangentially, then all z_n belong to some region $R_{M,\alpha}(1)$. Therefore

$$\varphi(z_n) \in R_{M\beta_\alpha,\alpha}(1) \qquad n = 1, 2, \ldots$$

and, since $z_n \to 1$, $\varphi(z_n) \to 1$. It follows that $f \circ \varphi(z_n) \to f(1)$, which establishes the equality $f \circ \varphi(1) = f(1)$. Since $(1-z)^\alpha L^2_a(\mathbb{D})$ is the subspace of S_α consisting of functions of null nontangential limit at 1, it follows that C_φ leaves that subspace invariant and hence $A_{\varphi,\alpha}$ is bounded. $\square$

Thus, if $\alpha > 1$, the symbols conjugated to symbols inducing bounded composition operators on $L^2_a(\mu_p)$, are a select subclass of those inducing bounded composition operators on $\mathcal{S}_\alpha$. Although interesting, it is beyond the scope of this paper embarking on a thorough study of composition operators on the spaces $\mathcal{S}_\alpha$.

References

[1] J. E. Brennan, *The integrability of the derivative in comformal mapping*, J. London Math. Soc. (2), 18(1978), 261-272.

[2] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, New York, London, Tokyo, 1995.

[3] Z. Čučković and R. Zhao, *Weighted composition operators on the Bergmann space*, J. London Math. Soc. 70(2004), no. 2, 499–511.

[4] D. Luecking, *Forward and Reverse Carleson inequalities for functions in Bergman spaces and their derivatives*, Amer. J. Math., 107(1985), 85–111.

[5] V. Matache, *Weighted composition operators on H^2 and applications*, Complex Analysis Operator Theory, 2(2008), 169–197.

[6] Ch. Pommerenke, *Boundary Behavior of Conformal Maps*, Springer-Verlag 1992.

[7] S. Richter, C. Sundberg, *A formula for the local Dirichlet integral*, Michigan Math. J. 38(1991), 355–379.

[8] W. Rudin, *Real and Complex Analysis*, third edition, MacGraw–Hill 1987.

[9] D. Sarason, J.-N. Nuno O. Silva, *Composition operators on a local Dirichlet space*, Journal d' analise mathématique 87(2002), 433–450.

[10] J. H. Shapiro, *Composition Operators and Classical Function Theory,* Springer Verlag 1993.

[11] J. H. Shapiro and W. Smith, *Hardy spaces that support no compact composition operators*, J. Functional Analysis, 205(2003), 62–89.

[12] S. Shimorin, *Weighted composition operators associated with conformal mappings. Quadrature domains and their applications*, Oper. Theory Adv. Appl., 156(2005), 217–237.

[13] W. Smith, *Brennan's conjecture for weighted composition operators*, Contemporary Math. 393(2006), 209–214.

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