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Abstract

Given a polynomial $f(x)$ of degree n, let $f^r(x)$ denote its *reciprocal*, i.e., $f^r(x) = x^n f(1/x)$. If a polynomial is equal to its reciprocal, we call it a *palindrome* since the coefficients are the same when read backwards or forwards. In this mathematical note we show that palindromes whose coefficients satisfy a certain magnitude-condition must have a root on the unit circle. More exactly our main result is the following. If a palindrome $f(x)$ of even degree n with real coefficients $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ satisfies the condition $|\epsilon_k| \geq |\epsilon_{n/2}| \cos(\pi/([\frac{n/2}{n/2-k}] + 2))$, for some $k \in \{0, 1, \ldots n/2 - 1\}$, then $f(x)$ has unimodular roots. In particular, palindromes with coefficients 0 and 1 always have a root on the unit circle.

Résumé

Soit $f(x)$ un polynôme de degré n. Soit $f^r(x) = x^n f(1/x)$. Le polynôme $f(x)$ s'appelle polynôme réciproque si $f^r(x) = f(x)$. Dans cet article nous prouvons que les polynômes réciproques dont les coefficients possèdent une certaine propriété, ont des zéros sur le cercle unité, c'est à dire ont des zeros de valeur absolue 1. Notre principal résultat est le théorème suivant. Soit $f(x)$ un polynôme réciproque dont le degré n est pair et dont les coefficients $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ sont des nombres réels tels que $|\epsilon_k| \geq |\epsilon_{n/2}| \cos(\pi/([\frac{n/2}{n/2-k}] + 2))$, pour au moins une valeur de $k \in \{0, 1, \ldots n/2 - 1\}$. Tel pôlynome $f(x)$ posède des zéros de valeur absolue 1. Pour cosequence, chaque polynôme réciproque avec des coefficients 0 et 1 a des zéros sur le cercle unité.

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1 INTRODUCTION

We arrived at an interesting geometric property of palindrome-polynomials, i.e. of polynomials whose coefficients are the same when read backwards or forwards, by investigating polynomials with coefficients 0 and 1 or $(0,1)$-polynomials. They are interesting because of their applications in various areas of pure and applied mathematics including algebra, number theory, combinatorics, and coding theory. Computer-assisted experiments lead us to conjecture that $(0,1)$-palindromes necessarily have at least one unimodular root, that is, have a root of absolute value 1 . In the sequel we will prove not only that this conjecture is true but that palindromes with real coefficients often have roots on the unit circle. This introductory section is dedicated to setting up notations and terminology. The second section contains the main results of the article. We begin by defining the notion of palindrome. Given a polynomial $f(x)$ of degree n, let $f^r(x)$ denote its *reciprocal*, i.e., $f^r(x) = x^n f(1/x)$. If a polynomial is equal to its reciprocal, we call it a *palindrome-polynomial* or simply a *palindrome*. In the next section we establish rather easily the following. *Let $f(x) \in \mathbb{R}[X]$ be a palindrome having even degree n and null middle coefficient. Then the polynomial $f(x)$ has unimodular roots,* (Lemma 2). Note that the existence of unimodular roots is interesting only for palindromes of even degree, because if the degree is odd, obviously -1 is a root. By $\mathbb{R}[X]$ we denote the ring of polynomials of one variable with coefficients in $\mathbb{R}$, the field of real numbers. If the middle coefficient is not null, a palindrome with real coefficients may lack unimodular roots, (consider e.g. $f(x) = x^2 - 3x + 1$). It turns out that the relative size of the middle coefficient with respect to the other coefficients is important for the existence of unimodular roots, only the proof of this fact relies on deeper results on cosine polynomials going back to Féjer, Riesz, and Szegö. For each real x let $[x]$ denote the largest integer which is less than or equal to x. The main result proved in Section 2 is the following.

Let $f(x) = \sum_{k=0}^{n} \epsilon_k x^k$, with $\epsilon_j = \epsilon_{n-j}$, $\forall j = 0, 1, \ldots, n/2$ be a palindrome having even degree n. If there exists $k \in \{0, 1, \ldots, n/2 - 1\}$ such that

$$|\epsilon_k| \geq |\epsilon_{n/2}| \cos\left(\frac{\pi}{[\frac{n/2}{n/2-k}] + 2}\right),$$

then $f(x)$ has unimodular roots (Theorem 1).

In particular, $(0,1)$-palindromes of any degree have roots on the unit circle, (Corollary 1).

Another consequence of Theorem 1 is the following.

Let $f(x)$ be a palindrome with real coefficients, having even degree n. If $2|\epsilon_n| \geq |\epsilon_{n/2}|$ then $f(x)$ has unimodular roots, (Corollary 2).

2 PALINDROMES AND THEIR ZEROS

Our approach is based on the following classical construction. Denoting $u = x + 1/x$ we associate to each palindrome $f(x) \in \mathbb{R}[X]$ having even degree n the polynomial $g(u)$ uniquely determined by the following identity.

$$f(x) = x^{n/2} g(u) \tag{1}$$

Several comments are in order here. For each $k = 1, 2, \ldots$ we denote $\sigma_k(x) = x^k + 1/x^k$. For $k = 0$ we set $\sigma_0(x) = 1$. Observe that $\sigma_1(x) = u$ and that the following identity holds.

$$u^k = \sigma_k(x) + \sum_{1 \le j \le k/2} \sigma_{k-2j}(x) \binom{k}{j} \qquad k = 2, 3, \ldots \tag{2}$$

Let $f(x) = \sum_{k=0}^{n} \epsilon_k x^k$, with $\epsilon_j = \epsilon_{n-j}$, $\forall j = 0, 1, \ldots, n/2$. Note that

$$f(x) = x^{n/2} h(x) \qquad \text{where} \qquad h(x) = \sum_{k=0}^{n/2} \epsilon_k \sigma_{n/2-k}(x). \tag{3}$$

Equations (2) and (3) prove that $g(u)$ exists and is uniquely determined. After these preparations we can prove the following lemma, which appears also in [3], in essentially the same form. We include it with proof, to make the paper self-contained.

Lemma 1 *Let $f(x) \in \mathbb{R}[X]$ and $g(u) \in \mathbb{R}[U]$ be as above. The polynomial $f(x)$ has a unimodular root if and only if the polynomial $g(u)$ has a real root in the interval $[-2, 2]$, respectively if and only if the following cosine polynomial has real zeros.*

$$\varphi(x) = \epsilon_{n/2} + \sum_{k=0}^{n/2-1} 2\epsilon_k \cos((n/2 - k)x) \tag{4}$$

Proof. If $z = e^{i\theta}$ is a unimodular root of $f(x)$ then $u = 2\cos\theta$ is a root of $g(u)$ and clearly $-2 \le u \le 2$. Conversely if $g(u)$ has a root, $u \in [-2, 2]$, then there exists $\theta \in \mathbb{R}$ such that $2\cos\theta = u$. Let $z = e^{i\theta}$, then z is a unimodular root of $f(x)$. The fact that the existence of unimodular roots for the palindrome under consideration is equivalent to the existence of zeros for the cosine polynomial in (4) is an immediate consequence of the considerations above, equality (3), and the fact that if $x = e^{i\theta}$ then $\sigma_k(x) = 2\cos k\theta$ if $k \ge 1$. □

Based on the previous remarks we can prove the existence of unimodular roots for arbitrary palindromes with real coefficients, having even degree, and null middle-term. More exactly the following is true.

Lemma 2 *Let $f(x) \in \mathbb{R}[X]$ be a palindrome having even degree n and such that $\epsilon_{n/2} = 0$. Then the polynomial $f(x)$ has unimodular roots.*

Proof. By Lemma 1 it will suffice to show that a cosine polynomial of the form

$$\varphi(x) = \sum_{k=0}^{n/2-1} \epsilon_k \cos((n/2-k)x)$$

with real coefficients ϵ_k, necessarily has a zero in the interval $[0, 2\pi]$. The later is an immediate consequence of the fact that the integral of $\varphi(x)$ on $[0, 2\pi]$ equals 0 and $\varphi(x)$ is a continuous function. □

As we observed in the introduction, if the middle coefficient is not null, a palindrome with real coefficients may fail to have unimodular roots. However, the relative size of the middle coefficient with respect to the other coefficients matters here. The crucial inequality needed in the proof of our next and main result is sometimes known as the *Egerváry-Szász Inequality*. It states that a non-negative cosine polynomial of the form

$$\frac{1}{2} + \sum_{k=1}^{n} a_k \cos kx \quad a_n \neq 0$$

has the property

$$|a_k| \leq \cos\left(\frac{\pi}{[n/k]+2}\right) \qquad k \in \{1, 2, \ldots, n\}$$

(see **[1]**, **[4]**).

Theorem 1 *Let $f(x)$ be a palindrome with real coefficients, having even degree n, such that for some $k \in \{0, 1, \ldots, n/2-1\}$ the following condition is satisfied*

$$|\epsilon_k| \geq \cos\left(\frac{\pi}{[\frac{n/2}{n/2-k}]+2}\right) |\epsilon_{n/2}|. \tag{5}$$

Such a palindrome has unimodular roots.

Proof. The case $\epsilon_{n/2} = 0$ is covered by Lemma 2. Let us consider the case $\epsilon_{n/2} \neq 0$. Since $f(x)$ and $-f(x)$ have the same roots we can assume without loss of generality that $\epsilon_{n/2} > 0$. Denote by $\varphi(x)$ the cosine polynomial in equality (4). Since the integral of $\varphi(x)$ on $[0, 2\pi]$ is positive, it follows that $\varphi(x)$ attains positive values on that interval. Thus, the only way it can have no zeros would be if it is a positive cosine polynomial. Assume by contradiction that this is the case, and observe that the following cosine polynomial, $\phi(x)$, is also positive.

$$\phi(x) = 1/2 + \sum_{k=0}^{n/2-1} \frac{\epsilon_k}{\epsilon_{n/2}} \cos((n/2-k)x)$$

Let $\delta = \min\{\phi(x) : x \in [0, 2\pi]\}$. Observe that $0 < \delta < 1/2$. Indeed, $\phi(x) - 1/2$ is not the null function on $[0, 2\pi]$ and has null integral on that interval, thus $\phi(x) - 1/2$, being a continuous function, must assume both positive and negative values. Therefore the following cosine polynomial is non-negative on $\mathbb{R}$

$$1/2 + \sum_{k=0}^{n/2-1} \frac{\epsilon_k}{2\epsilon_{n/2}(1/2-\delta)} \cos((n/2-k)x).$$

By the Egerváry-Szász Inequality, in such a case one should have the following inequality satisfied for each $k \in \{0, 1, \ldots, n/2-1\}$

$$\left| \frac{\epsilon_k}{\epsilon_{n/2}(1-2\delta)} \right| \leq \cos\left(\frac{\pi}{[\frac{n/2}{n/2-k}] + 2} \right).$$

This leads to the fact that for each $k \in \{0, 1, \ldots, n/2-1\}$ one can write

$$\left| \frac{\epsilon_k}{\epsilon_{n/2}} \right| \leq \cos\left(\frac{\pi}{[\frac{n/2}{n/2-k}] + 2} \right)(1-2\delta) < \cos\left(\frac{\pi}{[\frac{n/2}{n/2-k}] + 2} \right),$$

which is contradictory under our assumptions. □

Corollary 1 *Let $f(x)$ be a palindrome with real coefficients, having even degree n, such that*

$$\max\{|\epsilon_k| : k \in \{0, 1, \ldots, n/2-1\}\} \geq |\epsilon_{n/2}|.$$

Such a palindrome has unimodular roots. In particular, $(0,1)$-palindromes of any degree have roots on the unit circle.

Proof. The statement above is a direct consequence of Theorem 1 and the well-known fact that

$$\cos\left(\frac{\pi}{[\frac{n/2}{n/2-k}] + 2} \right) \leq 1, \quad k \in \{0, 1, \ldots, n/2-1\}.$$

□

Corollary 2 *Let $f(x)$ be a palindrome with real coefficients, having even degree n. If*

$$2|\epsilon_n| \geq |\epsilon_{n/2}| \tag{6}$$

then $f(x)$ has unimodular roots.

Proof. Enter $k = 0$ in inequality (5) and recall that $\epsilon_n = \epsilon_0$. □

The author of [2] proves that palindromes of the form

$$l(z^n + z^{n-1} + \cdots + z + 1) + \sum_{k=1}^{\left[\frac{n}{2}\right]} a_k(z^{n-k} + z^k) \tag{7}$$

have all their roots on the unit circle if their coefficients satisfy the following condition

$$|l| \geq 2\sum_{k=1}^{\left[\frac{n}{2}\right]} |a_k|. \tag{8}$$

Palindromes of the form (7) are important because of their role in the investigation of spectral properties of the Coxeter transformation of certain oriented graphs. If one needs that palindromes of form (7) have some, (not necessarily all), roots on the unit circle, then the following less restrictive version of condition (8) can be proved.

Corollary 3 *Even degree palindromes of the form* (7) *have unimodular roots if*

$$|l| \geq 2|a_{n/2}|. \tag{9}$$

Proof. Note that the middle coefficient of such a palindrome is $2a_{n/2} + l$ and that if (9) holds then

$$|2a_{n/2} + l| \leq 2|a_{n/2}| + |l| \leq 2|l|,$$

that is (6) holds and hence the palidrome under consideration must have unimodular roots. □

The statement in Theorem 1 does not extend to palindromes with complex coefficients. Indeed, consider $f(z) = z^2 + iz + 1$. The condition in Theorem 1 is sufficient, but not necessary, for the existence of unimodular roots. Indeed, consider $h(x) = -2x^4 - 2x^3 + 5x^2 - 2x - 2$ and use Lemma 1 to show that it has unimodular roots. Obviously this palindrome does not satisfy the condition in Theorem 1.

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