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The Discontinuous Galerkin Finite Element Method for Ordinary Differential Equations

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Chapter 2

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Additional information is available at the end of the chapter

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Abstract

We present an analysis of the discontinuous Galerkin (DG) finite element method for nonlinear ordinary differential equations (ODEs). We prove that the DG solution is $(p + 1)$th order convergent in the L^2-norm, when the space of piecewise polynomials of degree p is used. A $(2p+1)$th order superconvergence rate of the DG approximation at the downwind point of each element is obtained under quasi-uniform meshes. Moreover, we prove that the DG solution is superconvergent with order $p+2$ to a particular projection of the exact solution. The superconvergence results are used to show that the leading term of the DG error is proportional to the $(p + 1)$-degree right Radau polynomial. These results allow us to develop a residual-based *a posteriori* error estimator which is computationally simple, efficient, and asymptotically exact. The proposed *a posteriori* error estimator is proved to converge to the actual error in the L^2-norm with order $p+2$. Computational results indicate that the theoretical orders of convergence are optimal. Finally, a local adaptive mesh refinement procedure that makes use of our local *a posteriori* error estimate is also presented. Several numerical examples are provided to illustrate the global superconvergence results and the convergence of the proposed estimator under mesh refinement.

Keywords: discontinuous Galerkin finite element method, ordinary differential equations, a priori error estimates, superconvergence, a posteriori error estimates, adaptive mesh refinement

1. Introduction

In this chapter, we introduce and analyze the discontinuous Galerkin (DG) method applied to the following first-order initial-value problem (IVP)

$$\frac{d\vec{u}}{dt} = \vec{f}(t,\vec{u}), \quad t \in [0,T], \quad \vec{u}(0) = \vec{u}_0, \tag{1}$$

where $\vec{u} : [0,T] \to \mathbb{R}^n$, $\vec{u}_0 \in \mathbb{R}^n$, and $\vec{f} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$. We assume that the solution exists and is unique and we would like to approximate it using a discontinuous piecewise polynomial space. According to the ordinary differential equation (ODE) theory, the condition $\vec{f} \in C^1([0,T] \times \mathbb{R}^n)$ is sufficient to guarantee the existence and uniqueness of the solution to (1). We note that a general nth-order IVP of the form $y^{(n)} = g(t, y, y', \ldots, y^{(n-1)})$ with initial conditions $y(0) = a_0$, $y'(0) = a_1, \ldots, y^{(n-1)}(0) = a_{n-1}$ can be converted into a system of equations in the form (1), where $\vec{u} = [y, y', \ldots, y^{(n-1)}]^t$, $\vec{f}(t, \vec{u}) = [u_2, u_3, \ldots, u_n, g(t, u_1, \ldots, u_n)]^t$, and $\vec{u}_0 = [a_0, a_1, \ldots, a_{n-1}]^t$.

The high-order DG method considered here is a class of finite element methods (FEMs) using completely discontinuous piecewise polynomials for the numerical solution and the test functions. The DG method was first designed as an effective numerical method for solving hyperbolic conservation laws, which may have discontinuous solutions. Here, we will discuss the algorithm formulation, stability analysis, and error estimates for the DG method solving nonlinear ODEs. DG method combines the best proprieties of the classical continuous finite element and finite volume methods such as consistency, flexibility, stability, conservation of local physical quantities, robustness, and compactness. Recently, DG methods become highly attractive and popular, mainly because these methods are high-order accurate, nonlinear stable, highly parallelizable, easy to handle complicated geometries and boundary conditions, and capable to capture discontinuities without spurious oscillations. The original DG finite element method (FEM) was introduced in 1973 by Reed and Hill [1] for solving steady-state first-order linear hyperbolic problems. It provides an effective means of solving hyperbolic problems on unstructured meshes in a parallel computing environment. The discontinuous basis can capture shock waves and other discontinuities with accuracy [2, 3]. The DG method can easily handle adaptivity strategies since the h refinement (mesh refinement and coarsening) and the p refinement (method order variation) can be done without taking into account the continuity restrictions typical of conforming FEMs. Moreover, the degree of the approximating polynomial can be easily changed from one element to the other [3]. Adaptivity is of particular importance in nonlinear hyperbolic problems given the complexity of the structure of the discontinuities and geometries involved. Due to local structure of DG methods, physical quantities such as mass, momentum, and energy are conserved locally through DG schemes. This property is very important for flow and transport problems. Furthermore, the DG method is highly parallelizable [4, 5]. Because of these nice features, the DG method has been analyzed and extended to a wide range of applications. In particular, DG methods have been used to solve ODEs [6–9], hyperbolic [5, 6, 10–19] and diffusion and convection diffusion [20–23] partial differential equations (PDEs), to mention a few. For transient problems, Cockburn and Shu [17] introduced and developed the so-called Runge-Kutta discontinuous Galerkin (RKDG) methods. These numerical methods use DG discretizations in space and combine it with an explicit Runge-Kutta time-marching algorithm. The proceedings of Cockburn et al. [24] and

Shu [25] contain a more complete and current survey of the DG method and its applications. Despite the attractive advantages mentioned above, DG methods have some drawbacks. Unlike the continuous FEMs, DG methods produce dense and ill-conditioned matrices increasing with the order of polynomial degree[23].

Related theoretical results in the literature including superconvergence results and error estimates of the DG methods for ODEs are given in [7–9, 26–28]. In 1974, LaSaint and Raviart [9] presented the first error analysis of the DG method for the initial-value problem (1). They showed that the DG method is equivalent to an implicit Runge-Kutta method and proved a rate of convergence of $\mathcal{O}(h^p)$ for general triangulations and of $\mathcal{O}(h^{p+1})$ for Cartesian grids. Delfour et al. [7] investigated a class of Galerkin methods which lead to a family of one-step schemes generating approximations up to order $2p+2$ for the solution of an ODE, when polynomials of degree p are used. In their proposed method, the numerical solution u_h at the discontinuity point t_n is defined as an average across the jump, i.e., $\alpha_n u_h(t_n^-) + (1-\alpha_n)u_h(t_n^+)$. By choosing special values of α_n, one can obtain the original DG scheme of LeSaint and Raviart [9] and Euler's explicit, improved, and implicit schemes. Delfour and Dubeau [27] introduced a family of discontinuous piecewise polynomial approximation schemes. They presented a more general framework of one-step methods such as implicit Runge-Kutta and Crank-Nicholson schemes, multistep methods such as Adams-Bashforth and Adams-Moulton schemes, and hybrid methods. Later, Johnson [8] proved new optimal *a priori* error estimates for a class of implicit one-step methods for stiff ODEs obtained by using the discontinuous Galerkin method with piecewise polynomials of degree zero and one. Johnson and Pitkaränta [29] proved a rate of convergence of $\mathcal{O}(h^{p+1/2})$ for general triangulations and Peterson [19] confirmed this rate to be optimal. Richter [30] obtained the optimal rate of convergence $\mathcal{O}(h^{p+1})$ for some structured two-dimensional non-Cartesian grids. We also would like to mention the work of Estep [28], where the author outlined a rigorous theory of global error control for the approximation of the IVP (1). In [6], Adjerid et al. showed that the DG solution of one-dimensional hyperbolic problems exhibit an $\mathcal{O}(h^{p+2})$ superconvergence rate at the roots of the right Radau polynomial of degree $p+1$. Furthermore, they obtained a $(2p+1)$th order superconvergence rate of the DG approximation at the downwind point of each element. They performed a local error analysis and showed that the local error on each element is proportional to a Radau polynomial. They further constructed implicit residual-based *a posteriori* error estimates but they did not prove their asymptotic exactness. In 2010, Deng and Xiong [31] investigated a DG method with interpolated coefficients for the IVP (1). They proved pointwise superconvergence results at Radau points. More recently, the author [12, 15, 26, 32–39] investigated the global convergence of the several residual-based *a posteriori* DG and local DG (LDG) error estimates for a variety of linear and nonlinear problems.

This chapter is organized as follows: In Section 2, we present the discrete DG method for the classical nonlinear initial-value problem. In Section 3, we present a detailed proof of the optimal *a priori* error estimate of the DG scheme. We state and prove our main superconvergence results in Section 4. In Section 5, we present the *a posteriori* error estimation procedure and prove that these error estimates converge to the true errors under mesh refinement. In Section 6, we propose an adaptive algorithm based on the local *a posteriori* error estimates. In Section 7, we present several numerical examples to validate our theoretical results. We conclude and discuss our results in Section 8.

2. The DG scheme for nonlinear IVPs

The error analysis of nonlinear scalar and vector initial-value problems (IVPs) having smooth solutions is similar. For this, we restrict our theoretical discussion to the following nonlinear initial-value problem (IVP)

$$u' = f(t,u), \quad t \in [0,T], \quad u(0) = u_0, \tag{2}$$

where $f(t,u): [0,T] \times \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function with respect to the variables t and u. More precisely, we assume that $\left|f_u(t,u)\right| \le M_1$ on the set $D = [0,T] \times \mathbb{R} \subset \mathbb{R}^2$, where M_1 is a positive constant. We note that the assumption $\left|f_u(t,u)\right| \le M_1$ is sufficient to ensure that $f(t,u)$ satisfies a Lipschitz condition in u on the convex set D with Lipschitz constant M_1

$$| f(t,u) - f(t,v) | \le M_1 | u - v |, \quad \text{for any (t,u) and (t,v)} \in \text{D}. \tag{3}$$

Next, we introduce the DG method for the model problem (2). Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the interval $\Omega = [0,T]$. We denote the mesh by $I_j = [t_{j-1}, t_j]$, $j = 1, \ldots, N$. We denote the length of I_j by $h_j = t_j - t_{j-1}$. We also denote $h = \max_{1 \le j \le N} h_j$ and $h_{\min} = \min_{1 \le j \le N} h_j$ as the length of the largest and smallest subinterval, respectively. Here, we consider regular meshes, that is, $\frac{h}{h_{\min}} \le \lambda$, where $\lambda \ge 1$ is a constant (independent of h) during mesh refinement. If $\lambda = 1$, then the mesh is uniformly distributed. In this case, the nodes and mesh size are defined by $t_j = j\,h$, $j = 0, 1, \ldots, N$, $h = T/N$.

Throughout this work, we define $v(t_j^-)$ and $v(t_j^+)$ to be the left limit and the right limit of the function v at the discontinuity point t_j, i.e., $v(t_j^-) = \lim_{s \to 0^-} v(t_j + s)$ and $v(t_j^+) = \lim_{s \to 0^+} v(t_j + s)$. To simplify the notation, we denote by $[v](t_j) = v(t_j^+) - v(t_j^-)$ the jump of v at the point t_j.

If we multiply (2) by an arbitrary test function v, integrate over the interval I_j, and integrate by parts, we get the DG weak formulation

$$\int_{I_j} v'u\,dt + \int_{I_j} f(t,u)v\,dt - u(t_j)v(t_j) + u(t_{j-1})v(t_{j-1}) = 0. \tag{4}$$

We denote by V_h^p the finite element space of polynomials of degree at most p in each interval I_j, i.e.,

$$V_h^p = \{v : v|_{I_j} \in P^p(I_j),\ j = 1,\dots,N\},$$

where $P^p(I_j)$ denotes the set of all polynomials of degree no more than p on I_j. We would like to emphasize that polynomials in V_h^p are allowed to have discontinuities at the nodes t_j.

Replacing the exact solution $u(t)$ by a piecewise polynomial $u_h(t) \in V_h^p$ and choosing $v \in V_h^p$, we obtain the DG scheme: Find $u_h \in V_h^p$ such that $\forall\ v \in V_h^p$ and $j = 1, \dots, N$,

$$\int_{I_j} v'u_h\,dt + \int_{I_j} f(t,u_h)v\,dt - \hat{u}_h(t_j)v(t_j^-) + \hat{u}_h(t_{j-1})v(t_{j-1}^+) = 0, \tag{5a}$$

where $\hat{u}_h(t_j)$ is the so-called numerical flux which is nothing but the discrete approximation u at the node $t = t_j$. We remark that u_h is not necessarily continuous at the nodes.

To complete the definition of the DG scheme, we still need to define $\hat{u}_h$ on the boundaries of I_j. Since for IVPs, information travel from the past into the future, it is reasonable to take $\hat{u}_h$ as the classical upwind flux

$$\hat{u}_h(t_0) = u_0, \quad \text{and} \quad \hat{u}_h(t_j) = u_h(t_j^-), \quad j = 1,\dots,N. \tag{5b}$$

2.1. Implementation

The DG solution $u_h(t)$ can be efficiently obtained in the following order: first, we compute $u_h(t)$ in the first element I_1 using (5a) and (5b) with $j = 1$ since $u_h(t_0^-) = u_0$ is known. Then, we can find $u_h(t)$ in I_2 since $u_h(t)$ in I_1 is already available. We can repeat the same process to compute $u_h(t)$ in $I_3, \dots, I_N$. More specifically, $u_h(t)$ can be obtained locally for each I_j using the following two steps: (i) express $u_h(t)$ as a linear combination of orthogonal basis $L_{i,j}(t)$, $i = 0, \dots, p$, where $L_{i,j}$ is the ith degree Legendre polynomial on I_j, i.e., $u_h(t) = \sum_{i=0}^{p} c_{i,j} L_{i,j}(t)$, $t \in I_j$, where

$\left\{L_{i,j}(t)\right\}_{i=0}^{i=p}$ is a local basis of $P^p(I_j)$, and (ii) choose the test functions $v = L_{k,j}(t)$, $k = 0, ..., p$. Thus, on each I_j, we get a $(p+1) \times (p+1)$ system of nonlinear algebraic equations, which can be solved for the unknown coefficients $c_{0,j}, ..., c_{p,j}$ using, e.g., Newton's method for nonlinear systems. Once we obtain the DG solution on all elements I_j, $j = 1, ..., N$, we get the DG solution which is a piecewise discontinuous polynomial of degree $\le p$. We refer to [7–9] for more details about DG methods for ODEs as well as their properties and applications.

2.2. Linear stability for the DG method

Let us now establish a stability result for the DG method applied to the linear case, i.e., $f(t,u) = \lambda u$. Taking $v = u_h$ in the discrete weak formulation (5a), we get

$$\frac{u_h^2(t_j^-)}{2} + \frac{u_h^2(t_{j-1}^+)}{2} - u_h(t_{j-1}^-)u_h(t_{j-1}^+) = \lambda \int_{I_j} u_h^2 dt,$$

which is equivalent to

$$\frac{u_h^2(t_j^-)}{2} - \frac{u_h^2(t_{j-1}^-)}{2} + \frac{1}{2}\left(u_h(t_{j-1}^-) - u_h(t_{j-1}^+)\right)^2 = \lambda \int_{I_j} u_h^2 dt.$$

Summing over all elements, we get the equality

$$\frac{u_h^2(T^-)}{2} - \frac{u_0^2}{2} + \frac{1}{2}\sum_{j=1}^{N}\left(u_h(t_{j-1}^-) - u_h(t_{j-1}^+)\right)^2 = \lambda \int_{\Omega} u_h^2 dt.$$

Consequently, $\frac{u_h^2(T^-)}{2} - \frac{u_0^2}{2} \le \lambda \int_{\Omega} u_h^2 dt$, which gives the stability result $u_h^2(T^-) \le u_0^2$ provided that $\lambda \le 0$.

3. *A priori* error analysis

We begin by defining some norms that will be used throughout this work. We define the L^2 inner product of two integrable functions, u and v, on the interval $I_j = [t_{j-1}, t_j]$ as $(u,v)_{I_j} = \int_{I_j} u(t)v(t)dt$. Denote $\left\|u\right\|_{0,I_j} = (u,u)_{I_j}^{1/2}$ to be the standard L^2 norm of u on I_j. Moreover, the standard L^∞ norm of u on I_j is defined by

$\left\|u\right\|_{\infty, I_j} = \sup_{t \in I_j} \left|u(t)\right|$. Let $H^s(I_j)$, where $s = 0, 1, \ldots$, denote the standard Sobolev space of square integrable functions on I_j with all derivatives $u^{(k)}$, $k = 0, 1, \ldots, s$ being square integrable on I_j, i.e., $H^s(I_j) = \left\{u : \int_{I_j} |u^{(k)}(t)|^2 dt < \infty,\ 0 \le k \le s\right\}$, and equipped with the norm $\left\|u\right\|_{s, I_j} = \left(\sum_{k=0}^{s} \left\|u^{(k)}\right\|^2_{0, I_j}\right)^{1/2}$. The $H^s(I_j)$ seminorm of a function u on I_j is given by $|u|_{s, I_j} = \left\|u^{(s)}\right\|_{0, I_j}$. We also define the norms on the whole computational domain Ω as follows:

$$\left\|u\right\|_{0,\Omega} = \left(\sum_{j=1}^{N} \left\|u\right\|^2_{0,I_j}\right)^{1/2}, \quad \left\|u\right\|_{\infty,\Omega} = \max_{1\le j\le N} \left\|u\right\|_{\infty,I_j}, \quad \left\|u\right\|_{s,\Omega} = \left(\sum_{j=1}^{N} \left\|u\right\|^2_{s,I_j}\right)^{1/2}.$$

The seminorm on the whole computational domain Ω is defined as $|u|_{s,\Omega} = \left(\sum_{j=1}^{N} |u|^2_{s, I_j}\right)^{1/2}$. We note that if $u \in H^s(\Omega)$, $s = 1, 2, \ldots$, then the norm $\left\|u\right\|_{s,\Omega}$ on the whole computational domain is the standard Sobolev norm $\left(\sum_{k=0}^{s} \left\|u^{(k)}\right\|^2_{0,\Omega}\right)^{1/2}$. For convenience, we use $\left\|u\right\|_{I_j}$ and $\left\|u\right\|$ to denote $\left\|u\right\|_{0,I_j}$ and $\left\|u\right\|_{0,\Omega}$, respectively.

For $p \ge 1$, we consider two special projection operators, $P_h^{\pm}$, which are defined as follows: For a smooth function u, the restrictions of $P_h^{+}u$ and $P_h^{-}u$ to I_j are polynomials in $P^p(I_j)$ satisfying

$$\int_{I_j} (P_h^- u - u) v dt = 0, \forall\, v \in P^{p-1}(I_j), \quad \text{and} \quad (P_h^- u - u)(t_j^-) = 0, \tag{6a}$$

$$\int_{I_j} (P_h^+ u - u) v dt = 0, \forall\, v \in P^{p-1}(I_j), \quad \text{and} \quad (P_h^+ u - u)(t_{j-1}^+) = 0. \tag{6b}$$

These two particular Gauss-Radau projections are very important in the proofs of optimal L^2 error estimates and superconvergence results. We note that the special projections $P_h^{\pm}u$ are mainly utilized to eliminate the jump terms at the cell boundaries in the error estimate in order to achieve the optimal order of accuracy [22].

For the projections mentioned above, it is easy to show that for any $u \in H^{p+1}(I_j)$ with $j = 1, ..., N$, there exists a constant C independent of the mesh size h such that (see, e.g., [40])

$$\left\|u - P_h^{\pm}u\right\|_{I_j} \leq Ch_j^{p+1}\left|u\right|_{p+1,I_j}, \quad \left\|(u - P_h^{\pm}u)'\right\|_{I_j} \leq Ch_j^{p}\left|u\right|_{p,I_j}. \tag{7}$$

Moreover, we recall the inverse properties of the finite element space V_h^p that will be used in our error analysis: For any $v_h \in V_h^p$, there exists a positive constant C independent of v_h and h, such that, $\forall\ j = 1, ..., N$,

$$\left\|v_h^{(k)}\right\|_{I_j} \leq Ch_j^{-k} \| v_h \|_{I_j}, \quad k \geq 1, \quad \left|v_h(t_{j-1}^{+})\right| + \left|v_h(t_j^{-})\right| \leq Ch_j^{-1/2} \| v_h \|_{I_j}. \tag{8}$$

From now on, the notation C, C_1, C_2, etc. will be used to denote generic positive constants independent of h, but may depend upon the exact solution of (1) and its derivatives. They also may have different values at different places.

Throughout this work, let us denote $e = u - u_h$ to be the error between the exact solution of (2) and the DG solution defined in (5a) and (5b), $\varepsilon = u - P_h^{-}u$ to be the projection error, and $\bar{e} = P_h^{-}u - u_h$ to be the error between the projection of the exact solution $P_h^{-}u$ and the DG solution u_h. We observe that the actual error can be written as $e = (u - P_h^{-}u) + (P_h^{-}u - u_h) = \varepsilon + \bar{e}$.

Now, we are ready to prove our optimal error estimates for e in the L^2 and H^1 norms.

Theorem 3.1. *Suppose that the exact solution of (2) is sufficiently smooth with bounded derivatives, i.e., $\left\|u\right\|_{p+1,\Omega}$ is bounded. We also assume that $\left|f_u(t,u)\right| \leq M_1$ on $D = [0,T] \times \mathbb{R}$. Let $p \geq 0$ and u_h be the DG solution of (5a) and (5b), then, for sufficiently small h, there exists a positive constant C independent of h such that,*

$$\left\|e\right\| \leq C\, h^{p+1}, \tag{9}$$

$$\sum_{j=1}^{N}\left\|e'\right\|_{I_j}^2 \leq Ch^{2p}, \quad \left\|e\right\|_{1,\Omega} \leq Ch^{p}. \tag{10}$$

Proof. We first need to derive some error equations which will be used repeatedly throughout this and the next sections. Subtracting (5a) from (4) with $v \in V_h^p$ and using the numerical flux (5b), we obtain the following error equation: $\forall\ v \in V_h^p$,

$$\int_{I_j} v'edt + \int_{I_j} (f(t,u) - f(t,u_h))vdt + e(t_{j-1}^-)v(t_{j-1}^+) - e(t_j^-)v(t_j^-) = 0. \tag{11}$$

By integration by parts, we get

$$\int_{I_j} e'vdt - \int_{I_j} \big(f(t,u) - f(t,u_h)\big)vdt + [e](t_{j-1})v(t_{j-1}^+) = 0. \tag{12}$$

Applying Taylor's series with integral remainder in the variable u and using the relation $u - u_h = e$, we write

$$f(t,u) - f(t,u_h) = \theta(u - u_h) = \theta e, \quad \text{where } \theta = \int_0^1 f_u(t, u + s(u_h - u))ds = \int_0^1 f_u(t, u - se)ds. \tag{13}$$

Substituting (13) into (12), we arrive at

$$\int_{I_j} \big(e' - \theta e\big)vdt + [e](t_{j-1})v(t_{j-1}^+) = 0, \quad \forall\ v \in V_h^p. \tag{14}$$

To simplify the notation, we introduce the bilinear operator $\mathcal{A}_j(e;V)$ as

$$\mathcal{A}_j(e;V) = \int_{I_j} (e' - \theta e)Vdt + [e](t_{j-1})V(t_{j-1}^+). \tag{15}$$

Thus, we can write (14) as

$$\mathcal{A}_j(e;v) = 0, \quad \forall\ v \in V_h^p. \tag{16}$$

A direct calculation from integration by parts yields

$$\mathcal{A}_j(e;V) = \int_{I_j} (-V' - \theta V)edt + e(t_j^-)V(t_j^-) - e(t_{j-1}^-)V(t_{j-1}^+). \tag{17}$$

On the other hand, if we add and subtract $P_h^+ V$ to V then we can write (15) as

$$\mathcal{A}_j(e;V) = \mathcal{A}_j(e;V - P_h^+ V) + \mathcal{A}_j(e;P_h^+ V). \tag{18}$$

Combining (18) and (16) with $v = P_h^+ V \in P^p(I_j)$ and applying the property of the projection P_h^+, i.e., $(V - P_h^+ V)(t_{j-1}^+) = 0$, we obtain

$$\mathcal{A}_j(e;V) = \int_{I_j} (e' - \theta e)(V - P_h^+ V)dt + [e](t_{j-1})(V - P_h^+ V)(t_{j-1}^+) = \int_{I_j} (e' - \theta e)(V - P_h^+ V)dt. \tag{19}$$

If v is a polynomial of degree at most p then v' is a polynomial of degree at most $p-1$. Therefore, by the property of the projection P_h^+, we immediately see

$$\int_{I_j} v'(V - P_h^+ V)dt = 0, \quad \forall\, v \in P^p(I_j). \tag{20}$$

Substituting the relation $e = \varepsilon + \bar{e}$ into (19) and invoking (20) with $v = \bar{e}$, we get

$$\mathcal{A}_j(e;V) = \int_{I_j} (\varepsilon' - \theta e)(V - P_h^+ V)dt + \int_{I_j} \bar{e}'(V - P_h^+ V)dt = \int_{I_j} (\varepsilon' - \theta e)(V - P_h^+ V)dt. \tag{21}$$

Now, we are ready to prove the theorem. We construct the following auxiliary problem: find φ such that

$$-\varphi' - \theta\varphi = e, \quad t \in [0,T] \quad \text{subject to} \quad \varphi(T) = 0. \tag{22}$$

where $\theta = \theta(t) = \int_0^1 f_u(t, u(t) - se(t))ds$. Clearly, the exact solution to (22) is given by the explicit formula

$$\varphi(t) = \frac{1}{\Theta(t)} \int_t^T \Theta(y)e(y)dy, \quad \text{where} \quad \Theta(t) = \exp\left(-\int_t^T \theta(s)ds\right). \tag{23}$$

Next, we prove some regular estimates which will be needed in our error analysis. Using the assumption $\left|f_u(t,u)\right| \le M_1$, we see that $\theta(t)$, $t \in [0,T]$ is bounded by M_1

$$|\theta(t)| \le \int_0^1 |f_u(t,u(t)-se(t))|\, ds \le \int_0^1 M_1 ds = M_1, \quad \forall\, t \in [0,T]. \tag{24a}$$

Using the definition of Θ and the estimate (24a), we have

$$0 \le \Theta(t) \le \exp\left(\int_0^T |\theta(s)|\, ds\right) \le \exp\left(\int_0^T M_1 ds\right) = \exp\left(M_1 T\right) = C_1. \tag{24b}$$

Similarly, we can easily estimate $\frac{1}{\Theta(t)}$ as follows

$$0 \le \frac{1}{\Theta(t)} = \exp\left(\int_t^T \theta(s) ds\right) \le \exp\left(\int_0^T |\theta(s)|\, ds\right) \le \exp\left(\int_0^T M_1 ds\right) = \exp\left(M_1 T\right) = C_1. \tag{24c}$$

Applying the estimates (24b), (24c), and the Cauchy-Schwarz inequality, we get

$$|\varphi(t)| \le \frac{1}{\Theta(t)} \int_t^T \Theta(y)\, |e(y)|\, dy \le C_1 \int_t^T C_1 |e(y)|\, dy \le C_1^2 \int_0^T |e(y)|\, dy \le C_1^2 T^{1/2} \|e\|, \quad t \in [0,T].$$

Squaring both sides and intergrading over Ω yields

$$\|\varphi\|^2 \le C_1^4 T^2 \|e\|^2 = C_2 \|e\|^2. \tag{25a}$$

We also need to obtain an estimate of $|\varphi|_{1,\Omega}$. Using (22) and (24a) gives

$$|\varphi'| = |\theta\varphi + e| \le M_1 |\varphi| + |e|, \quad t \in [0,T].$$

Squaring both sides, applying the inequality $(a+b)^2 \le 2a^2 + 2b^2$, integrating over the computational domain Ω, and using (25a), we get

$$|\varphi|_{1,\Omega}^2 = \int_0^T |\varphi'|^2\, dt \le 2(M_1^2 \|\varphi\|^2 + \|e\|^2) \le 2(M_1^2 C_2 + 1)\|e\|^2 \le C_3 \|e\|^2. \tag{25b}$$

Applying the projection result and the estimate (3.20b) yields

$$\|\varphi - P_h^+ \varphi\| \le C_4 h |\varphi|_{1,\Omega} \le C_5 h \|e\|. \tag{25c}$$

Now, we are ready to show (9). Using (17) with $V = \varphi$ and (22), we obtain

$$\mathcal{A}_j(e;\varphi) = \int_{I_j} (-\varphi' - \theta\varphi) e dt - e(t_{j-1}^-)\varphi(t_{j-1}) + e(t_j^-)\varphi(t_j) = \int_{I_j} e^2 dt - e(t_{j-1}^-)\varphi(t_{j-1}) + e(t_j^-)\varphi(t_j).$$

Summing over the elements and using the fact that $\varphi(T) = e(t_0^-) = 0$ yields

$$\sum_{j=1}^{N} \mathcal{A}_j(e;\varphi) = \|e\|^2 - e(t_0^-)\varphi(t_0^-) + e(T^-)\varphi(T^-) = \|e\|^2. \tag{26}$$

On the other hand, if we choose $V = \varphi$ in (21) then we get

$$\mathcal{A}_j(e;\varphi) = \int_{I_j} (\varepsilon' - \theta e)(\varphi - P_h^+\varphi) dt. \tag{27}$$

Summing over all elements and applying the Cauchy-Schwarz inequality, we get

$$\sum_{j=1}^{N} \mathcal{A}_j(e;\varphi) \le (\|\varepsilon'\| + M_1\|e\|)\|\varphi - P_h^+\varphi\|.$$

Using the estimate (25c), we deduce that

$$\sum_{j=1}^{N} \mathcal{A}_j(e;\varphi) \le (C_0 h^p |u|_{p+1,\Omega} + M_1\|e\|) C_1 h \|e\| \le C(h^{p+1} + h\|e\|)\|e\|. \tag{28}$$

Combining (26) and (28), we conclude that

$$\|e\| \le C h^{p+1} + C h \|e\|. \tag{29}$$

Thus, $(1 - Ch)\|e\| \le C\, h^{p+1}$, where C is a positive constant independent of h. Therefore, for sufficiently small h, e.g., $h \le \frac{1}{2C}$, we obtain $\frac{1}{2}\|e\| \le (1 - Ch)\|e\| \le C\, h^{p+1}$, which yields $\|e\| \le 2C\, h^{p+1}$ for h small. Thus, we completed the proof of (9).

To show (10), we use $e = \bar{e} + \varepsilon$, the classical inverse inequality (8), the estimate (9), and the projection result (7) to obtain

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$$\|e'\| = \|\overline{e}' + \varepsilon'\| \le \|\overline{e}'\| + \|\varepsilon'\| \le C_1 h^{-1} \|\overline{e}\| + C_2\, h^p \le C_1 h^{-1}\left(\|e\| + \|\varepsilon\|\right) + C_2\, h^p \le C_3 h^p + C_2\, h^p \le Ch^p.$$

We note that $\|e\|_{1,\Omega}^2 = \|e\|^2 + \|e'\|^2$. Applying (9) and the estimate $\|e'\| \le Ch^p$ yields $\|e\|_{1,\Omega}^2 \le C_1 h^{2p+2} + C_2 h^{2p} = O(h^{2p})$, which completes the proof of the theorem.

4. Superconvergence error analysis

In this section, we study the superconvergence properties of the DG method. We first show a $(2p+1)$th order superconvergence rate of the DG approximation at the downwind point of each element. Then, we apply this superconvergence result to show that the DG solution converges to the special projection of the exact solution $P_h^- u$ at $O(h^{p+2})$. This result allows us to prove that the leading term of the DG error is proportional to the $(p+1)$ degree right Radau polynomial.

First, we define some special polynomials. The pth degree Legendre polynomial can be defined by Rodrigues formula [41]

$$\tilde{L}_p(\xi) = \frac{1}{2^p p!}\frac{d^p}{d\xi^p}\left((\xi^2-1)^p\right), \quad -1 \le \xi \le 1.$$

It satisfies the following important properties: $\tilde{L}_p(1) = 1$, $\tilde{L}_p(-1) = (-1)^p$, and the orthogonality relation

$$\int_{-1}^{1} \tilde{L}_p(\xi)\tilde{L}_q(\xi)d\xi = \frac{2}{2p+1}\delta_{pq}, \quad \text{where } \delta_{pq} \text{ is the Kronecker symbol.} \tag{30}$$

One can easily write the $(p+1)$ degree Legendre polynomial on $[-1,1]$ as

$$\tilde{L}_{p+1}(\xi) = \frac{(2p+2)!}{2^{p+1}((p+1)!)^2}\xi^{p+1} + \tilde{q}_p(\xi), \quad \text{where } \tilde{q}_p \in P^p([-1,1]).$$

The $(p+1)$ degree right Radau polynomial on $[-1,1]$ is defined as $\tilde{R}_{p+1}(\xi) = \tilde{L}_{p+1}(\xi) - \tilde{L}_p(\xi)$. It has $p+1$ real distinct roots, $-1 < \xi_0 < \cdots < \xi_p = 1$.

Mapping I_j into the reference element $[-1,1]$ by the linear transformation $t = \frac{t_j + t_{j-1}}{2} + \frac{h_j}{2}\xi$, we obtain the shifted Legendre and Radau polynomials on I_j:

$$L_{p+1,j}(t) = \tilde{L}_{p+1}\left(\frac{2t - t_j - t_{j-1}}{h_j}\right), \quad R_{p+1,j}(t) = \tilde{R}_{p+1}\left(\frac{2t - t_j - t_{j-1}}{h_j}\right).$$

Next, we define the monic Radau polynomial, $\psi_{p+1,j}(t)$, on I_j as

$$\psi_{p+1,j}(t) = \frac{h_j^{p+1}[(p+1)!]^2}{(2p+2)!} R_{p+1,j}(t) = c_p h_j^{p+1} R_{p+1,j}(t), \quad \text{where } c_p = \frac{((p+1)!)^2}{(2p+2)!}. \tag{31}$$

Throughout this work the roots of $R_{p+1,j}(t)$ are denoted by $t_{j,i} = \frac{t_j + t_{j-1}}{2} + \frac{h_j}{2}\xi_i,\ i = 0, 1, \ldots, p$.

In the next lemma, we recall the following results which will be needed in our error analysis [32].

Lemma 4.1. *The polynomials $L_{p,j}$ and $\psi_{p+1,j}$ satisfy the following properties*

$$\left\|L_{p,j}\right\|_{I_j}^2 = \frac{h_j}{2p+1}, \quad \int_{I_j} \psi'_{p+1,j}\psi_{p+1,j} dt = -k_1 h_j^{2p+2}, \quad \left\|\psi_{p+1,j}\right\|_{I_j}^2 = (2p+2)k_2 h_j^{2p+3}, \tag{32}$$

where $k_1 = 2c_p^2$, $k_2 = \frac{k_1}{(2p+1)(2p+3)}$, and $c_p = \frac{((p+1)!)^2}{(2p+2)!}$.

Now, we are ready to prove the following superconvergence results.

Theorem 4.1. *Suppose that the assumptions of Theorem 1 are satisfied. Also, we assume that f_u is sufficiently smooth with respect to t and u (for example, $h(t) = f_u(t, u(t)) \in C^p([0,T])$ is enough.). Then there exists a positive constant C such that*

$$\left|e(t_k^-)\right| \le Ch^{2p+1}, \quad k = 1, \ldots, N, \tag{33}$$

$$\left|\bar{e}(t_k^-)\right| \le Ch^{2p+1}, \quad k = 1, \ldots, N, \tag{34}$$

$$\|\bar{e}'\| \le C\, h^{p+1}, \tag{35}$$

$$\|\bar{e}\| \le C\, h^{p+2}. \tag{36}$$

Proof. To prove (33), we proceed by the duality argument. Consider the following auxiliary problem:

$$W' + \theta W = 0, \quad t \in [0, t_k] \quad \text{subject to} \quad W(t_k) = 1, \tag{37}$$

where $1 \le k \le N$ and $\theta = \theta(t) = \int_0^1 f_u(t, u(t) - se(t))ds$. The exact solution of this problem is $W(t) = \exp\left(\int_t^{t_k} \theta(s)ds\right)$, $t \in \Omega_k = [0, t_k]$. Using the assumption $h(t) = f_u(t, u(t)) \in C^p([0, T])$ and the estimate (24a), we can easily show that there exists a constant C such that

$$\|W\|_{p+1,\Omega_k} \le C. \tag{38}$$

Using (17) and (37), we get

$$\mathcal{A}_j(e; W) = \int_{I_j} (-W' - \theta W) e dt + -e(t_{j-1}^-)W(t_{j-1}) + e(t_j^-)W(t_j) = -e(t_{j-1}^-)W(t_{j-1}) + e(t_j^-)W(t_j).$$

Summing over the elements I_j, $j = 1, \ldots, k$, using $W(t_k) = 1$, and the fact that $e(t_0^-) = 0$, we obtain

$$\sum_{j=1}^{k} \mathcal{A}_j(e; W) = -e(t_0^-)W(t_0) + e(t_k^-)W(t_k) = e(t_k^-). \tag{39}$$

Now, taking $V = W$ in (21) yields

$$\mathcal{A}_j(e; W) = \int_{I_j} (\varepsilon' - \theta e)(W - P_h^+ W) dt.$$

Summing over all elements I_j, $j = 1, \ldots, k$ with $k = 1, \ldots, N$ and applying (39), we arrive at

$$e(t_k^-) = \sum_{j=1}^{k} \int_{I_j} (\varepsilon' - \theta e)(W - P_h^+ W) dt.$$

Using (24a) and applying the Cauchy-Schwarz inequality, we obtain

$$\left| e(t_k^-) \right| \le (\|\varepsilon'\|_{0,\Omega_k} + M_1 \|e\|_{0,\Omega_k}) \|W - P_h^+ W\|_{0,\Omega_k} \le (\|\varepsilon'\| + M_1 \|e\|) \|W - P_h^+ W\|_{0,\Omega_k}.$$

Invoking the estimates (7), (9), and (38), we conclude that

$$\left| e(t_k^-) \right| \le (C_0 h^p |u|_{p+1,\Omega} + M_1 C_1 h^{p+1}) C_2 h^{p+1} |W|_{p+1,\Omega_k} \le C(h^p + h^{p+1}) h^{p+1} = \mathcal{O}(h^{2p+1}), \tag{40}$$

for all $k = 1, \ldots, N$, which completes the proof of (33).

In order to prove (34), we use the relation $e = \bar{e} + \varepsilon$, the property of the projection P_h^-, i.e., $\varepsilon(t_k^-) = 0$, and the estimate (33) to get

$$\left| \bar{e}(t_k^-) \right| = \left| e(t_k^-) - \varepsilon(t_k^-) \right| = \left| e(t_k^-) \right| = \mathcal{O}(h^{2p+1}).$$

Next, we will derive optimal error estimate for $\|\bar{e}'\|$. By the property of P_h^-, we have

$$\int_{I_j} \varepsilon v' dt = 0, \quad \forall\, v \in P^p(I_j), \quad \text{and} \quad \varepsilon(t_j^-) = 0, \quad j = 1, \ldots, N. \tag{41}$$

Using the relation $e = \varepsilon + \bar{e}$, applying (41) and (11) yields

$$\int_{I_j} v' \bar{e} dt + \int_{I_j} (f(t,u) - f(t,u_h)) v dt + \bar{e}(t_{j-1}^-) v(t_{j-1}^+) - \bar{e}(t_j^-) v(t_j^-) = 0.$$

By integration by parts on the first term, we obtain

$$\int_{I_j} \left(\bar{e}' - f(t,u) + f(t,u_h) \right) v dt + [\bar{e}](t_{j-1}) v(t_{j-1}^+) = 0. \tag{42}$$

Choosing $v(t) = \bar{e}'(t) - (-1)^p \bar{e}'(t_{j-1}^+) L_{p,j}(t) \in P^p(I_j)$ in (42), we have, by the property $\tilde{L}_p(-1) = (-1)^p$ and the orthogonality relation (30), $v(t_{j-1}^+) = 0$ and

$$\begin{aligned}\int_{I_j} (\bar{e}')^2 dt &= (-1)^p \bar{e}'(t_{j-1}^+) \int_{I_j} L_{p,j} \bar{e}' dt + \int_{I_j} (f(t,u) - f(t,u_h))(\bar{e}' - (-1)^p \bar{e}'(t_{j-1}^+) L_{p,j}) dt \\ &= \int_{I_j} (f(t,u) - f(t,u_h))(\bar{e}' - (-1)^p \bar{e}'(t_{j-1}^+) L_{p,j}) dt.\end{aligned} \tag{43}$$

Using (3) and applying the Cauchy-Schwarz inequality gives

$$\begin{aligned}\|\bar{e}'\|_{I_j}^2 &\leq \int_{I_j} |f(t,u) - f(t,u_h)| \left(|\bar{e}'| + |\bar{e}'(t_{j-1}^+)| |L_{p,j}|\right) dt \leq M_1 \int_{I_j} |e| \left(|\bar{e}'| + |\bar{e}'(t_{j-1}^+)| |L_{p,j}|\right) dt \\ &\leq M_1 \|e\|_{I_j} \left(\|\bar{e}'\|_{I_j} + |\bar{e}'(t_{j-1}^+)| \|L_{p,j}\|_{I_j} \right).\end{aligned} \tag{44}$$

Combining (44) with (8) and (32), we obtain

$$\|\bar{e}'\|_{I_j}^2 \leq M_1 \|e\|_{I_j} \left(\|\bar{e}'\|_{I_j} + \left(C_1 h_j^{-1/2} \|\bar{e}'\|_{I_j} \right) \left(\frac{h_j^{1/2}}{(2p+1)^{1/2}} \right) \right) \leq C \|e\|_{I_j} \|\bar{e}'\|_{I_j}.$$

Consequently, $\|\bar{e}'\|_{I_j} \leq C \|e\|_{I_j}$. Taking the square of both sides, summing over all elements, and using (9), we conclude that

$$\|\bar{e}'\|^2 \leq C \|e\|^2 \leq C h^{2p+2}. \tag{45}$$

Finally, we will estimate $\|\bar{e}\|$. Using the fundamental theorem of calculus, we write

$$|\bar{e}(t)| = |\bar{e}(t_j^-) + \int_{t_j}^{t} \bar{e}'(s) ds| \leq |\bar{e}(t_j^-)| + \int_{I_j} |\bar{e}'(s)| ds, \quad \forall\, t \in I_j.$$

Taking the square of both sides, applying the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, and applying the Cauchy-Schwartz inequality, we get

$$|\bar{e}(t)|^2 \le 2|\bar{e}(t_j^-)|^2 + 2\left(\int_{I_j} |\bar{e}'(s)|\,ds\right)^2 \le 2|\bar{e}(t_j^-)|^2 + 2h_j\int_{I_j} |\bar{e}'(s)|^2\,ds = 2|\bar{e}(t_j^-)|^2 + 2h_j\left\|\bar{e}'\right\|_{I_j}^2.$$

Integrating this inequality with respect to t and using the estimate (34), we get

$$\left\|\bar{e}\right\|_{I_j}^2 \le 2h_j|\bar{e}(t_j^-)|^2 + 2h_j^2\left\|\bar{e}'\right\|_{I_j}^2 \le 2Ch_j^{4p+3} + 2h_j^2\left\|\bar{e}'\right\|_{I_j}^2.$$

Summing over all elements and using the estimate (35) and the fact that $h = \max h_j$, we obtain

$$\left\|\bar{e}\right\|^2 \le C_1h^{4p+2} + 2h^2\left\|\bar{e}'\right\|^2 \le C_1h^{4p+2} + 2C_2h^{2p+4} = \mathcal{O}(h^{2p+4}), \tag{46}$$

where we used $4p + 2 \ge 2p + 4$ for $p \ge 1$. This completes the proof of the theorem.

The previous theorem indicates that the DG solution u_h is closer to P_h^-u than to the exact solution u. Next, we apply the results of Theorem 2 to prove that the actual error e can be split into a significant part, which is proportional to the $(p + 1)$ degree right Radau polynomial, and a less significant part that converges at $\mathcal{O}(h^{p+2})$ rate in the L^2 norm. Before we prove this result, we introduce two interpolation operators π and $\hat{\pi}$. The interpolation operator π is defined as follows: For smooth $u = u(t)$, $\pi u\big|_{I_j} \in P^p(I_j)$ and interpolates u at the roots of the $(p + 1)$ degree right Radau polynomial shifted to I_j, i.e., at $t_{j,i}$, $i = 0, 1, ..., p$,. The interpolation operator $\hat{\pi}$ satisfies $\hat{\pi}u\big|_{I_j} \in P^{p+1}(I_j)$ and $\hat{\pi}u|_{I_j}$ interpolates u at $t_{j,i}$, $i = 0, 1, ..., p$, and at an additional point $\bar{t}_j$ in I_j with $\bar{t}_j \ne t_{j,i}$, $i = 0, 1, ..., p$. The choice of the additional point is not important and can be chosen as $\bar{t}_j = t_{j-1}$.

Next, we recall the following results from [12] which will be needed in our analysis.

Lemma 4.2. *If $u \in H^{p+2}(I_j)$, then interpolation error can be split as*

$$u - \pi u = \phi_j + \gamma_j, \quad on \ \ I_j, \tag{47a}$$

where

$$\phi_j(t) = \alpha_j\psi_{p+1,j}(t), \quad \psi_{p+1,j}(t) = \prod_{i=0}^{p}(t - t_{j,i}), \quad \gamma_j = u - \hat{\pi}u, \tag{47b}$$

and α_j *is the coefficient of* t^{p+1} *in the* $(p+1)$ *degree polynomial* $\hat{\pi}u$. *Furthermore,*

$$\|\phi_j\|_{s,I_j} \le Ch_j^{p+1-s}\|u\|_{p+1,I_j}, \quad s=0,\ldots,p, \tag{47c}$$

$$\|\gamma_j\|_{s,I_j} \le Ch_j^{p+2-s}\|u\|_{p+2,I_j}, \quad s=0,\ldots,p+1. \tag{47d}$$

Finally,

$$\|\pi u - P_h^- u\|_{I_j} \le Ch_j^{p+2}\|u\|_{p+2,I_j}. \tag{48}$$

Proof. The proof of this lemma can be found in [12], more precisely in its Lemma 2.1.

The main global superconvergence result is stated in the following theorem.

Theorem 4.2. *Under the assumptions of Theorem 2, there exists a constant* C *independent of* h *such that*

$$\|u_h - \pi u\| \le Ch^{p+2}. \tag{49}$$

Moreover, the true error can be divided into a significant part and a less significant part as

$$e(t) = \alpha_j \psi_{p+1,j}(t) + \omega_j(t), \quad on \;\; I_j, \tag{50a}$$

where

$$\omega_j = \gamma_j + \pi u - u_h, \tag{50b}$$

and

$$\sum_{j=1}^{N}\|\omega_j\|_{I_j}^2 \le Ch^{2(p+2)}, \qquad \sum_{j=1}^{N}\|\omega_{j'}\|_{I_j}^2 \le Ch^{2(p+1)}. \tag{50c}$$

Proof. Adding and subtracting $P_h^- u$ to $u_h - \pi u$, we write $u_h - \pi u = u_h - P_h^- u + P_h^- u - \pi u = -\bar{e} + P_h^- u - \pi u$. Taking the L^2 norm and using the triangle inequality, we get

$$\|u_h - \pi u\| \le \|\bar{e}\| + \|P_h^- u - \pi u\|.$$

Applying the estimates (36) and (48), we deduce (49). Next, adding and subtracting πu to e, we write $e = u - \pi u + \pi u - u_h$. Moreover, one can split the interpolation error $u - \pi u$ on I_j as in (47a) to get

$$e = \phi_j + \gamma_j + \pi u - u_h = \phi_j + \omega_j, \quad \text{where} \quad \omega_j = \gamma_j + \pi u - u_h. \tag{51}$$

Next, we use the Cauchy-Schwarz inequality and the inequality $|ab| \le \frac{1}{2}(a^2 + b^2)$ to write

$$\begin{aligned}\|\omega_j\|_{I_j}^2 &= \left(\gamma_j + \pi u - u_h, \gamma_j + \pi u - u_h\right)_{I_j} = \|\gamma_j\|_{I_j}^2 + 2\left(\gamma_j, \pi u - u_h\right)_{I_j} + \|\pi u - u_h\|_{I_j}^2 \\ &\le 2\left(\|\gamma_j\|_{I_j}^2 + \|\pi u - u_h\|_{I_j}^2\right).\end{aligned}$$

Summing over all elements and applying (47d) with $s = 0$ and (49) yields the first estimate in (50c).

Using the Cauchy-Schwarz inequality and the inequality $|ab| \le \frac{1}{2}(a^2 + b^2)$, we write

$$\|\omega_{j'}\|_{I_j}^2 = \left(\gamma_{j'} + (\pi u - u_h)', \gamma_{j'} + (\pi u - u_h)'\right)_{I_j} \le 2\left(\|\gamma_{j'}\|_{I_j}^2 + \|(\pi u - u_h)'\|_{I_j}^2\right). \tag{52}$$

Using the inverse inequality (8), i.e., $\|(\pi u - u_h)'\|_{I_j} \le C\,h^{-1}\|(\pi u - u_h)\|_{I_j}$, we obtain the estimate

$$\|\omega_{j'}\|_{I_j}^2 \le C\left(\|\gamma_{j'}\|_{I_j}^2 + h^{-2}\|\pi u - u_h\|_{I_j}^2\right).$$

Summing over all elements and applying (49) and the estimate (47d) with $s = 1$, we establish the second estimate in (50c).

5. *A posteriori* error estimation

In this section, we use the superconvergence results from the previous section to construct a residual-based *a posteriori* error estimator which is computationally simple, efficient, and asymptotically exact. We will also prove its asymptotic exactness under mesh refinement. First, we present the weak finite element formulation to compute *a posteriori* error estimate for the nonlinear IVP (2).

In order to obtain a procedure for estimating the error e, we multiply (2) by arbitrary smooth function v and integrate over the element I_j to obtain

$$\int_{I_j} u' v dt = \int_{I_j} f(t,u) v dt. \tag{53}$$

Replacing u by $u_h + e$ and choosing $v = \psi_{p+1,j}(t)$, we obtain

$$\int_{I_j} e' \psi_{p+1,j} dt = \int_{I_j} \left(f(t,u_h + e) - u_h' \right) \psi_{p+1,j} dt. \tag{54}$$

Substituting (50a), i.e., $e(t) = \alpha_j \psi_{p+1,j}(t) + \omega_j(t)$, into the left-hand side of (54) yields

$$\alpha_j \int_{I_j} \psi'_{p+1,j} \psi_{p+1,j} dt = \int_{I_j} \left(f(t,u_h + e) - u_h' - \omega_j' \right) \psi_{p+1,j} dt. \tag{55}$$

Using (32) and solving for α_j, we obtain

$$\alpha_j = -\frac{1}{k_1 h_j^{2p+2}} \int_{I_j} \left(f(t,u_h + e) - u_h' - \omega_j' \right) \psi_{p+1,j} dt. \tag{56}$$

Our error estimate procedure consists of approximating the true error on each element I_j by the leading term as

$$e(t) \approx E(t) = a_j \psi_{p+1,j}(t), \quad t \in I_j, \tag{57a}$$

where the coefficient of the leading term of the error, a_j, is obtained from the coefficient α_j defined in (56) by neglecting the terms ω_j and e, i.e.,

$$a_j = -\frac{1}{k_1 h_j^{2p+2}} \int_{I_j} \left(f(t, u_h) - u_{h'} \right) \psi_{p+1,j} dt. \tag{57b}$$

We remark that our *a posteriori* error estimate is obtained by solving local problems with no boundary condition.

The global effectivity index, defined by $\sigma = \frac{\|E\|}{\|e\|}$, is an important criterion for evaluating the quality of an error estimator. The main results of this section are stated in the following theorem. In particular, we prove that the error estimate E, in the L^2 norm, converges to the actual error e. Moreover, we show that our *a posterior* error estimate is asymptotically exact by showing that the global effectivity index $\sigma \to 1$ as $h \to 0$.

Theorem 5.1. *Suppose that the assumptions of Theorem 2 are satisfied. If* $E(t) = a_j \psi_{p+1,j}(t)$, $t \in I_j$, *where* a_j, $j = 1, \ldots, N$, *are defined in (57b), then*

$$\|e - E\|^2 \leq C\, h^{2p+4}. \tag{58}$$

Thus, the post-processed approximation $u_h + E$ *yields* $O(h^{p+2})$ *superconvergent solution, i.e.,*

$$\|u - (u_h + E)\|^2 = \sum_{j=1}^{N} \left\| u - (u_h + a_j \psi_{p+1,j}) \right\|_{I_j}^2 \leq C\, h^{2p+4}. \tag{59}$$

Furthermore, then there exists a positive constant C *independent of* h *such that*

$$\left| \|e\|^2 - \|E\|^2 \right| \leq C\, h^{2p+4}. \tag{60}$$

Finally, if there exists a constant $C = C(u) > 0$ *independent of* h *such that*

$$\|e\| \geq C h^{p+1}, \tag{61}$$

then the global effectivity index in the L^2 *norm converges to unity at* $O(h)$ *rate, i.e.,*

$$\frac{\|E\|}{\|e\|} = 1 + \mathcal{O}(h). \tag{62}$$

Proof. First, we will prove (58) and (59). Since $e = \alpha_j \psi_{p+1,j} + \omega_j$ and $E = a_j \psi_{p+1,j}$ on I_j, we have

$$\|e - E\|_{I_j}^2 = \|(\alpha_j - a_j)\psi_{p+1,j} + \omega_j\|_{I_j}^2 \le 2(\alpha_j - a_j)^2 \|\psi_{p+1,j}\|_{I_j}^2 + 2\|\omega_j\|_{I_j}^2,$$

where we used the inequality $(a+b)^2 \le 2a^2 + 2b^2$. Summing over all elements yields

$$\|e - E\|^2 = \sum_{j=1}^{N} \|e - E\|_{I_j}^2 \le 2\sum_{j=1}^{N}(\alpha_j - a_j)^2 \|\psi_{p+1,j}\|_{I_j}^2 + 2\sum_{j=1}^{N}\|\omega_j\|_{I_j}^2. \tag{63}$$

Next, we will derive upper bounds for $\sum_{j=1}^{N}(\alpha_j - a_j)^2 \|\psi_{p+1,j}\|_{I_j}^2$. Subtracting (56) from (57b), we obtain

$$a_j - \alpha_j = \frac{1}{k_1 h_j^{2p+2}} \int_{I_j} \left(f(t, u_h + e) - f(t, u_h) - \omega_{j'} \right) \psi_{p+1,j} dt. \tag{64}$$

Thus,

$$\left|a_j - \alpha_j\right| \le \frac{1}{k_1 h_j^{2p+2}} \int_{I_j} \left(\left|f(t, u_h + e) - f(t, u_h)\right| + \left|\omega_j'\right| \right) \left|\psi_{p+1,j}\right| dt. \tag{65}$$

Using the Lipschitz condition (3) and applying the Cauchy-Schwarz inequality yields

$$\left|a_j - \alpha_j\right| \le \frac{1}{k_1 h_j^{2p+2}} \int_{I_j} \left(M_1 |e| + \left|\omega_j'\right| \right) \left|\psi_{p+1,j}\right| dt \le \frac{\|\psi_{p+1,j}\|_{I_j}}{k_1 h_j^{2p+2}} \left(M_1 \|e\|_{I_j} + \|\omega_j'\|_{I_j} \right). \tag{66}$$

Applying the inequality $(a+b)^2 \le 2(a^2 + b^2)$, we obtain

$$(a_j - \alpha_j)^2 \leq \frac{2\left\|\psi_{p+1,j}\right\|_{I_j}^2}{k_1^2 h_j^{4p+4}} \left(M_1^2 \left\|e\right\|_{I_j}^2 + \left\|\omega_j'\right\|_{I_j}^2 \right). \tag{67}$$

Multiplying by $\|\psi_{p+1,j}\|_{I_j}^2$ and using (32), i.e., $\|\psi_{p+1,j}\|_{I_j}^2 = (2p+2)k_2 h_j^{2p+3}$ yields

$$(a_j - \alpha_j)^2 \left\|\psi_{p+1,j}\right\|_{I_j}^2 \leq \frac{2\left\|\psi_{p+1,j}\right\|_{I_j}^4}{k_1^2 h_j^{4p+4}} \left(M_1^2 \left\|e\right\|_{I_j}^2 + \left\|\omega_j'\right\|_{I_j}^2 \right) \leq k_3 h_j^2 \left(\left\|e\right\|_{I_j}^2 + \left\|\omega_j'\right\|_{I_j}^2 \right), \tag{68}$$

where $k_3 = \dfrac{2(2p+2)^2 k_2^2}{k_1^2} \max(M_1^2, 1)$ is a constant independent of the mesh size.

Summing over all elements and using $h = \max\limits_{1 \leq j \leq N} h_j$, we arrive at

$$\sum_{j=1}^{N} (a_j - \alpha_j)^2 \left\|\psi_{p+1,j}\right\|_{I_j}^2 \leq k_3 h^2 \left[\left\|e\right\|^2 + \sum_{j=1}^{N} \left\|\omega_j'\right\|_{I_j}^2 \right].$$

Combining this estimate with (9) and (50c), we establish

$$\sum_{j=1}^{N} \left(a_j - \alpha_j\right)^2 \left\|\psi_{p+1,j}\right\|_{I_j}^2 \leq C h^{2p+4}. \tag{69}$$

Now, combining (63) and the estimates (50c) and (69) yields

$$\left\|e - E\right\|^2 \leq 2C_1 h^{2p+4} + 2C_2 h^{2p+4} = C h^{2p+4},$$

which completes the proof of (58). Using the relation $e = u - u_h$ and the estimate (58), we obtain

$$\sum_{j=1}^{N} \left\|u - (u_h + a_j \psi_{p+1,j})\right\|_{I_j}^2 = \left\|u - (u_h + E)\right\|^2 = \left\|e - E\right\|^2 \leq C h^{2p+4}.$$

Next, we will prove (60). Using the reverse triangle inequality, we have

$$\left| \|E\| - \|e\| \right| \le \|E - e\|, \tag{70}$$

which, after applying the estimate (58), completes the proof of (60).

In order to show (62), we divide (70) by $\|e\|$ to obtain $|\sigma - 1| \le \frac{\|E-e\|}{\|e\|}$. Applying the estimate (58) and the inverse estimate (61), we arrive at

$$|\sigma - 1| \le Ch.$$

Therefore, $\sigma = \frac{\|E\|}{\|e\|} = 1 + O(h)$, which establishes (62).

Remark 5.1. The previous theorem indicates that the computable quantity $\|E\|$ converges to $\|e\|$ at $O(h^{p+2})$ rate. This accuracy enhancement is achieved by adding the error estimate E to the DG solution u_h.

Remark 5.2. The performance of an error estimator σ is typically measured by the global effectivity index which is defined as the ratio of the estimated error $\|E\|$ to the actual error $\|e\|$. We say that the error estimator is asymptotically exact if $\sigma \to 1$ as $h \to 0$. The estimate (62) indicates that the global effectivity index in the L^2 norm converge to unity at $O(h)$ rate. Therefore, the proposed estimator $\|E\|$ is asymptotically exact. We would like to emphasize that E is a computable quantity since it only depends on the DG solution u_h and f. It provides an asymptotically exact *a posteriori* estimator on the actual error $\|e\|$. Finally, we would like to point out that our procedure for estimating the error e is computationally simple. Furthermore, our DG error indicator is obtained by solving a local problem with no boundary condition on each element. This makes it useful in adaptive computations. We demonstrate this in Section 6.

Remark 5.3. Our proofs are valid for any regular meshes and using piecewise polynomials of degree $p \ge 1$. If $p = 0$ then (46) gives $\|\bar{e}\| = O(h)$ which is the same as $\|e\| = O(h)$. Thus, our superconvergence results are not valid when using $p = 0$. Also, our error estimate procedure does not apply.

Remark 5.4. The assumption (61), which is used to prove the convergence of σ to unity at $O(h)$, requires that terms of order $O(h^{p+1})$ are present in the error. If not, E might not be a good approximation of e. We note that the exponent of h in the estimate (9) is optimal in the sense that it cannot be improved. In fact, for the $h$version finite element method one may show that provided that the $(p + 1)$th order derivatives of the exact solution u do not vanish identically over the whole domain, then an inverse estimate of the form $\|e\| \geq C(u)h^{p+1}$ is valid for some positive constant $C(u)$ depending only on u [42–44].

Remark 5. Our results readily extend to nonlinear systems of ODEs of the form

$$\frac{d\vec{u}}{dt} = \vec{f}(t,\vec{u}), \quad t \in [0,T], \quad \vec{u}(0) = \vec{u}_0,$$

where $\vec{u} = [u_1, \ldots, u_n]^t : [0,T] \to \mathbb{R}^n$, $\vec{u}_0 \in \mathbb{R}^n$, and $\vec{f} = [f_1, \ldots, f_n]^t : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$. The DG method for this problem consists of finding $\vec{u}_h \in \vec{V}_h^p = \left\{ \vec{v} : \vec{v}\big|_{I_j} \in (P^p(I_j))^n, \ j = 1, \ldots, N \right\}$ such that: $\forall\ \vec{v} \in \vec{V}_h^p$ and $j = 1, \ldots, N$,

$$\int_{I_j} (\vec{v}\,')^t \vec{u}_h dt + \int_{I_j} (\vec{v})^t \vec{f}(t,\vec{u}_h) dt - (\vec{v})^t (t_j^-) \vec{u}(t_j^-) + (\vec{v})^t (t_{j-1}^+) \vec{u}(t_{j-1}^-) = 0.$$

6. Application: adaptive mesh refinement (AMR)

A posteriori error estimates play an essential role in assessing the reliability of numerical solutions and in developing efficient adaptive algorithms. Adaptive methods based on *a posteriori* error estimates have become established procedures for computing efficient and accurate approximations to the solution of differential equations. The standard adaptive FEMs through local refinement can be written in the following loop

$$\text{SOLVE} \ \to \ \text{ESTIMATE} \ \to \ \text{MARK} \ \to \ \text{REFINE}.$$

The local *a posteriori* errors estimator of Section 5 can be used to mark elements for refinement.

Next, we present a simple DG adaptive algorithm based on the local *a posteriori* error estimator proposed in the previous section. The adaptive algorithm that we propose has the following steps:

1. Select a tolerance Tol and a maximum bound on the number of interval (say $N_{\max} = 1000$). Put $\|E\| = 1$.

2. Construct an initial coarse mesh with $N + 1$ nodes. For simplicity, we start with a uniform mesh having $N = 2$ elements.

3. While $N + 1 \leq N_{\max}$ and $\|E\| \geq Tol$ do

(a) Solve the DG scheme to obtain the solution u_h as described in Section 2.

(b) For each element, use (57a) and (57b) to compute the local error estimators $\|E\|_{I_j}$, $j = 1, \ldots, N$ as described in Section 5 and the global error estimator $\|E\| = \left(\sum_{j=1}^{N} \|E\|_{I_j}^2\right)^{1/2}$.

(c) For all elements I_j

i. Choose a parameter $0 \le \lambda \le 1$. If the estimated global error $\|E\|_{I_j} < \lambda \max_{j=1,\ldots,N} \|E\|_{I_j}$ then stop and accept the DG solution on the element I_j.

ii. Otherwise, reject the DG solution on I_j and divide the element I_j into two uniform elements by adding the coordinate of the midpoint of I_j to the list of nodes.

4. Endwhile.

Remark 6.1. *There are many possibilities for selecting the elements to be refined given the local error indicator $\|E\|_{I_j}$. In the above algorithm, we used the most popular fixed-rate strategy which consists of refining the element I_j if $\|E\|_{I_j} > \lambda \max_{j=1,\ldots,N} \|E\|_{I_j}$, where $0 \le \lambda \le 1$ is a parameter provided by the user. Note that the choice $\lambda = 0$ gives uniform refinement, while $\lambda = 1$ gives no refinement. Also, there are other stopping criteria that may be used to stop the adaptive algorithm.*

7. Computational results

In this section, we present several numerical examples to (i) validate our superconvergence results and the global convergence of the residual-based *a posteriori* error estimates, and (ii) test the above local adaptive mesh refinement procedure that makes use of our local *a posteriori* error estimate.

Example 7.1. *The test problem we consider is the following nonlinear IVP*

$$u' = -u - u^2, \quad t \in [0,1], \quad u(0) = 1.$$

Clearly, the exact solution is $u(t) = \dfrac{1}{2e^t - 1}$. We use uniform meshes obtained by subdividing the computational domain [0,1] into N intervals with N = 5, 10, 20, 30, 40, 50. This example is tested by using P^p polynomials with $p = 0 - 4$. **Figure 1** shows the L^2 errors $\|e\|$ and $\|\bar{e}\|$ with log-log scale as well as their orders of convergence. These results indicate that $\left\|e\right\| = \mathcal{O}(h^{p+1})$ and $\left\|\bar{e}\right\| = \mathcal{O}(h^{p+2})$. This example demonstrates that our theoretical convergence rates are optimal.

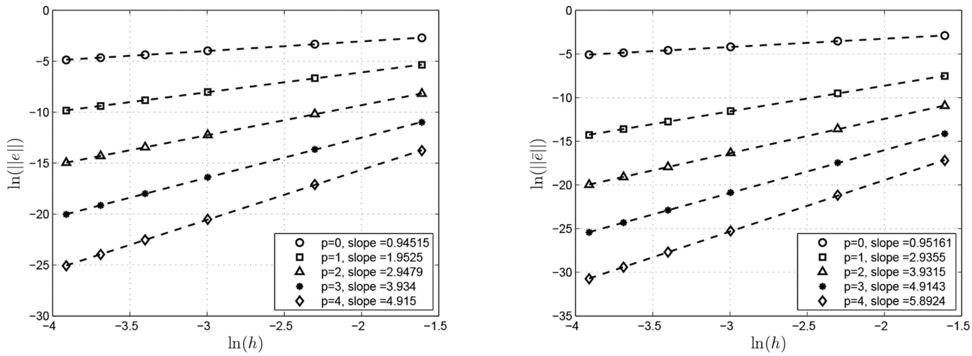

Figure 1. Log-log plots of $\left\|e\right\|$ (left) and $\left\|\bar{e}\right\|$ (right) versus mesh sizes h for Example 7.1 on uniform meshes having $N = 5, 10, 20, 30, 40, 50$ elements using $P^p, p = 0$ to 4.

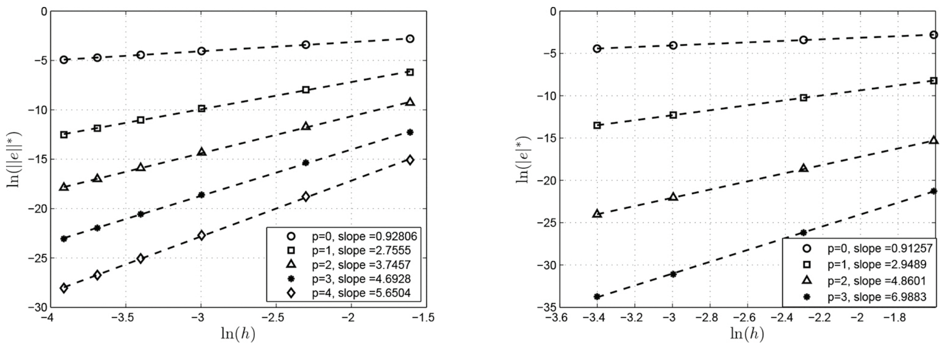

Figure 2. Log-log plots of $\left\|e\right\|^*$ (left) versus h for Example 7.1 using $N = 5, 10, 20, 30, 40, 50$ and $P^p, p = 0$ to 4. Log-log plots of $|e|^*$ (right) versus h using $N = 5, 10, 20, 30$ elements using $P^p, p = 0$ to 3.

Next, we compute the maximum error at the shifted roots of the $(p + 1)$ degree right Radau polynomial on each element I_j and then take the maximum over all elements. For simplicity, we use $\left\|e\right\|^*$ to denote $\max_{1 \le j \le N}\left(\max_{0 \le i \le p}\left|e(t_{j,i}^-)\right|\right)$, where $t_{j,i}$ are the roots of $R_{p+1,j}(t)$. Similarly, we compute the true error at the downwind point of each element and then we denote $|e|^*$ to be the maximum over all elements I_j, $j = 1, \ldots, N$, i.e.,

$|e|^* = \max_{1 \le j \le N} \left| e(t_j^-) \right|$. In **Figure 2**, we present the errors $\left\| e \right\|^*$, $|e|^*$ and their orders of convergence. We observe that $\left\| e \right\|^* = O(h^{p+2})$ and $|e|^* = O(h^{2p+1})$ as expected. Thus, the error at right Radau points converges at $O(h^{p+2})$. Similarly, the error at the downwind point of each element converge with an order $2p+1$. This is in full agreement with the theory.

Next, we use (57a) and (57b) to compute the *a posteriori* error estimate for the DG solution. The global errors $\left\| e - E \right\|$ and their orders of convergence, using the spaces P^p with $p = 1 - 4$, are shown in **Figure 3**. We observe that $\left\| e - E \right\| = O(h^{p+2})$. This is in full agreement with the theory. This example demonstrates that the convergence rate proved in this work is sharp. Since $\left\| e - E \right\| = \left\| u - (u_h + E) \right\| = O(h^{p+2})$, we conclude that the computable quantities $u_h + E$ converges to the exact solution u at $O(h^{p+2})$ rate in the L^2 norm. We would like to emphasize that this accuracy enhancement is achieved by adding the error estimate E to the DG solution u_h only once at the end of the computation. This leads to a very efficient computation of the postprocessed approximation $u_h + E$.

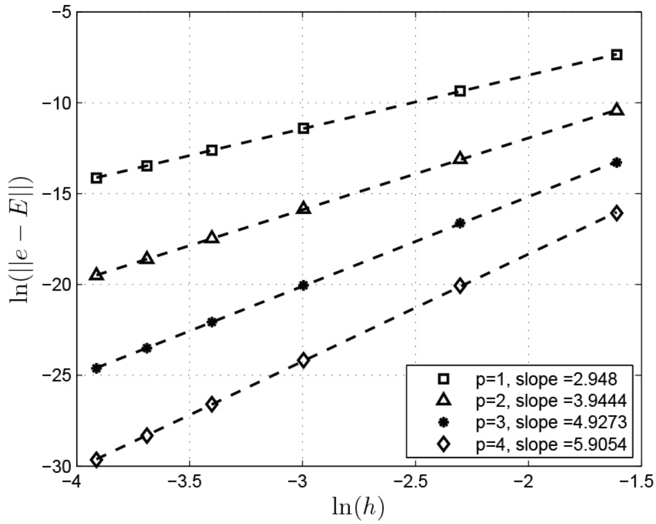

Figure 3. The errors $\left\| e - E \right\|$ and their orders of convergence for Example 1 on uniform meshes having $N = 5, 10, 20, 30, 40, 50$ elements using P^p, $p = 1$ to 4.

In **Table 1**, we present the actual L^2 errors and the global effectivity indices. These results demonstrate that the proposed *a posteriori* error estimates is asymptotically exact.

N	$p=1$		$p=2$		$p=3$		$p=4$	
	$\|e\|$	σ	$\|e\|$	σ	$\|e\|$	σ	$\|e\|$	σ
5	4.7637e-03	1.0362	2.7867e-04	1.0531	1.6847e-05	1.0637	1.0386e-06	1.0705
10	1.2750e-03	1.0179	3.7805e-05	1.0271	1.1742e-06	1.0326	3.7481e-08	1.0363
20	3.2849e-04	1.0089	4.8747e-06	1.0136	7.6227e-08	1.0164	1.2290e-09	1.0182
30	1.4736e-04	1.0059	1.4568e-06	1.0090	1.5201e-08	1.0109	1.6369e-10	1.0122
40	8.3262e-05	1.0044	6.1698e-07	1.0068	4.8296e-09	1.0082	3.9026e-11	1.0091
50	5.3429e-05	1.0035	3.1660e-07	1.0054	1.9827e-09	1.0066	1.2820e-11	1.0073

Table 1. The errors $\|e\|$ and the global effectivity indices for Example 7.1 on uniform meshes having $N = 5, 10, 20, 30, 40, 50$ elements using P^p, $p = 1$ to 4.

In **Figure 4**, we show the errors $\delta e = |\|e\| - \|E\||$ and $\delta\sigma = |\sigma - 1|$. We see that $\delta e = \mathcal{O}(h^{p+2})$ and $\delta\sigma = \mathcal{O}(h)$ as the theory predicts.

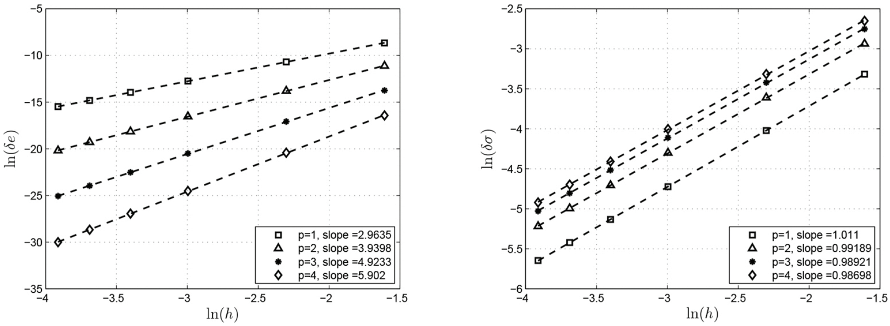

Figure 4. Convergence rates for δe (left) and $\delta\sigma$ (right) for Example 1 on uniform meshes having $N = 5, 10, 20, 30, 40, 50$ elements using P^p, $p = 1$ to 4.

Example 7.2. In this example we test our error estimation procedure presented in Section 6 on adaptively refined meshes. We consider the following model problem

$$u' = \beta u, \quad t \in [0,5], \quad u(0) = 1,$$

where the exact solution is simply $u(t) = e^{\beta t}$. We apply our adaptive algorithm using $\beta = 1$ (unstable), $\beta = -1$ (stable), and $\beta = -20$ (stiff). The DG solutions and the sequence of meshes obtained by applying our adaptive algorithm with $Tol = 10^{-2}$ for $p = 1 - 4$ are shown in **Figures 5–7** for $\beta = 1$, $\beta = -1$, and $\beta = -20$, respectively. As can be expected, the adaptive algorithm refines in the vicinity of the endpoint $t = 5$ with coarser meshes for increasing polynomial degree p. Furthermore, we observe that, when λ is closer to 0, we get more uniform refinement near the portion with high approximation error. When λ is near 1, we get less uniform refinement near the portion with high approximation error. We also observed that, both for $\lambda = 0.2$ and $\lambda = 0.9$, the optimal convergence rates are achieved asymptotically and that the global effectivity indices converge to unity with increasing polynomial degree p. Furthermore, we tested our adaptive algorithm on other problems and observed similar conclusions. These results are not included to save space.

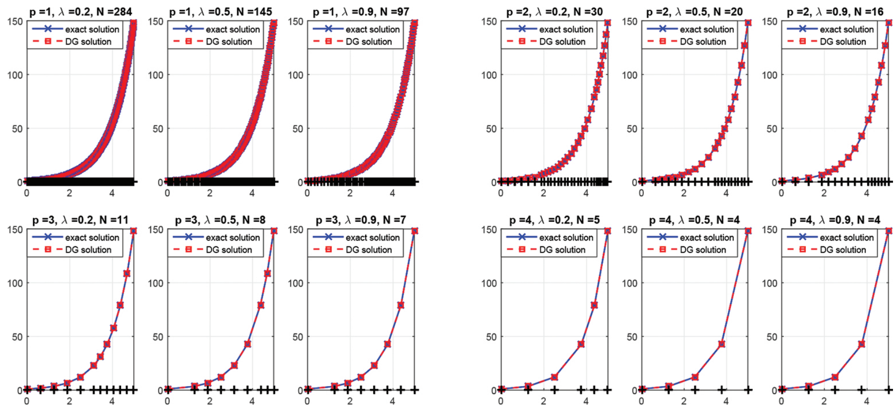

Figure 5. u, u_h, and final meshes for Example 7.2 with $\beta = -20$ using P^p, $p = 1$ to 4, and $Tol = 10^{-2}$.

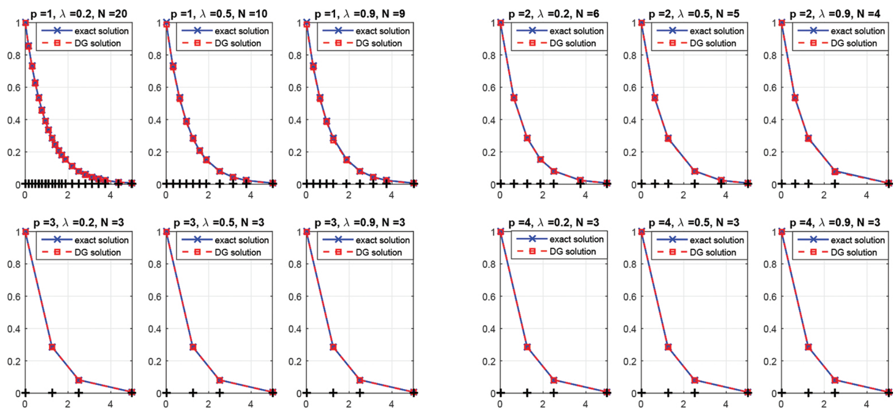

Figure 6. u, u_h, and final meshes for Example 7.2 with $\beta = -1$ using P^p, $p = 1$ to 4, and $Tol = 10^{-2}$.

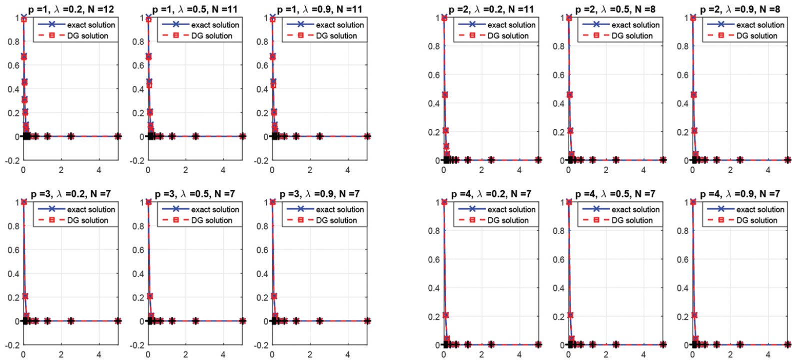

Figure 7. u, u_h, and final meshes for Example 7.2 with $\beta = -20$ using P^p, $p = 1$ to 4, and $Tol = 10^{-2}$.

8. Concluding remarks

In this chapter, we presented a detailed analysis of the original discontinuous Galerkin (DG) finite element method for the approximation of initial-value problems (IVPs) for nonlinear ordinary differential equations (ODEs). We proved several optimal error estimates and superconvergence results. In particular, we showed that the DG solution converges to the true solution with order $p + 1$, when the space of piecewise polynomials of degree p is used. We further proved the $(2p + 1)$th superconvergence rate at the downwind points. Moreover, we proved that the DG solution is $O(h^{p+2})$ superconvergent toward a particular projection of the exact solution. We used these results and showed that the leading term of the DG error is proportional to the $(p + 1)$ degree right Radau polynomial. This result allowed us to construct computationally simple, efficient, and asymptotically exact *a posteriori* error estimator. It is obtained by solving a local residual problem with no boundary condition on each element. Furthermore, we proved that the proposed *a posteriori* error estimator converges to the actual error in the L^2 norm. The order of convergence is proved to be $p + 2$. All proofs are valid for regular meshes and for P^p polynomials with $p \geq 1$. Finally, we presented a local adaptive procedure that makes use of our local *a posteriori* error estimate. Future work includes the study of superconvergence of DG method for nonlinear boundary-value problems.

Abbreviations

Symbols

AMR	Adaptive mesh refinement
DG	Discontinuous Galerkin
FEM, FEMs	Finite element method, finite element methods
IVP, IVPs	Initial-value problem, initial-value problems

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ODE, ODEs	Ordinary differential equation, ordinary differential equations
PDE, PDEs	Partial differential equation, partial differential equations
RKDG	Runge-Kutta discontinuous Galerkin
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n$	Real n-dimensional vector space
(a, b), $[a, b]$	$\{x \in \mathbb{R}: a < x < b\}$, $\{x \in \mathbb{R}: a \leq x \leq b\}$
$C(X)$	Set of all functions continuous on X
$C^m(X)$	Set of all functions having m continuous derivatives on X
$C^\infty(X)$	Space of functions infinitely differentiable on X
$\vec{v}$	Typical vector in $\mathbb{R}^n$ of the form $\vec{v} = [v_1, v_2, \ldots, v_n]^t$
δ_{ij}	Kronecker symbol
v', v''	First and second derivatives of v with respect to t
$v^{(n)}$	nth derivative of v with respect to t
	"big oh" asymptotic bound
$\left\vert x_n - L \right\vert = O(h^m)$	The sequence $\{x_n\}$ converges to L with order m
M_1	Lipschitz constant
I_j	Interval $[t_{j-1}, t_j]$, $j = 1, \ldots, N$
h_j	Length of interval I_j, $h_i = t_j - t_{j-1}$
h	Maximum of h_j, $h = \max_{1 \leq j \leq N} h_j$
$v(t_j^-)$	Left limit of the function v at the point t_j
$v(t_j^+)$	Right limit of the function v at the point t_j
$[v](t_j)$	Jump of v at the point t_j
p	Degree of a polynomial (integer)
$P^p(I_j)$	Set of all polynomials of degree no more than p on I_j
V_h^p	Finite element space of polynomials of degree at most p in the interval I_j
$\forall$	For all
$(v, w)_{I_j}$	L^2 inner product of v and w on the interval I_j, $(v, w)_{I_j} = \int_{I_j} v(t)w(t)dt$
$\left\Vert v \right\Vert_{0,I_j}$ and $\left\Vert v \right\Vert_{I_j}$	Standard L^2-norm of v on I_j
$\left\Vert v \right\Vert_{0,\Omega}$ and $\left\Vert v \right\Vert$	L^2-norm of v on $\Omega = \cup_{j=1}^{N} I_j$
$\left\Vert v \right\Vert_{\infty,I_j}$, $\left\Vert v \right\Vert_\infty$	Standard L^∞-norm of v on I_j and Ω, respectively

$H^s(I_j)$	Sobolev space $H^s(I_j) = \left\{ v : \int_{I_j} \lvert v^{(k)}(t) \rvert^2 dt < \infty,\ 0 \le k \le s \right\}$
$\lVert v \rVert_{s, I_j}$	Norm of $H^s(I_j)$
$\lvert v \rvert_{s, I_j}$	$H^s(I_j)$-seminorm of v on I_j
$\lvert v \rvert_{s, \Omega}$ and $\lvert v \rvert_s$	$H^s(I_j)$-seminorm of v on Ω
$P_h^{\pm} v$	Gauss-Radau projections of v onto V_h^p
$C,\ C_1,\ C_2$, etc	Generic positive constants independent of h
$u(t)$	Exact solution at time t
$u_h(t)$	DG solution at time t
$\hat{u}_h(t_j)$	$\hat{u}_h(t_j) = u_h(t_j^-)$
e	True error, $e = u - u_h$
ε	Projection error, $\varepsilon = u - P_h^- u$
$\bar{e}$	Error $\bar{e} = u_h - P_h^- u$.
$\tilde{L}_i(\xi),\ i = 0, 1, \ldots, p$	ith degree Legendre polynomial on $[-1,1]$
$\tilde{R}_i(\xi),\ i = 0, 1, \ldots, p$	ith degree right Radau polynomial on $[-1,1]$
$L_{i,j}(t),\ i = 0, 1, \ldots, p$	ith degree Legendre polynomial on I_j
$R_{i,j}(t),\ i = 0, 1, \ldots, p$	ith degree right Radau polynomial on I_j
$\psi_{i,j}(t),\ i = 0, 1, \ldots, p$	ith degree monic Radau polynomial on I_j
$\xi_i,\ i = 0, 1, \ldots, p$	Roots of $\tilde{R}_{p+1}(\xi)$
$t_{j,i},\ i = 0, 1, \ldots, p$	Roots of $R_{p+1,j}(t)$
πu	Interpolant, interpolates $u(t)$ at $t_{j,i},\ i = 0, 1, \ldots, p,$
$\hat{\pi} u$	Interpolant, interpolates $u(t)$ at $t_{j,i},\ i = 0, 1, \ldots, p,$ and at an additional point $\bar{t}_j$ in I_j with $\bar{t}_j \ne t_{j,i},\ i = 0, 1, \ldots, p$
γ_j	$u - \hat{\pi} u$
$\phi_j(t)$	$\alpha_j \psi_{p+1,j}(t)$, where α_j is the coefficient of t^{p+1} in the $(p+1)$ degree polynomial $\hat{\pi} u$
ω_j	$\gamma_j + \pi u - u_h$
$\sigma,\ \delta\sigma$	Global effectivity index, $\delta\sigma = \lvert \sigma - 1 \rvert$
Tol, N_{max}	Tolerance, maximum bound on the number of interval
$\lVert e \rVert^*$	Maximum error at Radau points
$\lvert e \rvert^*$	Maximum error at the downwind points

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References

[1] W. H. Reed, T. R. Hill, Triangular mesh methods for the neutron transport equation, Tech. Rep. LA-UR-73-479, Los Alamos Scientific Laboratory, Los Alamos, 1973.

[2] R. Biswas, K. Devine, J. E. Flaherty, Parallel adaptive finite element methods for conservation laws, Applied Numerical Mathematics 14 (1994): 255–284.

[3] J.-F. Remacle, J. E. Flaherty, M. S. Shephard, An adaptive discontinuous Galerkin technique with an orthogonal basis applied to compressible flow problems, SIAM Review 45 (2002): 53–72.

[4] J. Flaherty, R. Loy, M. Shephard, J. Teresco, Software for the parallel adaptive solution of conservation laws by discontinuous Galerkin methods, in: B. Cockburn, G. Karniadakis, C.-W. Shu (eds.), Discontinuous Galerkin Methods, Vol. 11 of Lecture Notes in Computational Science and Engineering, Springer, Berlin Heidelberg, 2000, pp. 113–123.

[5] J. E. Flaherty, R. Loy, M. S. Shephard, B. K. Szymanski, J. D. Teresco, L. H. Ziantz, Adaptive local refinement with octree load-balancing for the parallel solution of three-dimensional conservation laws, Journal of Parallel and Distributed Computing 47 (1997): 139–152.

[6] S. Adjerid, K. D. Devine, J. E. Flaherty, L. Krivodonova, *A posteriori* error estimation for discontinuous Galerkin solutions of hyperbolic problems, Computer Methods in Applied Mechanics and Engineering 191 (2002): 1097–1112.

[7] M. Delfour, W. Hager, F. Trochu, Discontinuous Galerkin methods for ordinary differential equation, Mathematics of Computation 154 (1981): 455–473.

[8] C. Johnson, Error estimates and adaptive time-step control for a class of one-step methods for stiff ordinary differential equations, SIAM Journal on Numerical Analysis 25 (1988): 908–926.

[9] P. Lesaint, P. Raviart, On a finite element method for solving the neutron transport equations, in: C. de Boor (ed.), Mathematical Aspects of Finite Elements in Partial Differential Equations, Academic Press, New York, 1974.

[10] S. Adjerid, M. Baccouch, The discontinuous Galerkin method for two-dimensional hyperbolic problems. Part I: Superconvergence error analysis, Journal of Scientific Computing 33 (2007): 75–113.

[11] S. Adjerid, M. Baccouch, The discontinuous Galerkin method for two-dimensional hyperbolic problems. Part II: *A posteriori* error estimation, Journal of Scientific Computing 38 (2009): 15–49.

[12] S. Adjerid, M. Baccouch, Asymptotically exact *a posteriori* error estimates for a one-dimensional linear hyperbolic problem, Applied Numerical Mathematics 60 (2010): 903–914.

[13] S. Adjerid, M. Baccouch, Adaptivity and error estimation for discontinuous Galerkin methods, in: X. Feng, O. Karakashian, Y. Xing (eds.), Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations, vol. 157 of The IMA Volumes in Mathematics and its Applications, Springer International Publishing, Switzerland, 2014, pp. 63–96.

[14] M. Baccouch, A local discontinuous Galerkin method for the second-order wave equation, Computer Methods in Applied Mechanics and Engineering 209–212 (2012): 129–143.

[15] M. Baccouch, *A posteriori* error estimates for a discontinuous Galerkin method applied to one-dimensional nonlinear scalar conservation laws, Applied Numerical Mathematics 84 (2014): 1–21.

[16] M. Baccouch, S. Adjerid, Discontinuous Galerkin error estimation for hyperbolic problems on unstructured triangular meshes, Computer Methods in Applied Mechanics and Engineering 200 (2010): 162–177.

[17] B. Cockburn, C. W. Shu, TVB Runge-Kutta local projection discontinuous Galerkin methods for scalar conservation laws II: General framework, Mathematics of Computation 52 (1989): 411–435.

[18] K. D. Devine, J. E. Flaherty, Parallel adaptive *hp*-refinement techniques for conservation laws, Computer Methods in Applied Mechanics and Engineering 20 (1996): 367–386.

[19] T. E. Peterson, A note on the convergence of the discontinuous Galerkin method for a scalar hyperbolic equation, SIAM Journal on Numerical Analysis 28 (1) (1991): 133–140.

[20] P. Castillo, A superconvergence result for discontinuous Galerkin methods applied to elliptic problems, Computer Methods in Applied Mechanics and Engineering 192 (2003): 4675–4685.

[21] F. Celiker, B. Cockburn, Superconvergence of the numerical traces for discontinuous Galerkin and hybridized methods for convection-diffusion problems in one space dimension, Mathematics of Computation 76 (2007): 67–96.

[22] Y. Cheng, C.-W. Shu, Superconvergence of discontinuous Galerkin and local discontinuous Galerkin schemes for linear hyperbolic and convection-diffusion

equations in one space dimension, SIAM Journal on Numerical Analysis 47 (2010): 4044–4072.

[23] B. Cockburn, C. W. Shu, The local discontinuous Galerkin method for time-dependent convection-diffusion systems, SIAM Journal on Numerical Analysis 35 (1998): 2440–2463.

[24] B. Cockburn, G. E. Karniadakis, C. W. Shu, Discontinuous Galerkin Methods Theory, Computation and Applications, Lecture Notes in Computational Science and Engineering, Vol. 11, Springer, Berlin, 2000.

[25] C.-W. Shu, Discontinuous Galerkin method for time-dependent problems: Survey and recent developments, in: X. Feng, O. Karakashian, Y. Xing (eds.), Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations, Vol. 157 of The IMA Volumes in Mathematics and its Applications, Springer International Publishing Switzerland, 2014, pp. 25–62.

[26] M. Baccouch, Analysis of a posteriori error estimates of the discontinuous Galerkin method for nonlinear ordinary differential equations, Applied Numerical Mathematics 106 (2016): 129–153.

[27] M. Delfour, F. Dubeau, Discontinuous polynomial approximations in the theory of one-step, hybrid and multistep methods for nonlinear ordinary differential equations, Mathematics of Computation 47 (1986): 169–189.

[28] D. Estep, *A posteriori* error bounds and global error control for approximation of ordinary differential equations, SIAM Journal on Numerical Analysis 32 (1995): 1–48.

[29] C. Johnson, J. Pitkaranta, An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation, Mathematics of Computation 47 (1986): 285–312.

[30] G. Richter, An optimal-order error estimate for discontinuous Galerkin method, Mathematics of Computation 50 (1988): 75–88.

[31] K. Deng, Z. Xiong, Superconvergence of a discontinuous finite element method for a nonlinear ordinary differential equation, Applied Mathematics and Computation 217 (2010): 3511–3515.

[32] M. Baccouch, Asymptotically exact *a posteriori* LDG error estimates for one-dimensional transient convection-diffusion problems, Applied Mathematics and Computation 226 (2014): 455–483.

[33] M. Baccouch, Global convergence of *a posteriori* error estimates for a discontinuous Galerkin method for one-dimensional linear hyperbolic problems, International Journal of Numerical Analysis and Modeling 11 (2014): 172–192.

[34] M. Baccouch, The local discontinuous Galerkin method for the fourth-order Euler-Bernoulli partial differential equation in one space dimension. Part II: *A posteriori* error estimation, Journal of Scientific Computing 60 (2014): 1–34.

[35] M. Baccouch, A superconvergent local discontinuous Galerkin method for the second-order wave equation on Cartesian grids, Computers and Mathematics with Applications 68 (2014): 1250–1278.

[36] M. Baccouch, Asymptotically exact *a posteriori* local discontinuous Galerkin error estimates for the one-dimensional second-order wave equation, Numerical Methods for Partial Differential Equations 31 (2015): 1461–1491.

[37] M. Baccouch, Asymptotically exact local discontinuous Galerkin error estimates for the linearized Korteweg-de Vries equation in one space dimension, International Journal of Numerical Analysis and Modeling 12 (2015): 162–195.

[38] M. Baccouch, Superconvergence and *a posteriori* error estimates of the DG method for scalar hyperbolic problems on Cartesian grids, Applied Mathematics and Computation 265 (2015): 144–162.

[39] M. Baccouch, B. Johnson, A high-order discontinuous Galerkin method for Itô stochastic ordinary differential equations, Journal of Computational and Applied Mathematics 308 (2016): 138–165.

[40] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland Pub. Co., Amsterdam/New York/Oxford, 1978.

[41] M. Abramowitz, I. A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1965.

[42] M. Ainsworth, J. T. Oden, *A posteriori* Error Estimation in Finite Element Analysis, John Wiley, New York, 2000.

[43] L. Schumaker, Spline Functions: Basic Theory, Cambridge University Press, Cambridge/New York, 2007.

[44] K. Segeth, A posteriori error estimation with the finite element method of lines for a nonlinear parabolic equation in one space dimension, Numerische Mathematik 83 (3) (1999): 455–475.