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A B S T R A C T

In this paper, we investigate the convergence and superconvergence properties of a local discontinuous Galerkin (LDG) method for nonlinear second-order two-point boundary-value problems (BVPs) of the form $u'' = f(x, u, u')$, $x \in [a, b]$ subject to some suitable boundary conditions at the endpoints $x = a$ and $x = b$. We prove optimal L^2 error estimates for the solution and for the auxiliary variable that approximates the first-order derivative. The order of convergence is proved to be $p + 1$, when piecewise polynomials of degree at most p are used. We further prove that the derivatives of the LDG solutions are superconvergent with order $p + 1$ toward the derivatives of Gauss-Radau projections of the exact solutions. Moreover, we prove that the LDG solutions are superconvergent with order $p + 2$ toward Gauss-Radau projections of the exact solutions. Finally, we prove, for any polynomial degree p, the $(2p + 1)$th superconvergence rate of the LDG approximations at the upwind or downwind points and for the domain average under quasi-uniform meshes. Our numerical experiments demonstrate optimal rates of convergence and superconvergence. Our proofs are valid for arbitrary regular meshes using piecewise polynomials of degree $p \geq 1$ and for the classical sets of boundary conditions. Several computational examples are provided to validate the theoretical results.

1. Introduction

The purpose of this paper is to study the convergence and superconvergence properties of the local discontinuous Galerkin (LDG) method for the nonlinear two-point second-order boundary-value problems (BVPs)

$$u'' = f(x, u, u'), \quad x \in [a, b], \tag{1.1a}$$

where $u : [a, b] \to \mathbb{R}$ and $f : D \to \mathbb{R}$ is a given smooth function on the set $D = [a, b] \times \mathbb{R}^2$. In this paper, we consider one of the following set of boundary conditions, which are commonly encountered in practice:

$$u(a) = \alpha_1, \quad u'(b) = \beta_1, \tag{1.1b}$$

$$u(a) = \alpha_1, \quad u(b) = \beta_1, \tag{1.1c}$$

$$u(a) = u(b), \quad u'(a) = u'(b), \tag{1.1d}$$

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where α_1, β_2 are given constants. In our analysis, we assume that the BVP (1.1) has one and only one solution. The conditions on f for the existence and uniqueness of the solution to the general BVP (1.1) are given in [22]. The nonlinear two-point BVP (1.1) arises in applied mathematics, theoretical physics, engineering, control and optimization theory; see *e.g.*, [3,28]. Since the analytic solution to (1.1) is difficult to obtain for general f, numerical techniques are often needed to approximate its solution. Many authors have designed numerical schemes to solve second-order BVPs. We refer to [30,17,22,3,25,20,5,19,11,23,27,2,30,21] for some numerical methods including the shooting method, the finite difference method, the collocation method, the monotone iterative method, and the quasilinearization method.

Superconvergent numerical methods of BVPs are necessary in many important scientific and engineering applications such as boundary layer theory, the study of stellar interiors, control and optimization theory, and flow networks in biology. A knowledge of superconvergence properties can be used to (i) construct simple and asymptotically exact *a posteriori* estimates of discretization errors and (ii) help detect discontinuities to find elements needing limiting, stabilization and/or refinement. Typically, *a posteriori* error estimators employ the known numerical solution to derive estimates of the actual solution errors. They are also used to steer adaptive schemes where either the mesh is locally refined (h-refinement) or the polynomial degree is raised (p-refinement). In the past several decades, there also has been considerable interest in studying superconvergence properties of numerical methods. In this paper, we present new superconvergence results of the LDG method for solving (1.1). Discontinuous Galerkin (DG) methods form a class of high order numerical methods for solving ordinary differential equations (ODEs) and partial differential equations (PDEs). They combine many attractive features of the finite element and finite volume methods. DG schemes have been successfully applied to many problems arising from a wide range of applications. The DG method is a finite element method using a completely discontinuous piecewise polynomial space for the numerical solution and the test functions. DG methods are becoming important techniques for the computational solution of many real-world problems. They are known to have good stability properties when applied to hyperbolic PDEs. Furthermore, DG methods have been successfully applied to hyperbolic, elliptic, and parabolic problems arising from a wide range of applications. DG methods are highly accurate numerical methods with the advantage that they can handle problems having discontinuities such as those that arise in hyperbolic problems, can handle problems with complex geometries, simplify adaptive $h-$, $p-$, and $r-$ refinement, and produce efficient parallel solution procedures. DG method was initially introduced by Reed and Hill in 1973 as a technique to solve neutron transport problems [29].

The local DG (LDG) methods are natural extension of the DG methods aimed at solving higher-order PDEs. The LDG method was first proposed by Cockburn and Shu in [16] for solving convection-diffusion problems. The LDG method consists of rewriting a higher order differential equation into a system of first-order equations and then discretizing it by the standard DG method. The success of LDG methods is due to the following properties: (i) they are robust and high order accurate, (ii) they can achieve stability without slope limiters, and (iii) they are element-wise conservative. This last feature is very important in the area of computational fluid dynamics, especially in situations where there are steep gradients or boundary layers or shocks. Moreover, LDG schemes are extremely flexible in the mesh-design. Thus, they can easily handle meshes with hanging nodes, elements of various types and shapes, and local spaces of different orders. Furthermore, they exhibit useful superconvergence properties that can be used to estimate the actual discretization errors. We refer the reader to *e.g.*, [8,15,13,7,14] and references therein for a more complete survey of several LDG methods.

Several authors designed and analyzed the LDG method for BVPs of the form (1.1), see *e.g.*, [9,24,33,36,32,31,37,4]. In [4], we proposed and analyzed a superconvergent and high order accurate LDG method for nonlinear two-point second-order BVPs of the form $u'' = f(x, u)$ subject to some suitable boundary conditions. We proved optimal L^2 error estimates for the solution and for the auxiliary variable that approximates its first-order derivative. The order of convergence is proved to be $p+1$, when piecewise polynomials of degree at most $p \geq 1$ are employed. We further proved that the derivatives of the LDG solutions are superconvergent with order $p+1$ toward the derivatives of Gauss-Radau projections of the exact solutions. Finally, we showed that the LDG solutions are superconvergent with order $p+3/2$ toward Gauss-Radau projections of the exact solutions, while computational results show higher $\mathcal{O}(h^{p+2})$ convergence rate. However, a theoretical proof of this property remains open. The main purpose of our current work is to use a different approach to prove the $(p+2)$th superconvergence rate and also the $(2p+1)$th superconvergence rate at upwind and downwind points and for the domain average. To the best of our knowledge, these results are original.

In this paper, we design a superconvergent LDG method for nonlinear BVPs of the form (1.1). We prove several optimal L^2 error estimates for the LDG solutions. In particular, we prove that the LDG solutions to approximate u and u' are $(p+1)$th order convergent in the L^2-norm, when the space of piecewise polynomials of degree $p \geq 1$ is used. We further show that the derivatives of the LDG solutions are superconvergent with order $p+1$ toward the derivatives of Gauss-Radau projections of the exact solutions. Moreover, we prove $(p+2)$th order superconvergence of the LDG solutions toward Gauss-Radau projections of the exact solutions. Finally, we show that the errors between the LDG solutions and the exact solutions are $(2p+1)$th order superconvergent at either the upwind point or downwind point in each element on regular meshes. Numerical experiments demonstrate that the theoretical orders of convergence and superconvergence are optimal. Our global error analysis is valid for any regular meshes and using piecewise polynomials of degree $p \geq 1$ and for the classical set of boundary conditions. We would like to mention that the proposed LDG method has several advantages over the standard methods due to the following nice features: (i) it achieves arbitrary high order accuracy, (ii) it exhibits optimal convergence properties for the solution and for the auxiliary variables that approximate the derivatives, (iii) it can easily handle meshes using local spaces of different orders, and (iv) achieves superconvergence results that can be used to construct asymptot-

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ically exact *a posteriori* error estimates by solving a local problem on each element. This will be discussed in a separate paper.

The rest of the paper is organized as follows: In section 2, we describe the LDG method for nonlinear second-order BVPs. We also present some preliminary results, which will be used in our error analysis. In section 3, we present a detailed proof of the optimal *a priori* error estimates of the LDG method. We state and prove our main superconvergence results in section 4. In section 5, we present several numerical examples to validate our theoretical results. Finally, we provide some concluding remarks in section 6.

2. The LDG scheme for nonlinear second-order BVPs

In order to define the LDG method, we introduce a new auxiliary variable $q = u'$ and convert (1.1a) into a first-order system of ODEs

$$q' = f(x, u, q), \quad u' = q. \tag{2.1}$$

To obtain the LDG weak formulation, we partition the computational domain $\Omega = [a, b]$ into a collection of non-overlapping elements $I_i = [x_{i-1}, x_i]$, $i = 1, \ldots, N$, where $x_0 = a$ and $x_N = b$. We denote the length of each interval I_i by $h_i = x_i - x_{i-1}$. We also define $h = \max\limits_{1\le i\le N} h_i$ and $h_{min} = \min\limits_{1\le i\le N} h_i$ to be the lengths of the largest and smallest cells, respectively. We assume that the mesh is regular in the sense that there exists a constant $K \ge 1$ independent of h such that $h \le K h_i$, $i = 1, \ldots, N$.

For simplicity, we use $v(x_i^-) = \lim\limits_{s\to 0^-} v(x_i + s)$ and $v(x_i^+) = \lim\limits_{s\to 0^+} v(x_i + s)$ to denote the left limit and the right limit of v at the discontinuity point x_i. We also use $[v](x_i) = v(x_i^+) - v(x_i^-)$ to denote the jump of v at x_i.

Multiplying the two equations in (2.1) by arbitrary test functions v_1 and v_2, integrating over the interval I_i, and using integration by parts, we get

$$-\int_{I_i} q v_1' dx + q(x_i)v_1(x_i) - q(x_{i-1})v_1(x_{i-1}) = \int_{I_i} f(x, u, q)v_1 dx, \tag{2.2a}$$

$$-\int_{I_i} u v_2' dx + u(x_i)v_2(x_i) - u(x_{i-1})v_2(x_{i-1}) = \int_{I_i} q v_2 dx. \tag{2.2b}$$

Next, we introduce the following discontinuous finite element approximation space

$$V_h^p = \{v : v|_{I_i} \in P^p(I_i),\ i = 1, \ldots, N\},$$

where $P^p(I_i)$ denotes the space of polynomials of degree at most p on I_i. We would like to emphasize that polynomials in the finite element space V_h^p are allowed to be completely discontinuous at the mesh points.

To obtain the LDG scheme, we replace the exact solutions u and q by piecewise polynomials of degree at most p and denote them by $u_h \in V_h^p$ and $q_h \in V_h^p$. We also choose the test functions v_1 and v_2 to be piecewise polynomials of degree at most p. The LDG scheme can now be defined as: find approximations $u_h,\ q_h \in V_h^p$ such that $\forall\ v_1,\ v_2 \in V_h^p$ and $\forall\ i = 1, \ldots, N$,

$$-\int_{I_i} q_h v_1' dx + \widehat{q}_h(x_i)v_1(x_i^-) - \widehat{q}_h(x_{i-1})v_1(x_{i-1}^+) = \int_{I_i} f(x, u_h, q_h)v_1 dx, \tag{2.3a}$$

$$-\int_{I_i} u_h v_2' dx + \widehat{u}_h(x_i)v_2(x_i^-) - \widehat{u}_h(x_{i-1})v_2(x_{i-1}^+) = \int_{I_i} q_h v_2 dx, \tag{2.3b}$$

where $\widehat{u}_h$ and $\widehat{q}_h$ are the so-called numerical fluxes, which are, respectively, the discrete approximations to u and q at the nodes. These numerical fluxes must be designed based on different guiding principles for different differential equations to ensure stability and optimal error estimates. To complete the definition of the LDG scheme, we only need to define $\widehat{u}_h$ and $\widehat{q}_h$ on the boundaries of I_i. It turns out that the following simple choices would guarantee the optimal convergence and superconvergence of our LDG scheme: For the mixed boundary conditions (1.1b), we take the following alternating numerical fluxes; see *e.g.*, [4]

$$\widehat{u}_h(x_i) = \begin{cases} \alpha_1, & i = 0, \\ u_h(x_i^-), & i = 1, 2, \ldots, N, \end{cases} \qquad \widehat{q}_h(x_i) = \begin{cases} q_h(x_i^+), & i = 0, 1, \ldots, N-1, \\ \beta_1, & i = N. \end{cases} \tag{2.4a}$$

If other boundary conditions are chosen, the numerical fluxes can be easily designed. For instance the numerical fluxes associated with the boundary conditions (1.1c) can be taken as

$$\widehat{u}_h(x_i)=\begin{cases}\alpha_1, & i=0,\\ u_h(x_i^-), & i=1,2,\dots,N-1,\\ \beta_1, & i=N,\end{cases}\qquad \widehat{q}_h(x_i)=\begin{cases}q_h(x_i^+), & i=0,1,\dots,N-1,\\ q_h(x_i^-)-\delta_1(u_h(x_i^-)-\beta_1), & i=N,\end{cases} \tag{2.4b}$$

where the stabilization parameter δ_1 for the LDG method is given by $\delta_1=\frac{p}{h_N^p}$.

For the periodic boundary conditions (1.1d), we choose the following alternating fluxes

$$\widehat{u}_h(x_i)=\begin{cases}u_h(x_N^-), & i=0,\\ u_h(x_i^-), & i=1,2,\dots,N,\end{cases}\qquad \widehat{q}_h(x_i)=\begin{cases}q_h(x_i^+), & i=0,1,\dots,N-1,\\ q_h(x_0^+), & i=N.\end{cases} \tag{2.4c}$$

Implementation: The LDG solution (u_h,q_h) can be obtained using the following steps:

(1) For $x\in I_i$, we choose $\{\phi_{k,i}(x)\}_{k=0}^{k=p}$ to be a local basis of $P^p(I_i)$ and we express u_h, q_h as

$$u_h(x)=\sum_{k=0}^{p}c_{k,i}\phi_{k,i}(x),\qquad q_h(x)=\sum_{k=0}^{p}c_{k+p+1,i}\phi_{k,i}(x).$$

In practice, we may choose $\phi_{k,i}=L_{k,i}$, where $L_{k,i}$ is the kth-degree Legendre polynomial on I_i.

(2) We choose the test functions $v_1=v_2=\phi_{j,i}(x),\ \ j=0,\dots,p$ to obtain $2N(p+1)\times 2N(p+1)$ system of nonlinear algebraic equations.

(3) We solve the nonlinear system for the unknown coefficients $c_{0,i},c_{1,i}\dots,c_{2p+1,i},\ \ i=1,\dots,N$ using *e.g.*, Newton's method for nonlinear systems. Once we solve for the unknown coefficients, we get the LDG solutions u_h and q_h, which are piecewise discontinuous polynomials of degree $\le p$.

Norms: We present some norms that will be used throughout the paper. Denote $\|u\|_{0,I_i}=\left(\int_{I_i}u^2(x)dx\right)^{1/2}$ to be the standard L^2-norm of the function u on I_i. For any natural integer s, the Sobolev space $H^s(I_i)$ consists of functions that have generalized derivatives of order s in the space $L^2(I_i)$. It is defined by

$$H^s(I_i)=\left\{u\in L^2(I_i)\ :\ D^k u\in L^2(I_i),\ \forall\, 0\le k\le s\right\}.$$

The norm of $H^s(I_i)$ is defined by $\|u\|_{s,I_i}=\left(\sum_{k=0}^{s}\left\|D^k u\right\|_{0,I_i}^2\right)^{1/2}$. We shall also use the following notation for the semi-norm $|u|_{s,I_i}=\left\|D^s u\right\|_{0,I_i}$. Finally, we define the norms on the whole computational domain Ω as follows:

$$|u|_{s,\Omega}=\left(\sum_{i=1}^{N}|u|_{s,I_i}^2\right)^{1/2},\qquad \|u\|_{s,\Omega}=\left(\sum_{i=1}^{N}\|u\|_{s,I_i}^2\right)^{1/2}.$$

For convenience, we use $\|u\|$, $\|u\|_s$, and $|u|_s$ to denote $\|u\|_{0,\Omega}$, $\|u\|_{s,\Omega}$, and $|u|_{s,\Omega}$, respectively. We would like to mention that if $u\in H^s(\Omega)$, $s=1,2,\dots$, then the Sobolev norm $\|u\|_{s,\Omega}$ on the whole computational domain Ω is the standard Sobolev norm defined by $\|u\|_{s,\Omega}=\left(\sum_{k=0}^{s}\left\|D^k u\right\|_{0,\Omega}^2\right)^{1/2}$.

Legendre polynomial: In our analysis we need the pth-degree Legendre polynomial defined by Rodrigues formula [1]

$$\tilde{L}_p(\xi)=\frac{1}{2^p p!}\frac{d^p}{d\xi^p}\left((\xi^2-1)^p\right),\quad -1\le\xi\le 1,$$

which satisfies the following properties: $\tilde{L}_p(1)=1$, $\tilde{L}_p(-1)=(-1)^p$, and the orthogonality relation

$$\int_{-1}^{1}\tilde{L}_p(\xi)\tilde{L}_q(\xi)d\xi=\frac{2}{2p+1}\delta_{pq},\quad \text{where } \delta_{pq} \text{ is the Kronecker symbol.} \tag{2.5}$$

Mapping the physical element $I_i=[x_{i-1},x_i]$ into a reference element $[-1,1]$ by the standard affine mapping

$$x(\xi,h_i)=\frac{x_i+x_{i-1}}{2}+\frac{h_i}{2}\xi, \tag{2.6}$$

we obtain the k-degree shifted Legendre polynomial, $L_{k,i}(x)=\tilde{L}_k\left(\frac{2x-x_i-x_{i-1}}{h_i}\right)$, on I_i.

Using the mapping (2.6) and the orthogonality relation (2.5), we obtain

$$\|L_{p,i}\|^2_{0,I_i} = \int_{I_i} L^2_{p,i}(x)dx = \frac{h_i}{2}\int_{-1}^{1} \tilde{L}^2_p(\xi)d\xi = \frac{h_i}{2}\frac{2}{2p+1} = \frac{h_i}{2p+1} \le h_i. \tag{2.7}$$

Gauss-Radau Projections: For $p \ge 1$, we introduce two special Gauss-Radau projections $P_h^{\pm}$. These projections are defined element-by-element as follows: For any integrable function u on Ω, $P_h^{\pm}u \in V_h^p$ and the restrictions of $P_h^{\pm}u$ to I_i are polynomials in $P^p(I_i)$ satisfying the conditions, see *e.g* [10]

$$\int_{I_i} (u - P_h^- u)v dx = 0, \quad \forall\, v \in P^{p-1}(I_i), \quad \text{and} \quad (u - P_h^- u)(x_i^-) = 0, \tag{2.8}$$

$$\int_{I_i} (u - P_h^+ u)v dx = 0, \quad \forall\, v \in P^{p-1}(I_i), \quad \text{and} \quad (u - P_h^+ u)(x_{i-1}^+) = 0. \tag{2.9}$$

By the scaling argument, we obtain the following projection results [12]: For any function $u \in H^{p+1}(\Omega)$, there exists a positive constant C independent of the mesh size h, such that

$$\left\|u - P_h^{\pm}u\right\| + h\left\|(u - P_h^{\pm}u)'\right\| \le Ch^{p+1}\,|u|_{p+1}\,. \tag{2.10}$$

Moreover, we recall the inverse properties of the finite element space V_h^p that will be used in our error analysis [26]: For any $v \in V_h^p$, there exists a positive constant C independent of v and h, such that

$$\|v'\| \le Ch^{-1}\|v\|\,, \quad \left(\sum_{i=1}^{N} v^2(x_{i-1}^+) + v^2(x_i^-)\right)^{1/2} \le Ch^{-1/2}\|v\|\,. \tag{2.11}$$

In the rest of the paper, we will not differentiate between various constants, and instead will use a generic constant C (or accompanied by lower indices) to represent a positive constant independent of the mesh size h, but which may depend upon the exact smooth solution of the BVP (1.1). They also may have different values at different places.

3. *A priori* error estimates

In this section, we derive optimal L^2 error estimates for the LDG method. For convenience, we use e_u and e_q to denote the errors between the exact solutions of (2.1) and the LDG solutions defined in (2.3), *i.e.*,

$$e_u = u - u_h, \quad e_q = q - q_h.$$

As the usual treatment in finite element analysis, we divide the errors into the form

$$e_u = \epsilon_u + \bar{e}_u, \quad e_q = \epsilon_q + \bar{e}_q, \tag{3.1}$$

where the projection errors are defined by

$$\epsilon_u = u - P_h^- u, \quad \epsilon_q = q - P_h^+ q,$$

and the errors between the numerical solutions and the projection of the exact solutions are defined by

$$\bar{e}_u = P_h^- u - u_h, \quad \bar{e}_q = P_h^+ q - q_h.$$

In our error analysis, we assume that the function f appearing in the right-hand side of (1.1a) is sufficiently differentiable function. More precisely, we assume that the f satisfies the following conditions:

Assumption A1. The functions f, f_u, and f_q are continuous on the set $D = \{(x,u,q) \mid x \in [a,b],\ u \in \mathbb{R},\ q \in \mathbb{R}\}$.

Assumption A2. For all $(x,u,q) \in D$ there exist constants $K_i,\ i = 1, 2$ such that

$$0 < f_u(x,u,q) \le K_1, \quad \left|f_q(x,u,q)\right| \le K_2. \tag{3.2}$$

Remark 3.1. The proofs of our theorems require that the function f is smooth, f_u and f_q are bounded on the set D. These assumptions are usually the hypotheses of the existence and uniqueness theorem of (1.1).

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In the next theorem, we prove *a priori* error estimates for e_u and e_q in the L^2-norm.

Theorem 3.1. *Let* (u, q) *be the exact solution of (2.1). We assume that* f *satisfies* Assumption A1 *and* Assumption A2. *Let* $p \ge 1$ *and* (u_h, q_h) *be the LDG solution of (2.3), then there exists a positive constant* C *independent of* h *such that*

$$\|e_u\| \le Ch^{p+1}. \tag{3.3}$$

$$\|e_q\| \le Ch^{p+1}. \tag{3.4}$$

Proof. First, we derive some error equations which will be used repeatedly throughout this paper. Subtracting (2.3) from (2.2) with $v_k \in V_h^p,\ k = 1, 2$ and using the numerical fluxes (2.4a), we obtain the error equations on I_i: $\forall\ v_1,\ v_2\ \in V_h^p$,

$$-\int_{I_i} e_q v_1' dx + e_q(x_i^+) v_1(x_i^-) - e_q(x_{i-1}^+) v_1(x_{i-1}^+) = \int_{I_i} (f(x, u, q) - f(x, u_h, q_h))\, v_1 dx, \tag{3.5a}$$

$$-\int_{I_i} e_u v_2' dx + e_u(x_i^-) v_2(x_i^-) - e_u(x_{i-1}^-) v_2(x_{i-1}^+) = \int_{I_i} e_q v_2 dx. \tag{3.5b}$$

Applying Taylor's Theorem with integral remainder and using (3.1), we write

$$f(x, u, q) - f(x, u_h, q_h) = R_u e_u + R_q e_q, \tag{3.6}$$

where

$$R_u = R_u(x) = \int_0^1 f_u\left(x, u(x) - t e_u(x), q(x) - t e_q(x)\right) dt,$$

$$R_q = R_q(x) = \int_0^1 f_q\left(x, u(x) - t e_u(x), q(x) - t e_q(x)\right) dt.$$

Under Assumption A2, we have

$$0 < R_u(x) \le K_1, \quad \left|R_q(x)\right| \le K_2, \quad \forall\, x \in [a, b]. \tag{3.7}$$

Using (3.6), we rewrite (3.5) as

$$\mathcal{A}_1^{(i)}(v_1) = \mathcal{A}_2^{(i)}(v_2) = 0, \quad \forall\ v_1,\ v_2\ \in V_h^p, \tag{3.8}$$

where the operators $\mathcal{A}_k^{(i)}:\ H^{p+1}(\Omega) \to \mathbb{R},\ k = 1 - 2$ are defined by

$$\mathcal{A}_1^{(i)}(V_1) = \int_{I_i} e_q V_1' dx - e_q(x_i^+) V_1(x_i^-) + e_q(x_{i-1}^+) V_1(x_{i-1}^+) + \int_{I_i} \left(R_u e_u + R_q e_q\right) V_1 dx, \tag{3.9a}$$

$$\mathcal{A}_2^{(i)}(V_2) = \int_{I_i} e_u V_2' dx - e_u(x_i^-) V_2(x_i^-) + e_u(x_{i-1}^-) V_2(x_{i-1}^+) + \int_{I_i} e_q V_2 dx. \tag{3.9b}$$

Adding the above equations, we get

$$\begin{aligned} \mathcal{A}_1^{(i)}(V_1) + \mathcal{A}_2^{(i)}(V_2) &= \int_{I_i} e_q \left(V_1' + R_q V_1 + V_2\right) dx + \int_{I_i} e_u \left(V_2' + R_u V_1\right) dx \\ &\quad - e_q(x_i^+) V_1(x_i^-) + e_q(x_{i-1}^+) V_1(x_{i-1}^+) - e_u(x_i^-) V_2(x_i^-) + e_u(x_{i-1}^-) V_2(x_{i-1}^+). \end{aligned} \tag{3.10}$$

Summing over all elements gives

$$\begin{aligned} \sum_{i=1}^{N} \left(\mathcal{A}_1^{(i)}(V_1) + \mathcal{A}_2^{(i)}(V_2)\right) &= \int_{\Omega} e_q \left(V_1' + R_q V_1 + V_2\right) dx + \int_{\Omega} e_u \left(V_2' + R_u V_1\right) dx \\ &\quad - e_q(x_N^+) V_1(x_N^-) + e_q(x_0^+) V_1(x_0^+) - e_u(x_N^-) V_2(x_N^-) + e_u(x_0^-) V_2(x_0^+). \end{aligned}$$

If the boundary conditions (1.1b) are used then $e_u(x_0^-) = e_q(x_N^+) = 0$. Thus, we have

$$\sum_{i=1}^{N}\left(\mathcal{A}_1^{(i)}(V_1)+\mathcal{A}_2^{(i)}(V_2)\right)=\int_\Omega e_q\left(V_1'+R_qV_1+V_2\right)dx+\int_\Omega e_u\left(V_2'+R_uV_1\right)dx+e_q(x_0^+)V_1(x_0^+)-e_u(x_N^-)V_2(x_N^-). \tag{3.11}$$

On the other hand, performing integration by parts, we write (3.9) as

$$\mathcal{A}_1^{(i)}(V_1)=\int_{I_i}\left(-e_q'+R_ue_u+R_qe_q\right)V_1dx-[e_q](x_i)V_1(x_i^-), \tag{3.12a}$$

$$\mathcal{A}_2^{(i)}(V_2)=\int_{I_i}\left(-e_u'+e_q\right)V_2dx-[e_u](x_{i-1})V_2(x_{i-1}^+). \tag{3.12b}$$

We note that, with the numerical fluxes (2.4a), the jumps of e_u and e_q at an interior point x_i are defined as

$$[e_u](x_i)=e_u(x_i^+)-e_u(x_i^-),\ [e_q](x_i)=e_q(x_i^+)-e_q(x_i^-).$$

Since $e_u(x_0^-)=e_q(x_N^+)=0$, the jumps at the endpoints of the computational domain are given by

$$[e_u](x_0)=e_u(x_0^+),\quad [e_q](x_N)=-e_q(x_N^-).$$

Adding and subtracting $P_h^-V_1$ to V_1 and $P_h^+V_2$ to V_2 and using (3.8) with $v_1=P_h^-V_1\in P^p(I_i)$ and $v_2=P_h^+V_2\in P^p(I_i)$, we get

$$\mathcal{A}_1^{(i)}(V_1)=\mathcal{A}_1^{(i)}(V_1-P_h^-V_1)+\mathcal{A}_1^{(i)}(P_h^-V_1)=\mathcal{A}_1^{(i)}(V_1-P_h^-V_1), \tag{3.13a}$$

$$\mathcal{A}_2^{(i)}(V_2)=\mathcal{A}_2^{(i)}(V_2-P_h^+V_2)+\mathcal{A}_2^{(i)}(P_h^+V_2)=\mathcal{A}_2^{(i)}(V_2-P_h^+V_2). \tag{3.13b}$$

Combining (3.13) and (3.12) and using the properties of the projections $P_h^\pm$, *i.e.*, $(V-P_h^-V)(x_i^-)=(V-P_h^+V)(x_{i-1}^+)=0$, we obtain

$$\mathcal{A}_1^{(i)}(V_1)=\int_{I_i}\left(-e_q'+R_ue_u+R_qe_q\right)(V_1-P_h^-V_1)dx,\quad \mathcal{A}_2^{(i)}(V_2)=\int_{I_i}\left(-e_u'+e_q\right)(V_2-P_h^+V_2)dx. \tag{3.14}$$

By the property of the projection $P_h^\pm$, we have

$$\int_{I_i} w'(V-P_h^\pm V)dx=0,\quad \forall\, w\in P^p(I_i), \tag{3.15}$$

since w is a polynomial of degree at most p and thus w' is a polynomial of degree at most $p-1$.

Substituting (3.1) into (3.14) and using (3.15) with $w=\bar{e}_u,\ \bar{e}_q\ \in P^p(I_i)$, we get

$$\mathcal{A}_1^{(i)}(V_1)=\int_{I_i}\left(-\epsilon_q'+R_ue_u+R_qe_q\right)(V_1-P_h^-V_1)dx,\quad \mathcal{A}_2^{(i)}(V_2)=\int_{I_i}\left(-\epsilon_u'+e_q\right)(V_2-P_h^+V_2)dx.$$

Adding these two equations, we obtain

$$\mathcal{A}_1^{(i)}(V_1)+\mathcal{A}_2^{(i)}(V_2)=\int_{I_i}\left(-\epsilon_q'+R_ue_u+R_qe_q\right)(V_1-P_h^-V_1)dx+\int_{I_i}\left(-\epsilon_u'+e_q\right)(V_2-P_h^+V_2)dx. \tag{3.16}$$

Summing over all elements, we arrive at

$$\sum_{i=1}^{N}\left(\mathcal{A}_1^{(i)}(V_1)+\mathcal{A}_2^{(i)}(V_2)\right)=\int_\Omega\left(-\epsilon_q'+R_ue_u+R_qe_q\right)(V_1-P_h^-V_1)dx+\int_\Omega\left(-\epsilon_u'+e_q\right)(V_2-P_h^+V_2)dx. \tag{3.17}$$

Combining (3.11) and (3.17) yields

$$\begin{aligned}&\int_\Omega e_q\left(V_1'+R_qV_1+V_2\right)dx+\int_\Omega e_u\left(V_2'+R_uV_1\right)dx+e_q(x_0^+)V_1(x_0^+)-e_u(x_N^-)V_2(x_N^-)\\&=\int_\Omega\left(-\epsilon_q'+R_ue_u+R_qe_q\right)(V_1-P_h^-V_1)dx+\int_\Omega\left(-\epsilon_u'+e_q\right)(V_2-P_h^+V_2)dx.\end{aligned} \tag{3.18}$$

The main idea behind the proof of the theorem is to construct the following adjoint problem: find W_1 and W_2 such that

$$W_1' + W_2 + R_q W_1 = e_q, \quad W_2' + R_u W_1 = e_u, \quad \text{for } x \in (a,b) \quad \text{subject to } W_1(a) = W_2(b) = 0. \tag{3.19}$$

The BVP (3.19) can be converted into the system of equations

$$\mathbf{W}' + A(x)\mathbf{W} = \mathbf{b}(x), \quad x \in \Omega \quad \text{subject to} \quad B_1\mathbf{W}(a) + B_2\mathbf{W}(b) = \mathbf{0}, \tag{3.20}$$

where

$$\mathbf{W} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \; A = \begin{bmatrix} R_q & 1 \\ R_u & 0 \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} e_q \\ e_u \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The solution to (3.20) can be expressed in terms of its fundamental matrix

$$\mathbf{W}(x) = M(x)\mathbf{W}(a) + M(x)\int_a^x M^{-1}(t)\mathbf{b}(t)dt, \tag{3.21}$$

where the 2×2 fundamental matrix $M(x)$ satisfies the following initial-value problem

$$M'(x) = -A(x)M(x), \quad M(a) = I, \tag{3.22}$$

with I the 2×2 identity matrix. It is possible theoretically to solve (3.20) directly by (3.21). This requires integrating (3.22) to obtain $M(x)$ and $M^{-1}(x)$ over the interval $[a,b]$. An equation for (3.21) is evaluated at $x = b$ and solved for the missing initial condition.

Under Assumption A1 and Assumption A2, the entries of the 2×2 matrix $A(x)$ are bounded on $[a,b]$. Using (3.20), we can deduce that there exists a constant C such that (see [18, Lemma 4.2])

$$\|\mathbf{W}\|_1^2 = \|W_1\|_1^2 + \|W_2\|_1^2 \le C\,\|\mathbf{b}\|^2 = C\left(\|e_u\|^2 + \|e_q\|^2\right),$$

which gives

$$\|W_k\|_1 \le C\left(\|e_u\| + \|e_q\|\right), \quad k = 1, 2. \tag{3.23}$$

Now, we are ready to prove (3.3)–(3.4). Taking $V_1 = W_1$ and $V_2 = W_2$ in (3.18) and using (3.19) gives

$$\|e_q\|^2 + \|e_u\|^2 = \int_\Omega \left(-\epsilon_q' + R_u e_u + R_q e_q\right)(W_1 - P_h^- W_1)dx + \int_\Omega \left(-\epsilon_u' + e_q\right)(W_2 - P_h^+ W_2)dx.$$

Using (3.7) and applying the Cauchy-Schwarz inequality yields

$$\|e_q\|^2 + \|e_u\|^2 \le \left(\|\epsilon_q'\| + K_1\|e_u\| + K_2\|e_q\|\right)\|W_1 - P_h^- W_1\| + \left(\|\epsilon_u'\| + \|e_q\|\right)\|W_1 - P_h^+ W_1\|.$$

Applying the standard interpolation error estimate (2.10), we get

$$\|e_q\|^2 + \|e_u\|^2 \le \left(Ch^p + K_1\|e_u\| + K_2\|e_q\|\right)Ch\,|W_1|_1 + \left(Ch^p + \|e_q\|\right)Ch\,|W_2|_1.$$

Applying the regularity estimate (3.23), we get

$$\|e_q\|^2 + \|e_u\|^2 \le C_1 h^{p+1}\left(\|e_q\| + \|e_u\|\right) + C_2 h\left(\|e_q\|^2 + \|e_u\|^2\right).$$

Thus, for sufficiently small h, *e.g.*, $\frac{1}{2} \le 1 - C_2 h$, or equivalently, $h \le \frac{1}{2C_2}$, we deduce

$$\|e_q\|^2 + \|e_u\|^2 \le 2C_1 h^{p+1}\left(\|e_q\| + \|e_u\|\right).$$

Invoking Young's inequality $ab \le \frac{a^2}{2} + \frac{1}{2}b^2$, we obtain

$$\|e_q\|^2 + \|e_u\|^2 \le Ch^{2p+2},$$

which completes the proof of (3.3)–(3.4). □

4. Superconvergence error analysis

In this section, we investigate the superconvergence properties of the proposed LDG method. We prove that the derivatives of the LDG solutions are superconvergent with order $p+1$ toward the derivatives of Gauss-Radau projections of the exact solutions. We further prove pointwise superconvergence results at the upwind and downwind points of each element. More precisely, we will prove that, for $i=1,2,\ldots,N$, $|e_u(x_i^-)| = \mathcal{O}(h^{2p+1})$ and $|e_q(x_{i-1}^+)| = \mathcal{O}(h^{2p+1})$. We will use these results to show that the p-degree LDG solutions u_h and q_h, respectively, converge in the L^2-norm to $P_h^- u$ and $P_h^+ q$ at $\mathcal{O}(h^{p+2})$.

4.1. Superconvergence for the derivative approximations

In the next theorem, we prove that the derivatives of the LDG solutions u_h' and q_h' are $\mathcal{O}(h^{p+1})$ super close to $(P_h^- u)'$ and $(P_h^+ q)'$, respectively.

Theorem 4.1. *Suppose that the assumptions of Theorem 3.1 are satisfied. Then there exists a positive constant C independent of h such that*

$$\|\bar{e}_u'\| \le Ch^{p+1}, \quad \|\bar{e}_q'\| \le Ch^{p+1}. \tag{4.1}$$

Proof. By the property of $P_h^{\pm}$, we have

$$\epsilon_u(x_i^-) = \epsilon_q(x_i^+) = 0, \quad \int_{I_i} \epsilon_q v_1' dx = \int_{I_i} \epsilon_u v_2' dx = 0, \quad \forall\, v_1,\ v_2 \in P^p(I_i), \quad i=1,\ldots,N. \tag{4.2}$$

Using (3.1) and applying (4.2), we rewrite (3.5) as

$$-\int_{I_i} \bar{e}_q v_1' dx + \bar{e}_q(x_i^+) v_1(x_i^-) - \bar{e}_q(x_{i-1}^+) v_1(x_{i-1}^+) = \int_{I_i} \left(R_u e_u + R_q e_q\right) v_1 dx, \tag{4.3a}$$

$$-\int_{I_i} \bar{e}_u v_2' dx + \bar{e}_u(x_i^-) v_2(x_i^-) - \bar{e}_u(x_{i-1}^-) v_2(x_{i-1}^+) = \int_{I_i} e_q v_2 dx. \tag{4.3b}$$

Using integration by parts, we write (4.3) as

$$\int_{I_i} \bar{e}_q' v_1 dx + [\bar{e}_q](x_i) v_1(x_i^-) = \int_{I_i} \left(R_u e_u + R_q e_q\right) v_1 dx, \tag{4.4a}$$

$$\int_{I_i} \bar{e}_u' v_2 dx + [\bar{e}_u](x_{i-1}) v_2(x_{i-1}^+) = \int_{I_i} e_q v_2 dx. \tag{4.4b}$$

Next we follow the idea in [34,35]. Choosing $v_1(x) = \bar{e}_q'(x) - \bar{e}_q'(x_i^-) L_{p,i}(x) \in P^p(I_i)$ in (4.4a), $v_2(x) = \bar{e}_u'(x) - (-1)^p \bar{e}_u'(x_{i-1}^+) L_{p,i}(x) \in P^p(I_i)$ in (4.4b), and applying (2.5) gives

$$\|\bar{e}_q'\|_{0,I_i}^2 = \int_{I_i} \left(R_u e_u + R_q e_q\right)\left(\bar{e}_q' - \bar{e}_q'(x_i^-) L_{p,i}\right) dx,$$

$$\|\bar{e}_u'\|_{0,I_i}^2 = \int_{I_i} e_q \left(\bar{e}_u' - (-1)^p \bar{e}_u'(x_{i-1}^+) L_{p,i}\right) dx,$$

since $v_1(x_i^-) = v_2(x_{i-1}^+) = 0$ and $\int_{I_i} \bar{e}_u' L_{p,i} dx = \int_{I_i} \bar{e}_q' L_{p,i} dx = 0$.

Applying the estimate (3.7), the Cauchy-Schwarz inequality, the inverse inequality, and the estimate (2.7) yields

$$\begin{aligned}
\|\bar{e}_q'\|_{0,I_i}^2 &\le \left(K_1 \|e_u\|_{0,I_i} + K_2 \|e_q\|_{0,I_i}\right)\left(\|\bar{e}_q'\|_{0,I_i} + |\bar{e}_q'(x_i^-)|\, \|L_{p,i}\|_{0,I_i}\right) \\
&\le \left(K_1 \|e_u\|_{0,I_i} + K_2 \|e_q\|_{0,I_i}\right)\left(\|\bar{e}_q'\|_{0,I_i} + C h_i^{-1/2} \|\bar{e}_q'\|_{0,I_i} h_i^{1/2}\right) \\
&\le C_1 \left(\|e_u\|_{0,I_i} + \|e_q\|_{0,I_i}\right) \|\bar{e}_q'\|_{0,I_i},
\end{aligned}$$

$$\|\bar{e}_u'\|_{0,I_i}^2 \le \|e_q\|_{0,I_i} \left(\|\bar{e}_u'\|_{0,I_i} + |\bar{e}_u'(x_{i-1}^+)|\, \|L_{p,i}\|_{0,I_i}\right)$$

$$\leq \left\| e_q \right\|_{0,I_i} \left(\left\| \bar{e}'_u \right\|_{0,I_i} + C h_i^{-1/2} \left\| \bar{e}'_u \right\|_{0,I_i} h_i^{1/2} \right)$$
$$\leq C_2 \left\| e_q \right\|_{0,I_i} \left\| \bar{e}'_u \right\|_{0,I_i}.$$

Consequently, we deduce

$$\left\| \bar{e}'_q \right\|_{0,I_i} \leq C_1 \left(\| e_u \|_{0,I_i} + \left\| e_q \right\|_{0,I_i} \right), \quad \left\| \bar{e}'_u \right\|_{0,I_i} \leq C_2 \left\| e_q \right\|_{0,I_i}.$$

Squaring both sides, using the inequality $(a_1 + a_2)^2 \leq 2(a_1^2 + a_2^2)$, summing over all elements, and using the estimates (3.3) and (3.4), we obtain

$$\left\| \bar{e}'_q \right\|^2 \leq 2C_1^2 \left(\| e_u \|^2 + \left\| e_q \right\|^2 \right) \leq C h^{2p+2}, \quad \left\| \bar{e}'_u \right\|^2 \leq C_2^2 \left\| e_q \right\|^2 \leq C h^{2p+2},$$

which completes the proof of (4.1). □

4.2. Pointwise superconvergence

First, we prove superconvergence results at the endpoints of the computational domain Ω. We state them in the following theorem.

Theorem 4.2. *Suppose that the assumptions of Theorem 3.1 are satisfied. We assume that the function $f \in C_b^p(D)$, where $C_b^m(D)$ is the set of real m-times continuously differentiable functions which are bounded together with their derivatives up to the mth order on the set $D = [a, b] \times \mathbb{R}^2$. Let $p \geq 1$ and (u_h, q_h) be the LDG solutions of (2.3), then there exists a positive constant C independent of h such that*

$$\left| e_u(x_N^-) \right| \leq C h^{2p+1}. \tag{4.5}$$
$$\left| e_q(x_0^+) \right| \leq C h^{2p+1}. \tag{4.6}$$

Proof. We construct the following adjoint problem: find U_1 and U_2 such that

$$U_1' + U_2 + R_q U_1 = 0, \quad U_2' + R_u U_1 = 0, \quad \text{for } x \in (a, b) \quad \text{subject to } U_1(a) = 0, \ U_2(b) = -1. \tag{4.7}$$

The BVP (4.7) can be transformed into the system of equations

$$\mathbf{U}' + A(x)\mathbf{U} = 0, \quad x \in (a, b), \quad B_1 \mathbf{U}(a) + B_2 \mathbf{U}(b) = \mathbf{U}_0, \tag{4.8}$$

where

$$\mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \ A = \begin{bmatrix} R_q & 1 \\ R_u & 0 \end{bmatrix}, \ \mathbf{U}_0 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The solution to (4.8) can be expressed in terms of its fundamental matrix

$$\mathbf{U}(x) = M(x)\mathbf{U}(a), \tag{4.9}$$

where the 2×2 fundamental matrix $M(x)$ satisfies the initial-value problem

$$M'(x) = -A(x)M(x), \quad M(a) = I. \tag{4.10}$$

If $f \in C_b^p(D)$ then the entries of $A(x)$ are in $H^p(\Omega)$. Differentiate (4.8) p times to express $\mathbf{U}^{(p+1)}$ in terms of $\mathbf{U}$, then replace $\mathbf{U}$ using (4.9). Thus, the BVP (4.7) satisfies the following regularity estimate

$$|U_k|_{p+1} \leq C, \quad k = 1, 2. \tag{4.11}$$

Taking $V_1 = U_1$ and $V_2 = U_2$ in (3.18) and applying (4.7), we get

$$e_u(x_N^-) = \int_\Omega \left(-\epsilon'_q + R_u e_u + R_q e_q \right) (U_1 - P_h^- U_1) dx + \int_\Omega \left(-\epsilon'_u + e_q \right) (U_2 - P_h^+ U_2) dx.$$

Using (3.7) and applying the Cauchy-Schwarz inequality yields

$$\left| e_u(x_N^-) \right| \leq \left(\left\| \epsilon'_q \right\| + K_1 \| e_u \| + K_2 \left\| e_q \right\| \right) \left\| U_1 - P_h^- U_1 \right\| + \left(\left\| \epsilon'_u \right\| + \left\| e_q \right\| \right) \left\| U_2 - P_h^+ U_2 \right\|.$$

Applying (3.3), (3.4), the standard interpolation error estimate (2.10) and using the regularity estimate (4.11), we get

$$\left|e_u(x_N^-)\right| \le \left(Ch^p\,|q|_{p+1} + K_1Ch^{p+1} + K_2Ch^{p+1}\right)Ch^{p+1}\,|U_1|_{p+1} + \left(Ch^p\,|u|_{p+1} + Ch^{p+1}\right)Ch^{p+1}\,|U_2|_{p+1}$$
$$\le Ch^{2p+1},$$

which completes the proof of (4.5).

Next, we will prove (4.6). We construct the following adjoint problem: find $U_k,\ k = 1-4$ such that

$$U_1' + U_2 + R_q U_1 = 0, \quad U_2' + R_u U_1 = 0, \quad \text{for } x \in (a,b) \quad \text{subject to } U_1(a) = 1, \quad U_2(b) = 0. \tag{4.12}$$

As before, the BVP (4.12) satisfies the regularity estimate

$$|U_k|_{p+1} \le C, \quad k = 1, 2. \tag{4.13}$$

Taking $V_k = U_k,\ k = 1, 2$ in (3.18) and applying (4.12), we get

$$e_q(x_0^-) = \int_\Omega \left(-\epsilon_q' + R_u e_u + R_q e_q\right)(U_1 - P_h^- U_1)dx + \int_\Omega \left(-\epsilon_u' + e_q\right)(U_2 - P_h^+ U_2)dx.$$

Following the steps used to prove (4.5), we establish (4.6). □

In the next theorem, we state and prove superconvergence results at the downwind and upwind points. More precisely, we prove the following theorem.

Theorem 4.3. *Suppose that the assumptions of Theorem 4.2 hold. Then there exists a constant C such that*

$$\max_{j=1,2,\ldots,N} \left|e_u(x_j^-)\right| \le C\,h^{2p+1}. \tag{4.14}$$

$$\max_{j=1,2,\ldots,N} \left|e_q(x_j^+)\right| \le C\,h^{2p+1}. \tag{4.15}$$

Proof. For $j = 1, 2, \ldots, N$, we consider the following terminal problem: find $U_1,\ U_2 \in H^{p+1}[a, x_j]$ such that

$$U_1' + U_2 + R_q U_1 = 0, \quad U_2' + R_u U_1 = 0, \quad \text{for } x \in \Omega_j = [a, x_j] \quad \text{subject to } U_1(x_j) = 0,\ U_2(x_j) = -1. \tag{4.16}$$

Under the assumption of the theorem, one can easily verify that (4.16) satisfies the following regularity estimate

$$|U_k(a)| \le C, \quad |U_k|_{p+1,\Omega_j} \le C, \quad k = 1, 2. \tag{4.17}$$

Taking $V_1 = U_1$ and $V_2 = U_2$ in (3.10), summing over the elements $I_i,\ i = 1, 2, \ldots, j$, using (4.16) and the fact that $e_u(x_0^-) = 0$, we obtain

$$\begin{aligned}\sum_{i=1}^{j}\left(\mathcal{A}_1^{(i)}(U_1) + \mathcal{A}_2^{(i)}(U_2)\right) &= \int_{\Omega_j} e_q\left(U_1' + R_q U_1 + U_2\right)dx + \int_{\Omega_j} e_u\left(U_2' + R_u U_1\right)dx \\ &\quad - e_q(x_j^+)U_1(x_j^-) + e_q(x_0^+)U_1(x_0^+) - e_u(x_j^-)U_2(x_j^-) \\ &= e_u(x_j^-) + e_q(x_0^+)U_1(a).\end{aligned} \tag{4.18}$$

On the other hand, taking $V_k = U_k,\ k = 1, 2$ in (3.16) and summing over the elements $I_i,\ i = 1, 2, \ldots, j$, we get

$$\sum_{i=1}^{j}\left(\mathcal{A}_1^{(i)}(U_1) + \mathcal{A}_2^{(i)}(U_2)\right) = \int_{\Omega_j}\left(-\epsilon_q' + R_u e_u + R_q e_q\right)(U_1 - P_h^- U_1)dx + \int_{\Omega_j}\left(-\epsilon_u' + e_q\right)(U_2 - P_h^+ U_2)dx. \tag{4.19}$$

Combining the two formulas (4.18) and (4.19) yields

$$e_u(x_j^-) = -e_q(x_0^+)U_1(a) + \int_{\Omega_j}\left(-\epsilon_q' + R_u e_u + R_q e_q\right)(U_1 - P_h^- U_1)dx + \int_{\Omega_j}\left(-\epsilon_u' + e_q\right)(U_2 - P_h^+ U_2)dx.$$

Using (3.7) and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|e_u(x_j^-)\right| &\le \left|e_q(x_0^+)\right| |U_1(a)| + \left(\|\epsilon_q'\|_{0,\Omega_j} + K_1\|e_u\|_{0,\Omega_j} + K_2\|e_q\|_{0,\Omega_j}\right)\|U_1 - P_h^- U_1\|_{0,\Omega_j}\\ &\quad + \left(\|\epsilon_u'\|_{0,\Omega_j} + \|e_q\|_{0,\Omega_j}\right)\|U_2 - P_h^+ U_2\|_{0,\Omega_j}\\ &\le \left|e_q(x_0^+)\right| |U_1(a)| + \left(\|\epsilon_q'\| + K_1\|e_u\| + K_2\|e_q\|\right)\|U_1 - P_h^- U_1\|_{0,\Omega_j}\\ &\quad + \left(\|\epsilon_u'\| + \|e_q\|\right)\|U_2 - P_h^+ U_2\|_{0,\Omega_j}.\end{aligned}$$

Using the standard interpolation error estimates (2.10), the estimates (3.3), (3.4), (4.6), and the regularity estimate (4.17), we obtain

$$\begin{aligned}\left|e_u(x_j^-)\right| &\le (Ch^{2p+1})C + (Ch^{2p+1})C + \left(Ch^p + K_1 Ch^{p+1} + K_2 Ch^{p+1}\right) Ch^{p+1} |U_1|_{p+1,\Omega_j}\\ &\quad + \left(Ch^p + Ch^{p+1}\right) Ch^{p+1} |U_2|_{p+1,\Omega_j}\\ &\le Ch^{2p+1},\end{aligned}$$

which completes the proof of (4.14).

Next, we will show (4.15). We construct the following terminal problem: find $U_1,\ U_2 \in H^{p+1}[x_{j-1}, b],\ \ j = 1, 2, \ldots, N$ such that

$$U_1' + U_2 + R_q U_1 = 0,\ U_2' + R_u U_1 = 0,\ \text{ for } x \in \bar{\Omega}_j = [x_{j-1}, b] \text{ subject to } U_1(x_{j-1}) = 1,\ U_2(x_{j-1}) = 0. \tag{4.20}$$

Under the assumptions of the theorem, one can easily verify that (4.20) satisfies the following regularity estimate

$$|U_k(b)| \le C, \quad |U_k|_{p+1,\bar{\Omega}_j} \le C, \quad k = 1, 2. \tag{4.21}$$

Taking $V_1 = U_1$ and $V_2 = U_2$ in (3.10), summing over the elements $I_i,\ i = j, \ldots, N$, using (4.20) and the fact that $e_q(x_N^+) = 0$, we obtain

$$\begin{aligned}\sum_{i=j}^{N}\left(\mathcal{A}_1^{(i)}(U_1) + \mathcal{A}_2^{(i)}(U_2)\right) &= \int_{\bar{\Omega}_j} e_q\left(U_1' + R_q U_1 + U_2\right)dx + \int_{\bar{\Omega}_j} e_u\left(U_2' + R_u U_1\right)dx\\ &\quad - e_q(x_N^+)U_1(x_N^-) + e_q(x_{j-1}^+)U_1(x_{j-1}^+) - e_u(x_N^-)U_2(x_N^-) + e_u(x_{j-1}^-)U_2(x_{j-1}^+)\\ &= e_q(x_{j-1}^+) - e_u(x_N^-)U_2(b).\end{aligned} \tag{4.22}$$

On the other hand, taking $V_k = U_k,\ k = 1, 2$ in (3.16) and summing over the elements $I_i,\ i = j, j+1, \ldots, N$, we get

$$\sum_{i=1}^{j}\left(\mathcal{A}_1^{(i)}(U_1) + \mathcal{A}_2^{(i)}(U_2)\right) = \int_{\bar{\Omega}_j}\left(-\epsilon_q' + R_u e_u + R_q e_q\right)(U_1 - P_h^- U_1)dx + \int_{\bar{\Omega}_j}\left(-\epsilon_u' + e_q\right)(U_2 - P_h^+ U_2)dx. \tag{4.23}$$

Combining the two formulas (4.22) and (4.23) yields

$$e_q(x_{j-1}^+) = e_u(x_N^-)U_2(b) + \int_{\bar{\Omega}_j}\left(-\epsilon_q' + R_u e_u + R_q e_q\right)(U_1 - P_h^- U_1)dx + \int_{\bar{\Omega}_j}\left(-\epsilon_u' + e_q\right)(U_2 - P_h^+ U_2)dx.$$

Using (3.7) and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|e_q(x_{j-1}^+)\right| &\le \left|e_u(x_N^-)\right| |U_2(b)| + \left(\|\epsilon_q'\|_{0,\bar{\Omega}_j} + K_1\|e_u\|_{0,\bar{\Omega}_j} + K_2\|e_q\|_{0,\bar{\Omega}_j}\right)\|U_1 - P_h^- U_1\|_{0,\bar{\Omega}_j}\\ &\quad + \left(\|\epsilon_u'\|_{0,\bar{\Omega}_j} + \|e_q\|_{0,\bar{\Omega}_j}\right)\|U_2 - P_h^+ U_2\|_{0,\bar{\Omega}_j}\\ &\le \left|e_u(x_N^-)\right| |U_2(b)| + \left(\|\epsilon_q'\| + K_1\|e_u\| + K_2\|e_q\|\right)\|U_1 - P_h^- U_1\|_{0,\bar{\Omega}_j}\\ &\quad + \left(\|\epsilon_u'\| + \|e_q\|\right)\|U_2 - P_h^+ U_2\|_{0,\bar{\Omega}_j}.\end{aligned}$$

Using the standard interpolation error estimates (2.10), the estimates (3.3), (3.4), (4.5), and the regularity estimate (4.21), we obtain

$$\begin{aligned}\left|e_q(x_{j-1}^+)\right| &\le (Ch^{2p+1})C + (Ch^{2p+1})C + \left(Ch^p + K_1 Ch^{p+1} + K_2 Ch^{p+1}\right) Ch^{p+1} |U_1|_{p+1,\bar{\Omega}_j}\\ &\quad + \left(Ch^p + Ch^{p+1}\right) Ch^{p+1} |U_2|_{p+1,\bar{\Omega}_j}\\ &\le Ch^{2p+1},\end{aligned}$$

which completes the proof of (4.15). □

Corollary 4.1. *Suppose that the assumptions of Theorem 4.2 hold. Then there exists a constant C such that*

$$\max_{j=1,2,\ldots,N}\left|\bar{e}_u(x_j^-)\right| \le C\,h^{2p+1}, \quad \max_{j=1,2,\ldots,N}\left|\bar{e}_q(x_j^+)\right| \le C\,h^{2p+1}. \tag{4.24}$$

Proof. Using (3.1) and the properties of the projections $P_h^{\pm}$, we have $e_u(x_j^-)=\bar{e}_u(x_j^-)$ and $e_q(x_j^+)=\bar{e}_q(x_j^+)$. Invoking the estimates (4.14) and (4.15), we establish (4.24). □

4.3. Superconvergence for average errors at downwind/upwind points

Next, we deduce the $(2p+1)$th superconvergence rate for the average errors at downwind/upwind points. More specifically, we have the following superconvergence results.

Corollary 4.2. *Suppose that the assumptions of Theorem 4.2 hold. Then there exists a constant C such that*

$$\left(\frac{1}{N}\sum_{j=1}^{N}\left|e_u(x_j^-)\right|^2\right)^{1/2} \le C\,h^{2p+1}, \quad \left(\frac{1}{N}\sum_{j=1}^{N}\left|e_q(x_j^+)\right|^2\right)^{1/2} \le C\,h^{2p+1}. \tag{4.25}$$

Proof. These results follow immediately from (4.14) and (4.15). □

4.4. Superconvergence toward Gauss-Radau projections

Next, we prove that the LDG solutions are superconvergent with order $p+2$ toward Gauss-Radau projections of the exact solutions.

Theorem 4.4. *Suppose that the assumptions of Theorem 3.1 are satisfied. We further assume that the function $f \in C_b^2(D)$, where $C_b^m(D)$ is the set of real m-times continuously differentiable functions which are bounded together with their derivatives up to the mth order on the set $D=[a,b]\times\mathbb{R}^2$. Let $p \ge 1$ and (u_h, q_h) be the LDG solutions of (2.3), then there exists a positive constant C independent of h such that*

$$\|\bar{e}_u\| \le Ch^{p+2}, \quad \|\bar{e}_q\| \le Ch^{p+2}. \tag{4.26}$$

Proof. Using the Fundamental Theorem of Calculus, we write

$$|\bar{e}_u(x)| = \left|\bar{e}_u(x_i^-) + \int_{x_i}^{x}\bar{e}_u'(s)ds\right| \le \left|\bar{e}_u(x_i^-)\right| + \int_{I_i}\left|\bar{e}_u'(s)\right|ds, \quad \forall\, x \in I_i,$$

$$\left|\bar{e}_q(x)\right| = \left|\bar{e}_q(x_{i-1}^+) + \int_{x_{i-1}}^{x}\bar{e}_q'(s)ds\right| \le \left|\bar{e}_q(x_{i-1}^+)\right| + \int_{I_i}\left|\bar{e}_q'(s)\right|ds, \quad \forall\, x \in I_i.$$

Taking the square of both sides, applying the inequality $(a+b)^2 \le 2a^2+2b^2$ and applying the Cauchy-Schwartz inequality, we get

$$|\bar{e}_u(x)|^2 \le 2\left|\bar{e}_u(x_i^-)\right|^2 + 2\left(\int_{I_i}\left|\bar{e}_u'(s)\right|ds\right)^2 \le 2\left|\bar{e}_u(x_i^-)\right|^2 + 2h_i\left\|\bar{e}_u'\right\|_{0,I_i}^2,$$

$$\left|\bar{e}_q(x)\right|^2 \le 2\left|\bar{e}_q(x_{i-1}^+)\right|^2 + 2\left(\int_{I_i}\left|\bar{e}_q'(s)\right|ds\right)^2 \le 2\left|\bar{e}_q(x_{i-1}^+)\right|^2 + 2h_i\left\|\bar{e}_q'\right\|_{0,I_i}^2.$$

Integrating these inequalities with respect to x, using the estimates in (4.24), and the fact that $h_i \le h$, we get

$$\|\bar{e}_u\|_{0,I_i}^2 \le 2h_i\left|\bar{e}_u(x_i^-)\right|^2 + 2h_i^2\left\|\bar{e}_u'\right\|_{0,I_i}^2 \le 2Ch^{4p+3} + 2h^2\left\|\bar{e}_u'\right\|_{0,I_i}^2,$$

$$\left\|\bar{e}_q\right\|_{0,I_i}^2 \le 2h_i\left|\bar{e}_q(x_{i-1}^+)\right|^2 + 2h_i^2\left\|\bar{e}_q'\right\|_{0,I_i}^2 \le 2Ch^{4p+3} + 2h^2\left\|\bar{e}_q'\right\|_{0,I_i}^2.$$

Summing over all elements and using the estimates in (4.1), we obtain

$$\|\bar{e}_u\|^2 \le CNh^{4p+3} + 2h^2 \left\|\bar{e}'_u\right\|^2 \le C_1 h^{4p+2} + 2C_2 h^{2p+4} = \mathcal{O}(h^{2p+4}), \tag{4.27a}$$

$$\left\|\bar{e}_q\right\|^2 \le CNh^{4p+3} + 2h^2 \left\|\bar{e}'_q\right\|^2 \le C_1 h^{4p+2} + 2C_2 h^{2p+4} = \mathcal{O}(h^{2p+4}), \tag{4.27b}$$

where we used $4p+2 \ge 2p+4$ for $p \ge 1$. This completes the proof of (4.26). □

Remark 4.1. The proof of the previous theorem is valid for any regular meshes and using piecewise polynomials of degree $p \ge 1$. When $p=0$, the estimate (4.27) gives $\|\bar{e}_u\| = \mathcal{O}(h)$ and $\left\|\bar{e}_q\right\| = \mathcal{O}(h)$, which converge at the same rate as the errors $\|e_u\|$ and $\left\|e_q\right\|$ (See Theorem 3.1). Thus, our superconvergence results towards Gauss-Radau projections are not valid for $p=0$.

5. Numerical examples

In this section, we present numerical examples to verify our theoretical findings.

Example 5.1. We consider the following nonlinear second-order BVP

$$u'' = \frac{1}{2}\left(1 - (u')^2 - u\sin(x)\right), \quad x \in [0, 2\pi], \tag{5.1a}$$

subject to the periodic boundary conditions

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \tag{5.1b}$$

The exact solution of (5.1) is $u = 2 + \sin(x)$. We solve (5.1) using the proposed LDG method on uniform meshes obtained by partitioning the computational domain $[0, 2\pi]$ into N subintervals with $N = 10, 15, 20, 25, 30, 35$ and using the spaces P^p with $p = 1 - 4$. The L^2-norm of the errors and their orders of convergence are shown in Fig. 1 on the log-log scale. These results suggest that the LDG method lead to optimal convergence orders as expected by the theory; see Theorem 3.1. The L^2-norm of the errors $\|\bar{e}_u\|$ and $\left\|\bar{e}_q\right\|$ presented in Fig. 2 indicate that the LDG solutions u_h and q_h, respectively, converge to the projections $P_h^- u$ and $P_h^+ q$ with order $p+2$. These computational results indicate that the observed numerical superconvergence order is in full agreement with the theoretical convergence order. We observe from Fig. 3 a convergence rate $p+1$ for $\left\|\bar{e}'_u\right\|$ and $\left\|\bar{e}'_q\right\|$. Finally, we present the maximum errors at downwind and upwind points in Fig. 4. We observe that the convergence rate of $\max_{j=1,2,\ldots,N} \left|e_u(x_j^-)\right|$ and $\max_{j=1,2,\ldots,N} \left|e_q(x_j^+)\right|$ is $2p+1$. These results confirm our theoretical findings in Theorem 4.2.

We repeat the previous experiment with all parameters kept unchanged except for the boundary conditions where we use (1.1b). To be more precise, we consider

$$u'' = \frac{1}{2}\left(1 - (u')^2 - u\sin(x)\right), \quad x \in [0, 3], \tag{5.2a}$$

subject to the mixed boundary conditions

$$u(0) = 2, \quad u'(3) = \cos(3). \tag{5.2b}$$

We display the all errors in Figures 5–8. These results suggest optimal convergence and superconvergence rates. Again these results are in full agreement with the theoretical results.

Finally, we solve the following BVP subject to the Dirichlet boundary conditions

$$u'' = \frac{1}{2}\left(1 - (u')^2 - u\sin(x)\right), \quad x \in [0, 3], \tag{5.3a}$$

$$u(0) = 2, \quad u(3) = 2 + \sin(3). \tag{5.3b}$$

We present the errors in Figures 9–12. Again, these results suggest optimal convergence and superconvergence rates when the Dirichlet boundary conditions are used.

Example 5.2. Consider the second-order nonlinear boundary-value problem [6]

$$u'' = \frac{1}{8}\left(32 + 2x^3 - uu'\right), \quad x \in [1, 2], \tag{5.4a}$$

$$u(1) = 17, \quad u'(2) = 0. \tag{5.4b}$$

The exact solution of (5.4) is given by $u(x) = x^2 + 16/x$. We solve (5.4) using the proposed LDG method with $p = 1 - 4$. We use uniform meshes obtained by subdividing $[1, 2]$ into N intervals with $N = 4, 6, 8, 10, 12, 14$. In Fig. 13 we display the

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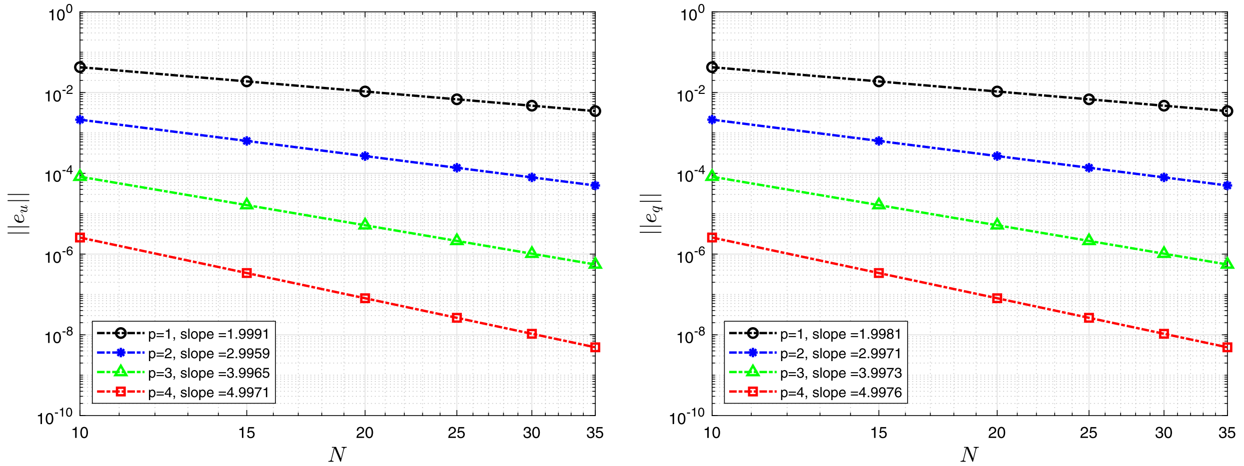

Fig. 1. Log-log plots of $\|e_u\|$ (left) and $\|e_q\|$ (right) versus N for the BVP (5.1) on uniform meshes having $N = 10, 15, 20, 25, 30, 35$ elements using P^p, $p = 1$ to 4.

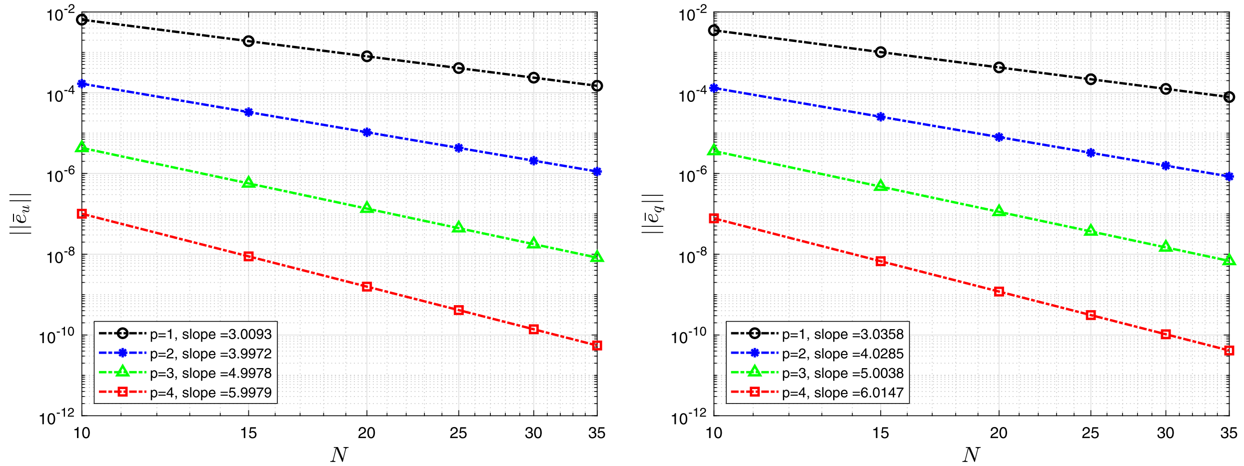

Fig. 2. Log-log plots of $\|\bar{e}_u\|$ (left) and $\|\bar{e}_q\|$ (right) versus mesh sizes N for the BVP (5.1) on uniform meshes having $N = 10, 15, 20, 25, 30, 35$ elements using P^p, $p = 1$ to 4.

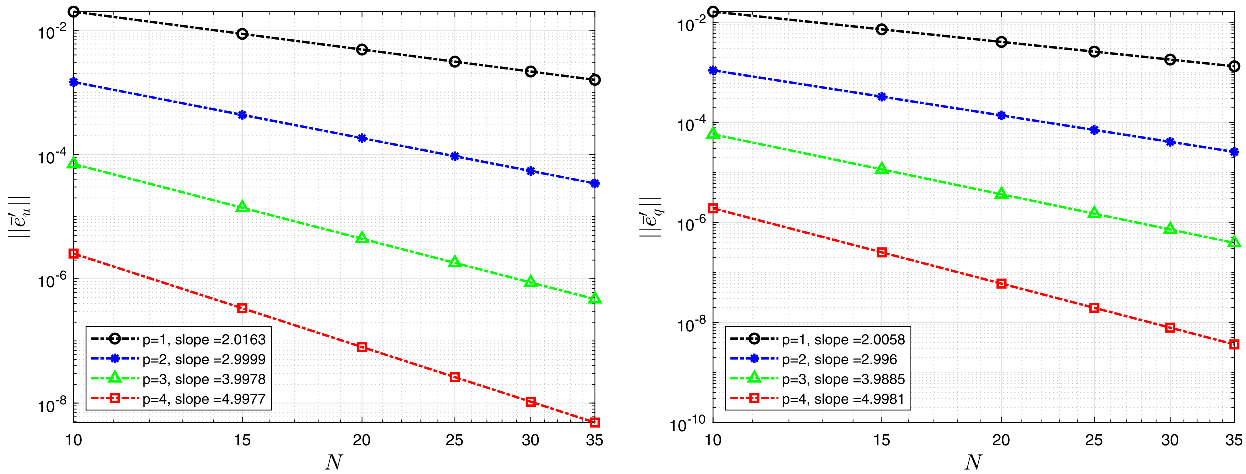

Fig. 3. Log-log plots of $\|\bar{e}'_u\|$ (left) and $\|\bar{e}'_q\|$ (right) versus N for the BVP (5.1) on uniform meshes having $N = 10, 15, 20, 25, 30, 35$ elements using P^p, $p = 1$ to 4.

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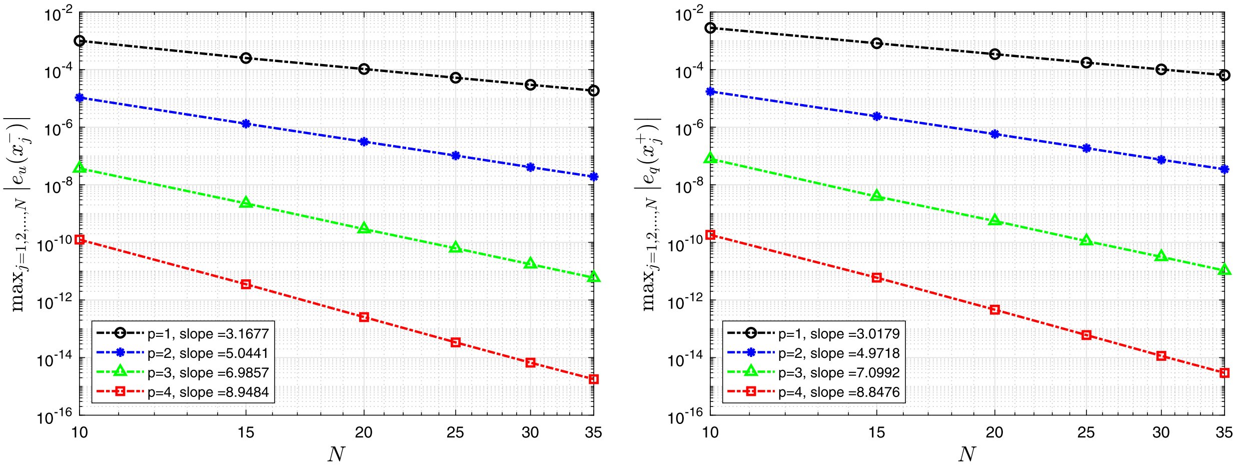

Fig. 4. Log-log plots of $\max_{j=1,2,\ldots,N} \left|e_u(x_j^-)\right|$ (left) and $\max_{j=1,2,\ldots,N} \left|e_q(x_j^+)\right|$ (right) versus N for the BVP (5.1) on uniform meshes having $N = 10, 15, 20, 25, 30, 35$ elements using P^p, $p = 1$ to 4.

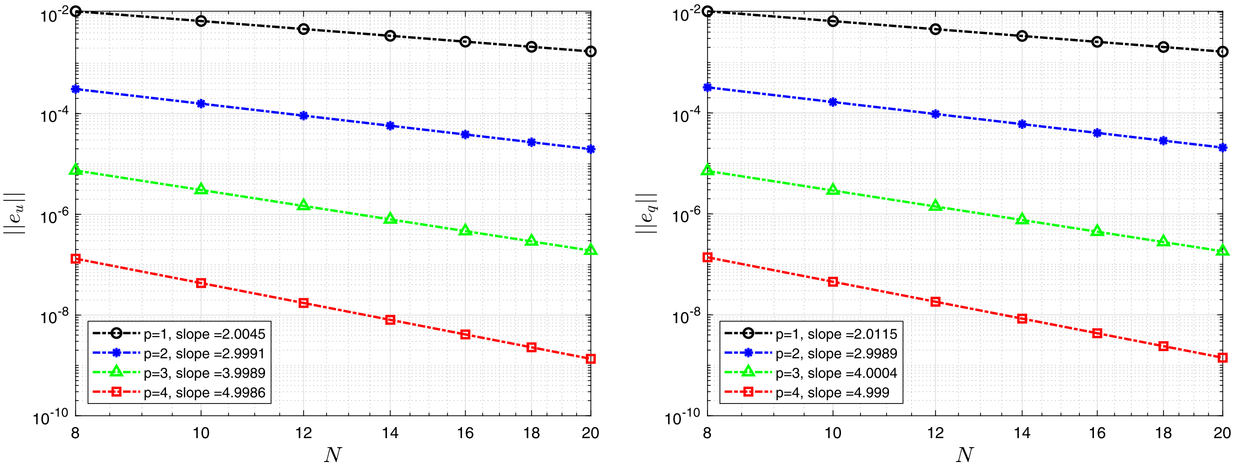

Fig. 5. Log-log plots of $\|e_u\|$ (left) and $\|e_q\|$ (right) versus N for the BVP (5.2) on uniform meshes having $N = 8, 10, 12, 14, 16, 18, 20$ elements using P^p, $p = 1$ to 4.

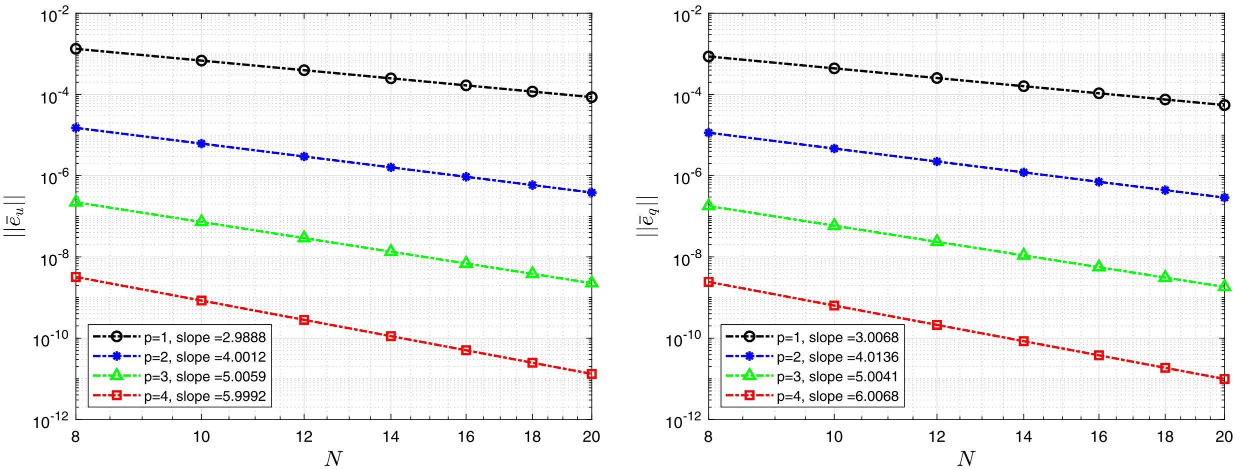

Fig. 6. Log-log plots of $\|\bar{e}_u\|$ (left) and $\|\bar{e}_q\|$ (right) versus N for the BVP (5.2) on uniform meshes having $N = 8, 10, 12, 14, 16, 18, 20$ elements using P^p, $p = 1$ to 4.

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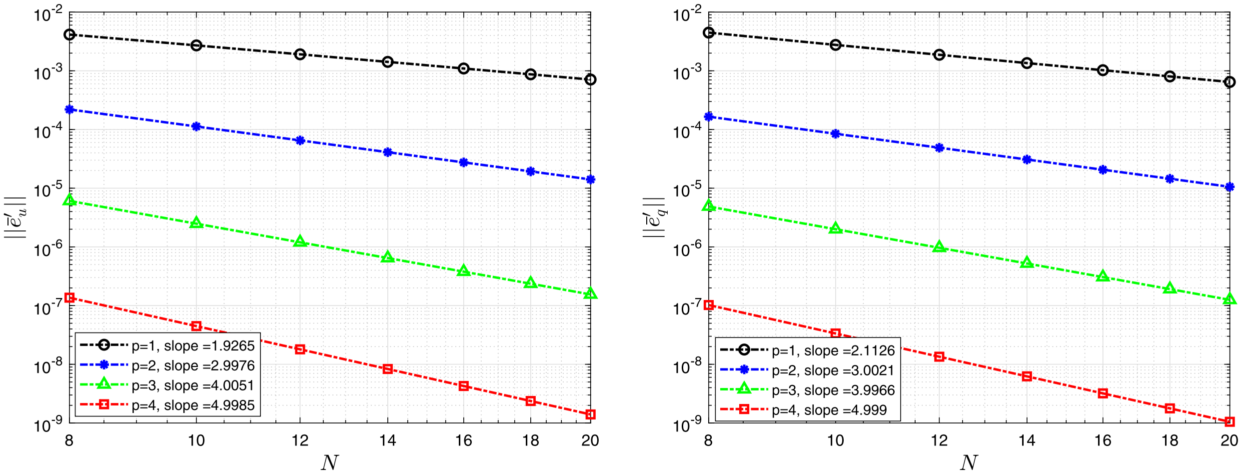

Fig. 7. Log-log plots of $\|\bar{e}_u'\|$ (left) and $\|\bar{e}_q'\|$ (right) versus N for the BVP (5.2) on uniform meshes having $N = 8, 10, 12, 14, 16, 18, 20$ elements using P^p, $p = 1$ to 4.

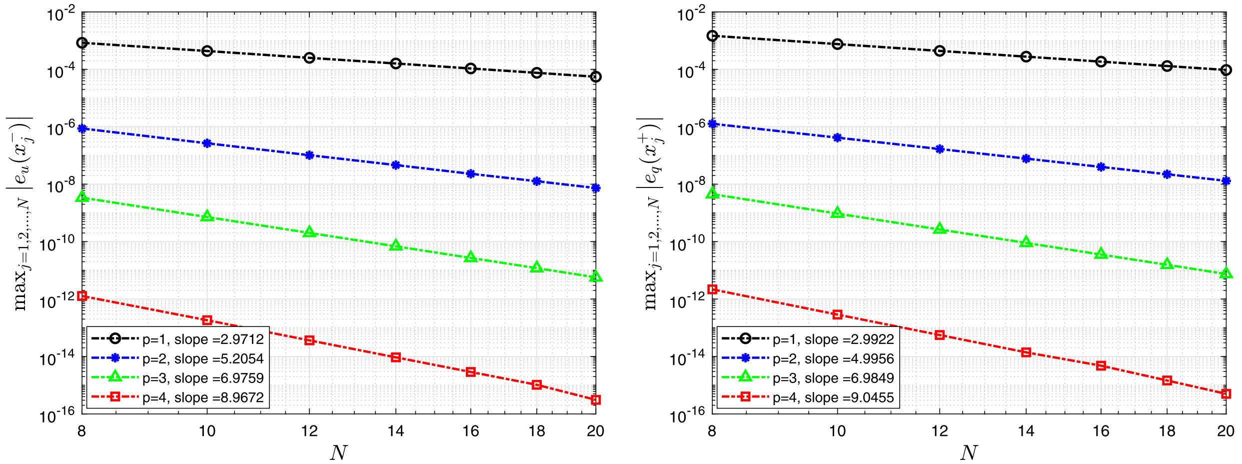

Fig. 8. Log-log plots of $\max_{j=1,2,\ldots,N} \left|e_u(x_j^-)\right|$ (left) and $\max_{j=1,2,\ldots,N} \left|e_q(x_j^+)\right|$ (right) versus N for the BVP (5.2) on uniform meshes having $N = 8, 10, 12, 14, 16, 18, 20$ elements using P^p, $p = 1$ to 4.

L^2-norm of the errors $\|e_u\|$ and $\|e_q\|$ with log-log scale. We also show their orders of convergence. These results indicate that $\|e_u\|$ and $\|e_q\|$ converge at $\mathcal{O}(h^{p+1})$. Thus, the error estimates proved in this paper are optimal in the exponent of the parameter h. In Fig. 14, we report the L^2-norm of the errors $\|\bar{e}_u\|$ and $\|\bar{e}_q\|$ and their orders of convergence. We observe that $\|\bar{e}_u\| = \mathcal{O}(h^{p+2})$ and $\|\bar{e}_q\| = \mathcal{O}(h^{p+2})$. Thus, the LDG solutions u_h and q_h are, respectively, superconvergent with order $p+2$ to the particular projections $P_h^- u$ and $P_h^+ q$. Finally, we report the maximum errors at downwind and upwind points in Fig. 15. We observe that the convergence rate of $\max_{j=1,2,\ldots,N} \left|e_u(x_j^-)\right|$ and $\max_{j=1,2,\ldots,N} \left|e_q(x_j^+)\right|$ is $2p+1$. These results confirm our theoretical findings in Theorem 4.2.

Example 5.3. We consider a two-point boundary-value problem subject to the mixed boundary conditions

$$u'' = \frac{(u')^2}{x^3} - 9\frac{u^2}{x^5} + 4x, \quad x \in [1, 2], \quad u(1) = 0, \quad u'(2) = 4 + 12\ln(2), \tag{5.5}$$

which admits the unique solution $u = x^3 \ln(x)$ [6]. We solve this problem using the LDG method on uniform meshes having $N = 5, 10, 15, 20, 25, 30, 35, 40$ elements and using the spaces P^p with $p = 1, 2, 3$, and 4. The L^2 errors as well as their order of convergence are shown in Fig. 16. These results indicate that the LDG method yields $\mathcal{O}(h^{p+1})$ convergent solutions. The rate of convergence is clearly optimal. The L^2-norm of the errors $||\bar{e}_u||$ and $||\bar{e}_q||$ shown in Fig. 17 indicate that the LDG solutions u_h and q_h are $\mathcal{O}(h^{p+2})$ super close to the projections $P_h^- u$ and $P_h^+ q$, respectively. Finally, we present the

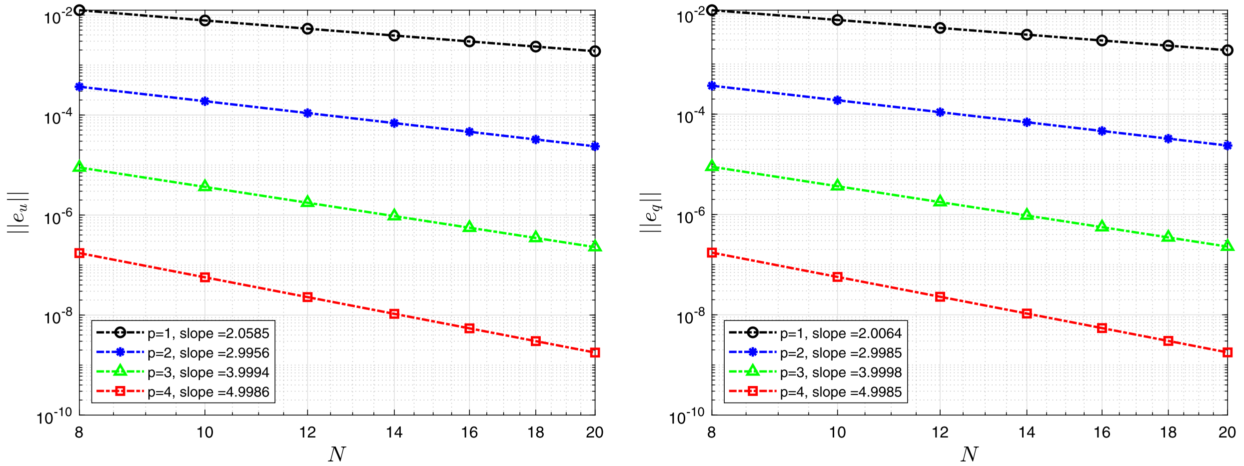

Fig. 9. Log-log plots of $\|e_u\|$ (left) and $\|e_q\|$ (right) versus N for the BVP (5.3) on uniform meshes having $N = 8, 10, 12, 14, 16, 18, 20$ elements using P^p, $p = 1$ to 4.

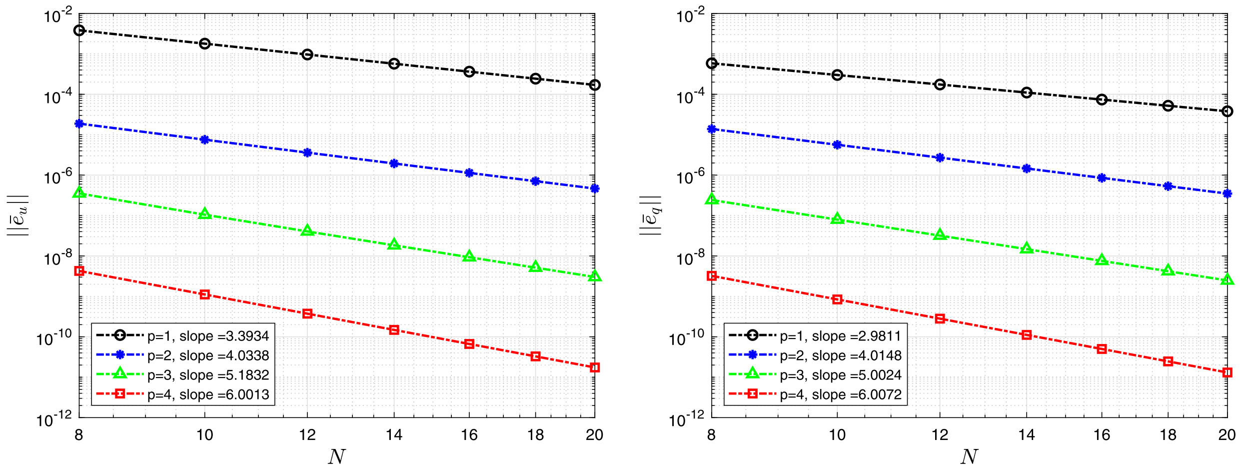

Fig. 10. Log-log plots of $\|\bar{e}_u\|$ (left) and $\|\bar{e}_q\|$ (right) versus N for the BVP (5.3) on uniform meshes having $N = 8, 10, 12, 14, 16, 18, 20$ elements using P^p, $p = 1$ to 4.

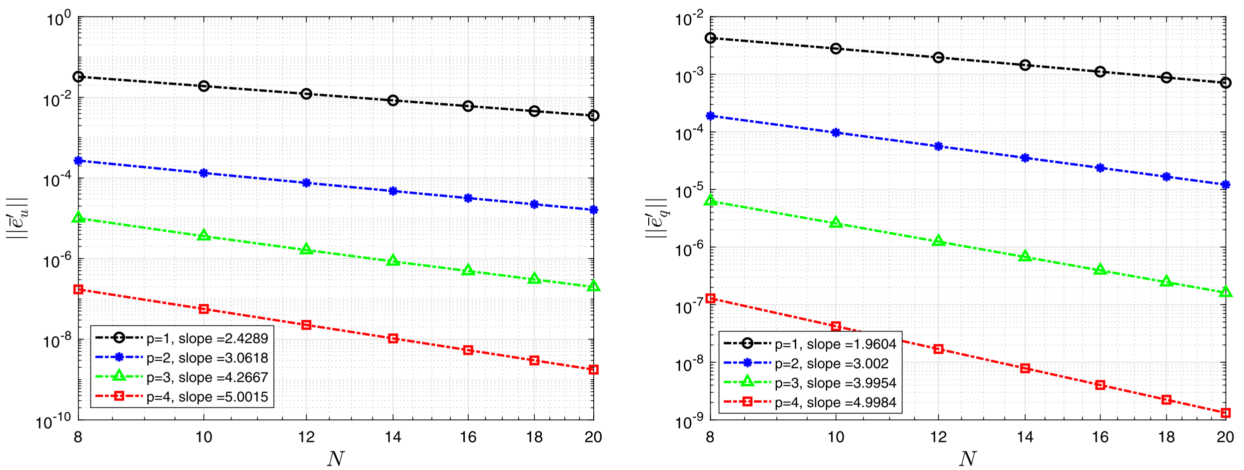

Fig. 11. Log-log plots of $\|\bar{e}_u'\|$ (left) and $\|\bar{e}_q'\|$ (right) versus N for the BVP (5.3) on uniform meshes having $N = 8, 10, 12, 14, 16, 18, 20$ elements using P^p, $p = 1$ to 4.

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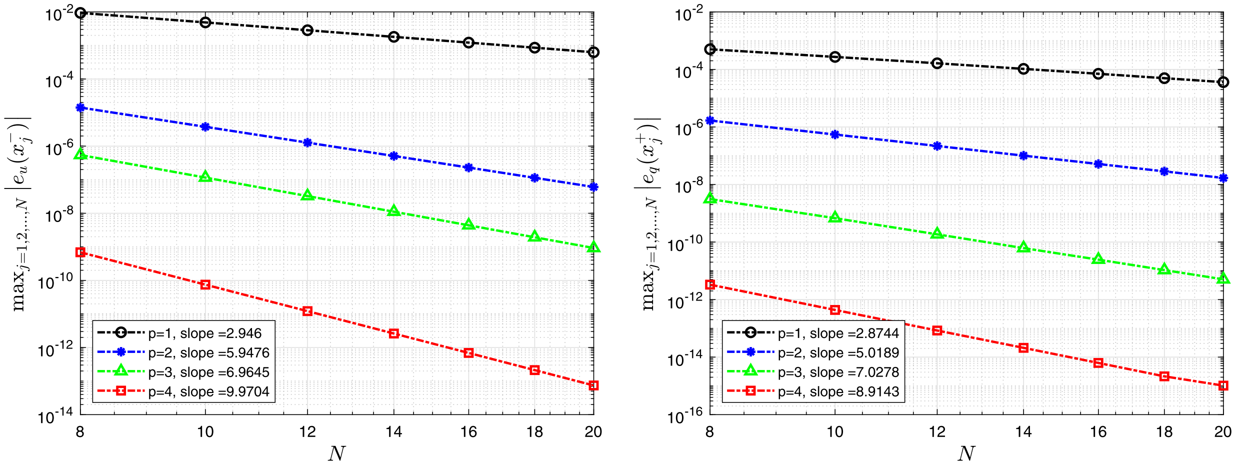

Fig. 12. Log-log plots of $\max_{j=1,2,...,N} \left|e_u(x_j^-)\right|$ (left) and $\max_{j=1,2,...,N} \left|e_q(x_j^+)\right|$ (right) versus N for the BVP (5.3) on uniform meshes having $N = 8, 10, 12, 14, 16, 18, 20$ elements using P^p, $p = 1$ to 4.

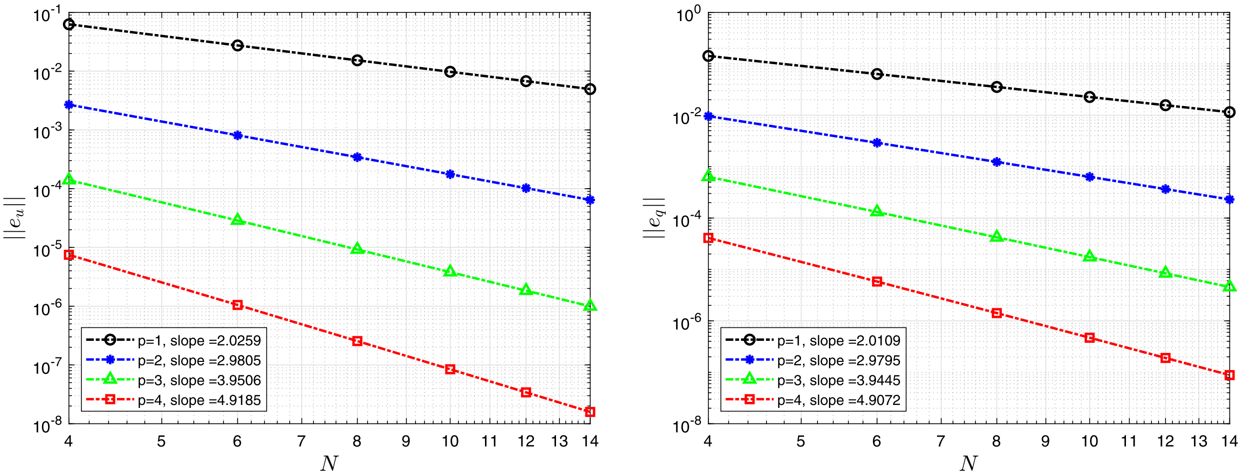

Fig. 13. Log-log plots of $\|e_u\|$ (left) and $\|e_q\|$ (right) versus N for Example 5.2 on uniform meshes having $N = 4, 6, 8, 10, 12, 14$ elements using P^p, $p = 1$ to 4.

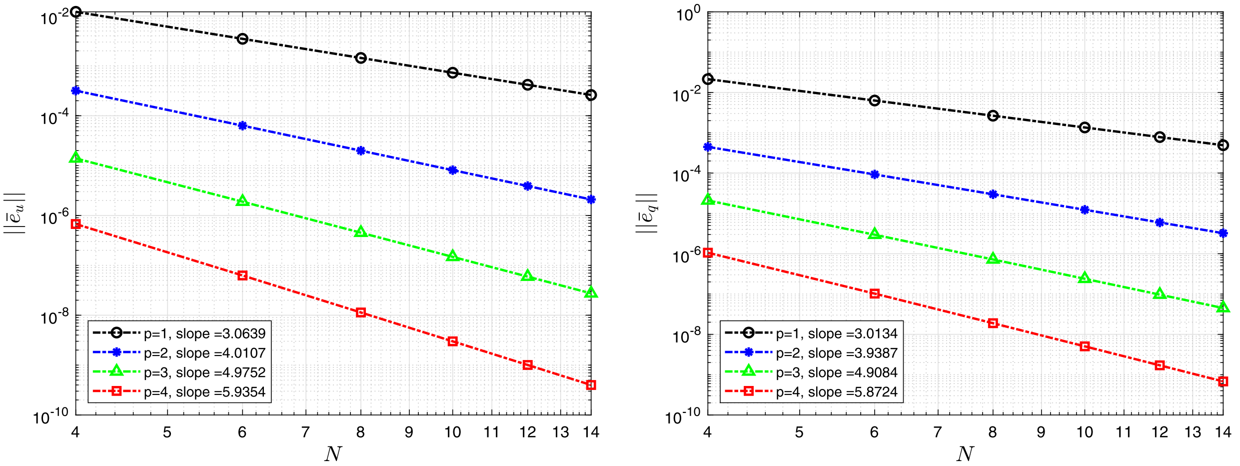

Fig. 14. Log-log plots of $\|\bar{e}_u\|$ (left) and $\|\bar{e}_q\|$ (right) versus N for Example 5.2 on uniform meshes having $N = 4, 6, 8, 10, 12, 14$ elements using P^p, $p = 1$ to 4.

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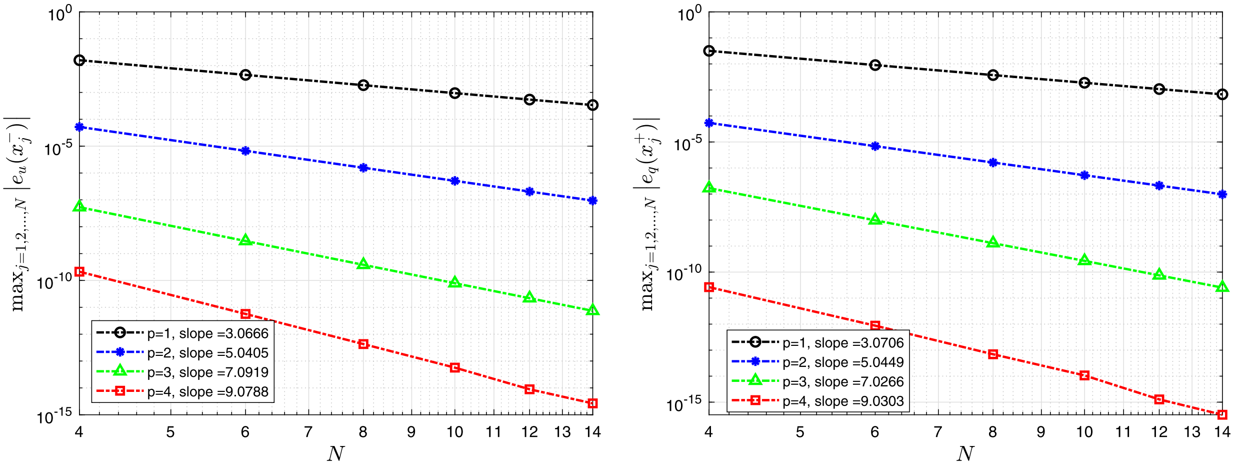

Fig. 15. Log-log plots of $\max_{j=1,2,\ldots,N} \left|e_u(x_j^-)\right|$ (left) and $\max_{j=1,2,\ldots,N} \left|e_q(x_j^+)\right|$ (right) versus N for Example 5.2 on uniform meshes having $N = 8$, 10, 12, 14, 16, 18, 20 elements using P^p, $p = 1$ to 4.

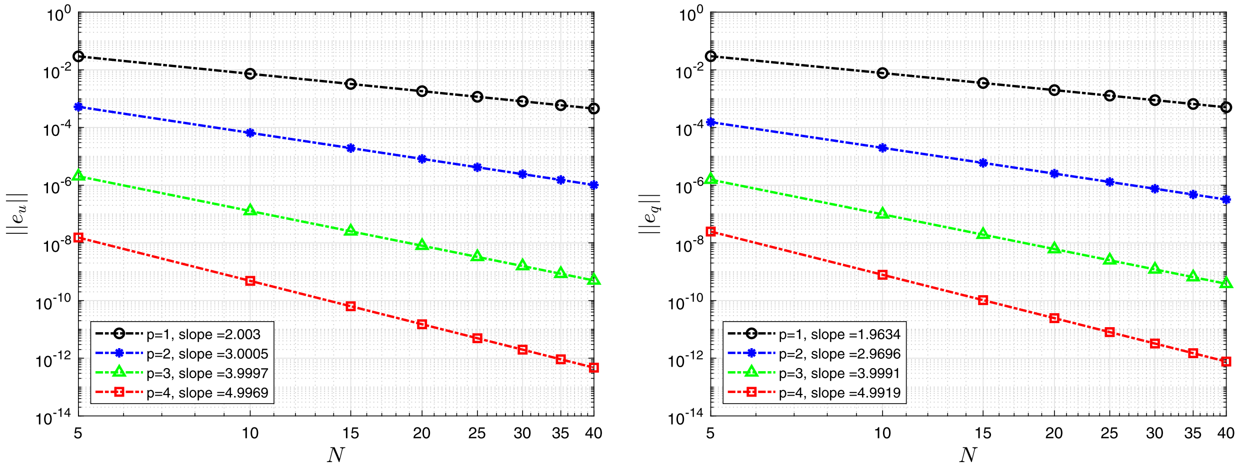

Fig. 16. Log-log plots of $\|e_u\|$ (left) and $\|e_q\|$ (right) versus N for Example 5.3 on uniform meshes having $N = 5$, 10, 15, 20, 25, 30, 35, 40 elements using P^p, $p = 1$ to 4.

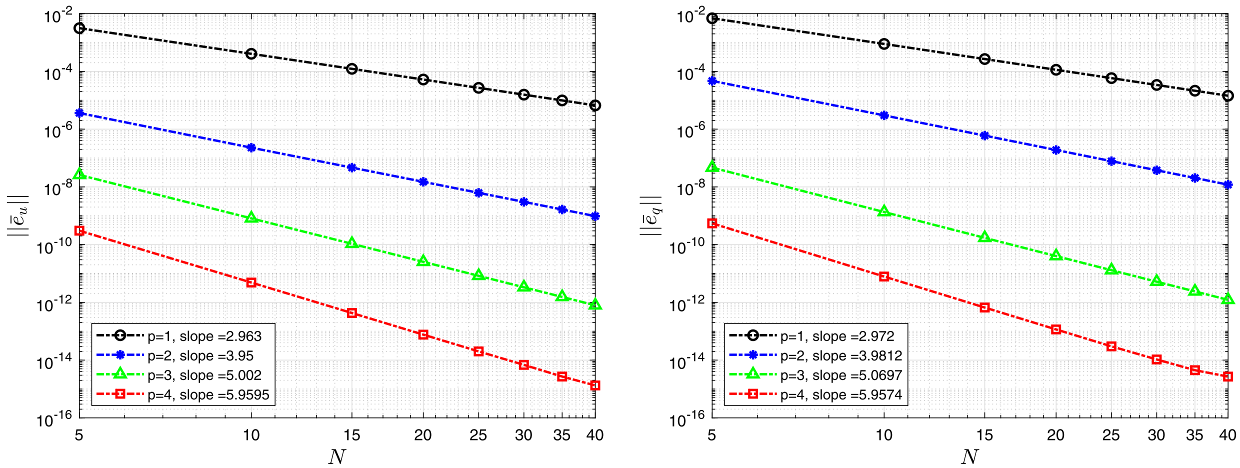

Fig. 17. Log-log plots of $\|\bar{e}_u\|$ (left) and $\|\bar{e}_q\|$ (right) versus N for Example 5.3 on uniform meshes having $N = 5$, 10, 15, 20, 25, 30, 35, 40 elements using P^p, $p = 1$ to 4.

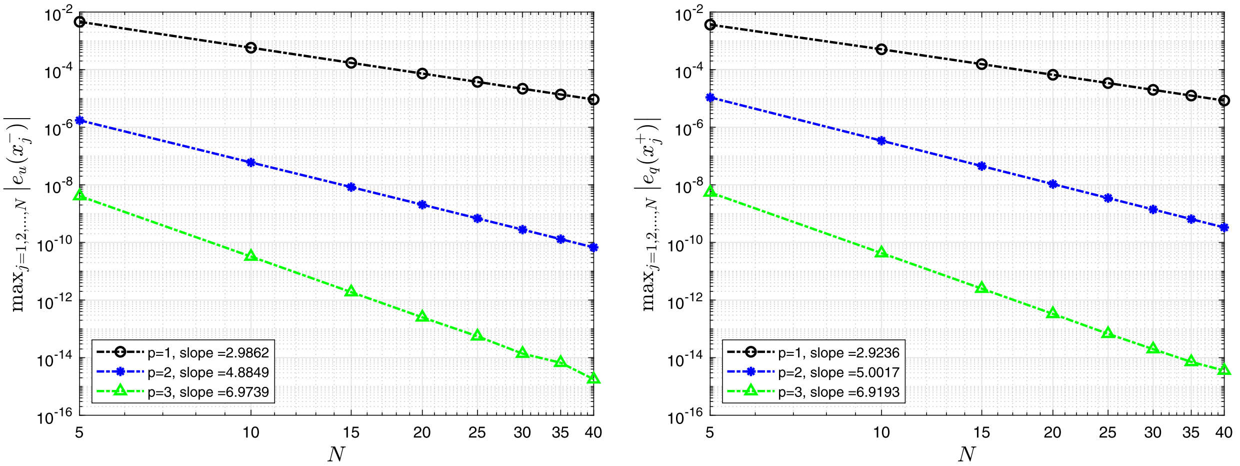

Fig. 18. Log-log plots of $\max\limits_{j=1,2,\ldots,N} \left|e_u(x_j^-)\right|$ (left) and $\max\limits_{j=1,2,\ldots,N} \left|e_q(x_j^+)\right|$ (right) versus N for Example 5.3 on uniform meshes having $N = 5, 10, 15, 20, 25, 30, 35, 40$ elements using P^p, $p = 1$ to 3.

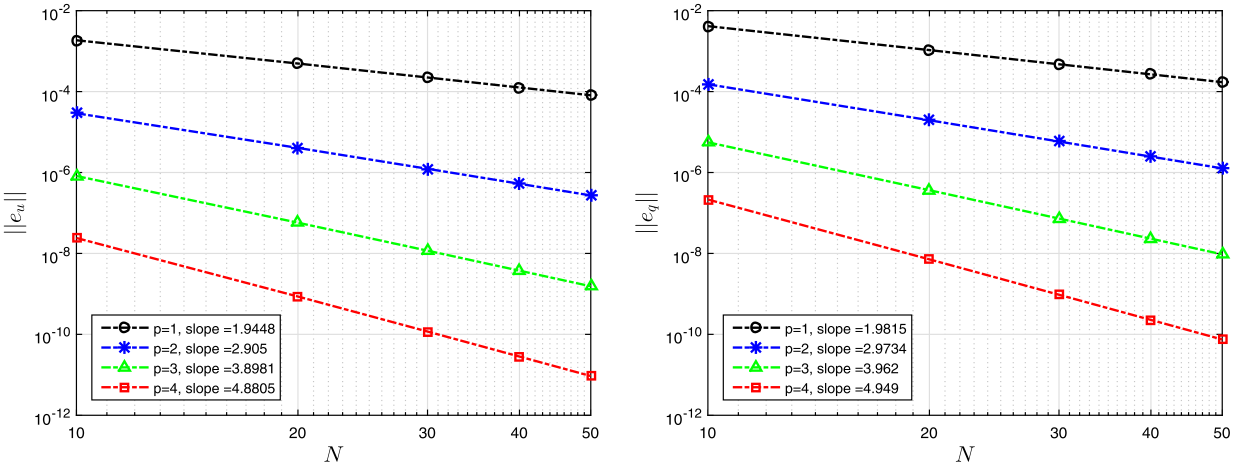

Fig. 19. Log-log plots of $\|e_u\|$ (left) and $\|e_q\|$ (right) versus N for Example 5.4 on uniform meshes having $N = 10, 20, 30, 40, 50$ elements using P^p, $p = 1$ to 4.

maximum errors at downwind and upwind points in Fig. 18. We observe that the convergence rate of $\max\limits_{j=1,2,\ldots,N} \left|e_u(x_j^-)\right|$ and $\max\limits_{j=1,2,\ldots,N} \left|e_q(x_j^+)\right|$ is again $2p+1$. These results confirm our theoretical findings in Theorem 4.2.

Example 5.4. Consider the second-order nonlinear Bratu problem

$$u'' - 2e^u = 0, \quad x \in [0,1], \quad u(0) = 0, \quad u(1) = -2\ln(\cos(1)), \tag{5.6}$$

where the exact solution is $u(x) = -2\ln(\cos(x))$. We use uniform meshes obtained by subdividing the computational domain $[0,1]$ into N intervals with $N = 10, 20, 30, 40, 50$. Fig. 19 shows the L^2 errors $\|e_u\|$ and $\|e_q\|$ and their orders of convergence. These results indicate that $\|e_u\|$ and $\|e_q\|$ are both $\mathcal{O}(h^{p+1})$. In Fig. 20, we report the L^2-norm of the errors $\|\bar{e}_u\|$ and $\|\bar{e}_q\|$ and their orders of convergence. We observe that $\|\bar{e}_u\| = \mathcal{O}(h^{p+2})$ and $\|\bar{e}_q\| = \mathcal{O}(h^{p+2})$. Thus, the LDG solutions u_h and q_h are, respectively, superconvergent with order $p+2$ to the particular projections $P_h^- u$ and $P_h^+ q$.

6. Concluding remarks

In this paper, we investigated the convergence and superconvergence of a local discontinuous Galerkin (LDG) finite element method for nonlinear second-order boundary-value problems (BVPs) for ordinary differential equations. We proved several L^2 error estimates, and superconvergence results toward special projections. To be more precise, we proved that the

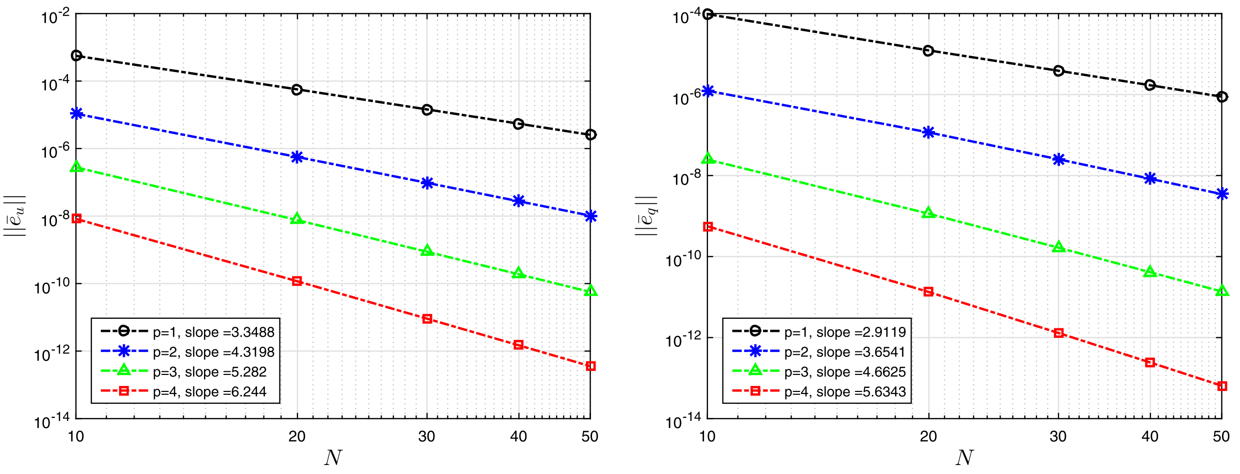

Fig. 20. Log-log plots of $\|\bar{e}_u\|$ (left) and $\|\bar{e}_q\|$ (right) versus N for Example 5.4 on uniform meshes having $N = 10, 20, 30, 40, 50$ elements using P^p, $p = 1$ to 4.

LDG solutions converge to the true solutions with order $p+1$, when the space of piecewise polynomials of degree $p \geq 1$ is used. Moreover, we showed that the derivatives of the LDG solutions converge with order $p+1$ toward the derivatives of Gauss-Radau projections of the exact solutions. In addition, we also proved that the LDG solutions are superconvergent with order $p+2$ to the Gauss-Radau projections of the exact solutions. Finally, we established superconvergence rate of order $2p+1$ for the maximum errors at the upwind or downwind points. Numerical experiments demonstrate that the error bounds are sharp. Our current and future works include two-dimensional elliptic, parabolic, and hyperbolic problems, which would be more challenging and interesting.

Declaration of Competing Interest

The author declares that he has no conflict of interest.

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