n–localization property

Andrzej Rosłanowski

University of Nebraska at Omaha, aroslanowski@unomaha.edu

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0. Introduction

The present paper is concerned with the \(n\)-localization property and its preservation in countable support (CS) iterations. This property was first introduced in Newelski and Rosłanowski [10, p. 826].

Definition 0.1. Let \(n\) be an integer greater than 1.

1. A tree \(T\) is an \(n\)-ary tree provided that \((\forall s \in T)(|\text{succ}_T(s)| \leq n)\).
2. A forcing notion \(P\) has the \(n\)-localization property if

\[\vdash_P \forall f \in \omega \exists T \in V (T \text{ is an } n\text{-ary tree and } f \in [T]).\]

In [10, Theorem 2.3] we showed that countable support products of the \(n\)-Sacks forcing notion \(D_n\) (see Definition 1.5(1)) have the \(n\)-localization property. That theorem was used to obtain some consistency results concerning cardinal characteristics of the ideal determined by unsymmetric games. Soon after this, the uniform \(n\)-Sacks forcing notion \(Q_n\) (see Definition 1.5(2)) was introduced in [11, §4] and applied in the proof of [11, Theorem 5.13]. The crucial property of \(Q_n\) which was used there is that the CS iterations of \(Q_n\) have the \(n\)-localization property, but in [11] we only stated that the proof is similar to that of [10, Theorem 2.3].

One of the difficulties with the \(n\)-localization property was that there was no “preservation theorem” for it. Geschke and Quickert [5] give full and detailed proofs of the 2-localization property for both CS products and CS iterations of the Sacks forcing \(D_2\) (and those proofs can be easily rewritten for \(n\)-localization property and \(D_n\)). And the same proof can be repeated for \(Q_n\), but a more general theorem has been missing.

Recently, the \(n\)-localization property, the \(\sigma\)-ideal generated by \(n\)-ary trees and \(n\)-Sacks forcing notion \(D_n\) have been found applicable to some questions concerning convexity numbers of closed subsets of \(\mathbb{R}^n\), see Geschke, Kojman, Kubiś and Schipperus [4], Geschke and Kojman [3] and most recently Geschke [2]. The latter paper is raison d’être for this note — when I read [2] to write a review for Mathematical Reviews I wanted to check as many technical details as I could. In [2, §2] an interesting forcing notion\(^1\) \(P_G\) was introduced and a proof was given that it has the \(n\)-localization property. However, the proof that the CS iteration of this forcing has the \(n\)-localization property was left to the reader as “similar to that for Sacks”.

\(^1\)we call it the Geschke forcing here, see Definition 1.5(4)
At first I was not sure about technical details of that proof, so I decided to look at $\mathbb{P}_G$ and $D_n$ together. Soon I have become convinced that a unifying theorem is needed and this note presents a result which has such character.

It was stated in [10, Theorem 2.3] that the same proof as for $D_n$ works also for CS iterations and products of the $n$–Silver forcing notions $S_n$ (see Definition 1.5(3)). Maybe some old wisdom got lost, but it does not look like that the same arguments work for the $n$–Silver forcing $S_n$. As a matter of fact, we believe that it is an open question if $S_n$ and its CS iterations have the $n$–localization property. Also, motivated by [1, Questions 3.3, 3.4] we asked if the iteration of two $2$–Silver forcing notions may add a $4$–Silver real, but because of the claim in [10, Corollary 2.4] we did not state the question explicitly in the final version of [1]. In the light of what we said above, it is only proper to pose this problem again.

**Problem 0.2.** (1) Can a finite iteration of $2$–Silver forcings $S_2$ add a generic real for the $4$–Silver forcing notion $S_4$?

(2) Does the $n$–Silver forcing $S_n$ have the $n$–localization property? The same about CS iterations of $n$–Silver forcings.

The author offers “all you can drink in 3 days” coffee/espresso in a place similar to Caffeine Dreams in Omaha for full solution to this problem. Partial solutions may be eligible for partial awards.

It may occur that the answer to the above problem is hidden in Shelah and Steprāns [16]. Let us note that Remark 4.6 suggests that if we can show that finite iterations of $S_n$ have the $n$–localization property, then we will be able to handle all CS iterations.

The following general question remains still unsolved.

**Problem 0.3.** Do CS iterations of proper forcing notions with $n$–localization property have $n$–localization property? What if we restrict ourselves to (s)nep forcing notions (see Shelah [15]) or even Suslin$^+$ (see Goldstern [9] or Kellner [8], [9])?

**Content of the paper:** In the first section we introduce several properties related to the $n$–localization property. The strongest one, $\oplus_n$–property, does imply the $n$–localization. However not all forcing notions around have the $\oplus_n$–property so this is why we have weaker relatives. We also remind definitions of the forcing notions that we are interested in and the basic facts on trees of conditions.

The following section shows that CS iterations of forcing notions with the $\oplus_n$–property have the $n$–localization (Theorem 2.1). Since we do not know if $Q_n$ has the $\oplus_n$–property, in the third section we somewhat weaken that property to cover more forcing notions. From the point of view of applications Theorems 3.3 and 3.4 are strongest and they include the result of the second section. Still, we think that the proof of 2.1 is somewhat easier and it is a good preparation for Section 3.

One should note that the proofs of our iteration theorems are very “not proper” in their form. We work with trees of conditions which were used in pre-proper era and our arguments resemble those of Roslanowski and Shelah [12, §2] and to some extend also [13, §A.2].

**Notation:** Our notation is rather standard and compatible with that of classical textbooks (like Jech [7]). In forcing we keep the older convention that a stronger condition is the larger one.
n–localization property

(1) $n$ is our fixed integer, $n \geq 2$. Ordinal numbers will be denoted by the lower case initial letters of the Greek alphabet ($\alpha, \beta, \gamma, \delta \ldots$) with possible sub- and superscripts. Natural number will be labeled by $i, j, k, \ell, m$ (also upper cases).

By $\chi$ we will denote a sufficiently large regular cardinal; $\mathcal{H}(\chi)$ is the family of all sets hereditarily of size less than $\chi$. Moreover, we fix a well ordering $\prec^*_\chi$ of $\mathcal{H}(\chi)$.

(2) For two sequences $\eta, \nu$ we write $\nu \subset \eta$ whenever $\nu$ is a proper initial segment of $\eta$, and $\nu \leq \eta$ when either $\nu \subset \eta$ or $\nu = \eta$. The length of a sequence $\eta$ is denoted by $\ellh(\eta)$.

(3) A tree is a family of finite sequences closed under initial segments. For a tree $T$ and $\eta \in T$ we define the successors of $\eta$ in $T$ and maximal points of $T$ by:

\[
\text{succ}_T(\eta) = \{\nu \in T : \eta \subset \nu \& -\exists \rho \in T)(\eta \subset \rho \subset \nu)\},
\]

\[
\text{max}(T) = \{\nu \in T : \text{there is no } \rho \in T \text{ such that } \nu \subset \rho\}.
\]

For a tree $T$ the family of all $\omega$–branches through $T$ is denoted by $[T]$.

(4) We will consider some games of two players. One player will be called Generic, and we will refer to this player as “she”. Her opponent will be called Antigeneric and will be referred to as “he”.

(5) For a forcing notion $\mathbb{P}$, $\Gamma_{\mathbb{P}}$ stands for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$. With this one exception, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted by a tilde below (e.g., $\tau, \chi$). The weakest element of $\mathbb{P}$ will be denoted by $\emptyset_{\mathbb{P}}$ (and we will always assume that there is one, and that there is no other condition equivalent to it). We will also assume that all forcing notions under considerations are atomless.

By “CS iterations” we mean iterations in which domains of conditions are countable. However, we will pretend that conditions in a CS iteration $\bar{Q} = (\mathbb{P}_\zeta, \mathbb{Q}_\zeta : \zeta < \gamma)$ are total functions on $\gamma$ and for $p \in \text{lim}(\bar{Q})$ and $\alpha < \gamma$ we have $\Vdash_{\mathbb{P}_\alpha} p(\alpha) \in \mathbb{Q}_\alpha$, and if $\alpha \in \gamma \setminus \text{Dom}(p)$ then $p(\alpha) = \emptyset_{\mathbb{Q}_\alpha}$.

1. Tools

In this section we introduce the main concepts and properties ans we show how they are related to various forcing notions. We also introduce the main tool for our forcing arguments: trees of conditions.

**Definition 1.1.** Let $\mathbb{P}$ be a forcing notion.

(1) For a condition $p \in \mathbb{P}$ we define a game $\mathcal{G}_n^\oplus(p, \mathbb{P})$ of two players, Generic and Antigeneric. A play of $\mathcal{G}_n^\oplus(p, \mathbb{P})$ lasts $\omega$ moves and during it the players construct a sequence $\langle(s_i, \bar{p}^i, \bar{q}^i) : i < \omega\rangle$ as follows. At a stage $i < \omega$ of the play, first Generic chooses a finite $n$–ary tree $s_i$ and a system $\bar{p}^i = \langle p^i_\eta : \eta \in \max(s_i)\rangle$ such that:

\[
(\alpha) \ |\max(s_0)| \leq n \text{ and if } i = j + 1 \text{ then } s_j \text{ is a subtree of } s_i \text{ such that:}
\]

\[
(\forall \eta \in \max(s_i))(\exists \ell < \ellh(\eta))(\eta|\ell \in \max(s_j)),
\]

and

\[
(\forall \nu \in \max(s_j))(0 < |\{\eta \in \max(s_i) : \nu \subset \eta\}| \leq n),
\]

\[
(\beta) \ p^i_\eta \in \mathbb{P} \text{ for all } \eta \in \max(s_i),
\]
Definition 1.2. Let \( K \subseteq \omega \) be infinite, \( p \in \mathbb{P} \). A strategy \( st \) for Generic in \( \mathcal{D}^\odot_n(p, \mathbb{P}) \) is said to be nice for \( K \) (or just \( K \)-nice) whenever

\( (\mathbb{E}^\odot_n) \) if so far Generic used \( st \) and \( s_i \) is given to her as a move at a stage \( i < \omega \), then

- \( s_i \subseteq \bigcup_{j \leq (i+1)} (n+1), \max(s_i) \subseteq (i+1)(n+1) \) and
- if \( \eta \in \max(s_i) \) and \( i \notin K \), then \( \eta(i) = n \), and
- if \( \eta \in \max(s_i) \) and \( i \in K \), then \( \operatorname{succ}_s(\eta[i]) = n \),
- if \( i \in K \), \( \eta[i] : \ell < k \) is an enumeration of \( \max(s_i) \) and \( (p'_\eta, q'_\eta : \ell < k) \) is the result of the subgame of level \( i \) in which Generic uses \( st \), then the conditions \( p'_\eta \) (for \( \ell < k \)) are pairwise incompatible.

In a similar way we define when a strategy \( st \) for Generic in \( \mathcal{D}^\odot_n(p, \mathbb{P}) \) or \( \mathcal{D}^\odot_n(p, \mathbb{P}) \) is nice for \( K \).

(2) We say that \( \mathbb{P} \) has the nice \( \odot_n \)-property if for every \( K \subseteq [\omega]^\omega \) and \( p \in \mathbb{P} \), Generic has a \( K \)-nice winning strategy in \( \mathcal{D}^\odot_n(p, \mathbb{P}) \).

Remark 1.3. (1) At a stage \( i < \lambda \) of a play of \( \mathcal{D}^\odot_n(p, \mathbb{P}) \), Alternategeneric may play stronger conditions, and we may require that if \( \vec{q}^i = \langle q^i_\eta : \eta \in \max(s_i) \rangle \) is his move, then the conditions \( q^i_\eta \) are pairwise incompatible. Thus the winning criterion \( (\odot) \) could be replaced by
Observation 1.4. For a forcing notion $\mathcal{P}$ the following implications hold:

\[ \oplus_n \text{-property} \Rightarrow \text{nice } \oplus_n \text{-property} \Rightarrow \ominus_n \text{-property} \Rightarrow \oplus_n \text{-property} \]

\[ n \text{-localization property} \]

Let us recall definitions of forcing notions that are main examples for the properties introduced in 1.1.

Definition 1.5. (1) The $n$–Sacks forcing notion $\mathcal{D}_n$ consists of perfect trees $p \subseteq \omega^{<}\omega$ such that

\[ (\forall \eta \in p)(\exists \nu \in p)(\eta \triangleleft \nu \land \text{succ}_p(\nu) = n). \]

The order of $\mathcal{D}_n$ is the reverse inclusion, i.e., $p \leq_{\mathcal{D}_n} q$ if and only if $q \subseteq p$.

(2) The uniform $n$–Sacks forcing notion $\mathcal{Q}_n$ consists of perfect trees $p \subseteq \omega^{<}\omega$ such that

\[ (\exists X \in [\omega]^\omega)(\forall \eta \in p)(\text{lh}(\eta) \in X \Rightarrow \text{succ}_p(\nu) = n). \]

The order of $\mathcal{Q}_n$ is the reverse inclusion, i.e., $p \leq_{\mathcal{Q}_n} q$ if and only if $q \subseteq p$.

(3) The $n$–Silver forcing notion $\mathcal{S}_n$ consists of partial functions $p$ such that $\text{Dom}(p) \subseteq \omega$, $\text{Rng}(p) \subseteq n$ and $\omega \setminus \text{Dom}(p)$ is infinite. The order of $\mathcal{S}_n$ is the inclusion, i.e., $p \leq_{\mathcal{S}_n} q$ if and only if $p \subseteq q$.

(4) Let us assume that $G = (V, E)$ is a hypergraph on a Polish space $V$ which is

- $(n + 1)$–regular open, that is $E \subseteq [V]^{n+1}$ is open in the topology inherited from $V^{n+1}$, and
- transitive, that is $(\forall e \in E)(\forall v \in V \setminus e)(\exists w \in e)((e \setminus \{w\}) \cup \{v\} \in E)$, and
- uncountably chromatic on every open set, that is for every non-empty open subset $U$ of $V$ and every countable family $\mathcal{F}$ of subsets of $U$, either $\bigcup \mathcal{F} \neq U$ or $|\mathcal{F}|^{n+1} \cap E \neq \emptyset$ for some $F \in \mathcal{F}$.

The Geschke forcing notion $\mathcal{P}_G$ for $G$ consists of all closed sets $C \subseteq V$ such that the hypergraph $(C, E \cap [C]^{n+1})$ is uncountably chromatic on every non-empty open subset of $C$. The order of $\mathcal{P}_G$ is the inverse inclusion, i.e., $C \leq_{\mathcal{P}_G} D$ if and only if $D \subseteq C$.

Observation 1.6. (1) The $n$–Sacks forcing notion $\mathcal{D}_n$ has the $\oplus_n$–property.

(2) The uniform $n$–Sacks forcing notion $\mathcal{Q}_n$ and the $n$–Silver forcing notion $\mathcal{S}_n$ have the nice $\ominus_n$–property.

(3) Assume that $G = (V, E)$ is a transitive $(n + 1)$–regular open hypergraph on a Polish space $V$ which is uncountably chromatic on every open set. Then the corresponding Geschke forcing notion $\mathcal{P}_G$ has the $\oplus_n$–property.

Proof. (1)–(3) Straightforward.

(4) This is included in the proof of [2] Lemma 2.8. ∎
The proofs of our theorems resemble arguments from the pre-proper era of iterated forcing and their crucial ingredients are trees of conditions. Let us first recall the relevant notions — in the definition below we follow the pattern that recently has been used in the context of iterations with uncountable supports.

**Definition 1.7** (cf [3] Def. A.1.7, [4] A.3.3, A.3.2). Let $\gamma$ be an ordinal and let $\bar{Q} = \langle \bar{P}_\xi, \bar{Q}_\xi : \xi < \gamma \rangle$ be a CS iteration.

1. Let $m < \omega$ and $w \subseteq \gamma$ be finite. A standard $(w, m)^\gamma$-tree is a pair $T = (T, \text{rk})$ such that
   - $(T, <)$ is a tree with root $\langle \rangle$, $\text{rk}: T \rightarrow w \cup \{\gamma\}$, and
   - if $t \in T$ and $\text{rk}(t) = \varepsilon$, then $t$ is a sequence $\langle (t)_\zeta : \zeta \in w \cap \varepsilon \rangle$, where each $(t)_\zeta$ is a sequence of length $m$.

   We will keep the convention that $\bar{T}^\gamma_x$ is $(T^\gamma_x, \text{rk}^\gamma_x)$.

2. Suppose that $w_0 \subseteq w_1$ are finite subsets of $\gamma$, $m_0 \leq m_1$, and $T_1 = (T_1, \text{rk}_1)$ is a standard $(w_1, m_1)^\gamma$-tree. The projection $\text{proj}^{(w_1, m_1)}_{(w_0, m_0)}(T_1)$ of $T_1$ onto $(w_0, m_0)$ is defined as a standard $(w_0, m_0)^\gamma$-tree $T_0 = \{\langle (t)_\zeta | m_0 : \zeta \in w_0 \cap \text{rk}_1(t) \rangle : t = \langle (t)_\zeta : \zeta \in w_1 \cap \text{rk}_1(t) \rangle \in T_1 \}$.

   The mapping
   $$T_1 \ni \langle (t)_\zeta : \zeta \in w_1 \cap \text{rk}_1(t) \rangle \longmapsto \langle (t)_\zeta | m_0 : \zeta \in w_0 \cap \text{rk}_1(t) \rangle \in T_0$$
   will be denoted $\text{proj}^{(w_1, m_1)}_{(w_0, m_0)}$ too.

3. A standard tree of conditions in $\bar{Q}$ is a system $\bar{p} = \langle p_t : t \in T \rangle$ such that
   - $(T, \text{rk})$ is a standard $(w, m)^\gamma$-tree for some finite set $w \subseteq \gamma$ and an integer $m < \omega$,
   - $p_t \in \mathbb{P}_{\text{rk}(t)}$ for $t \in T$, and
   - if $s, t \in T$, $s \not\prec t$, then $p_s = p_t | \text{rk}(s)$.

4. Let $\bar{p}^0, \bar{p}^1$ be standard trees of conditions in $\bar{Q}$, $\bar{p}^i = \langle p^i_t : t \in T_i \rangle$, where $T_0 = \text{proj}^{(w_1, m_1)}_{(w_0, m_0)}(T_1)$, $w_0 \subseteq w_1 \subseteq \gamma$, $m_0 \leq m_1$. We will write $\bar{p}^0 \leq_{w_1, m_1} \bar{p}^1$ (or just $\bar{p}^0 \leq \bar{p}^1$) whenever for each $t \in T_1$, letting $t' = \text{proj}^{(w_1, m_1)}_{(w_0, m_0)}(t) \in T_0$, we have $p^0_t | \text{rk}_1(t) \leq p^1_{t'}$.

**Lemma 1.8.** Assume that

- $\bar{Q} = \langle \bar{P}_\xi, \bar{Q}_\xi : \xi < \gamma \rangle$ is a CS iteration,
- $(T, \text{rk})$ is a standard $(w, m)^\gamma$-tree, $w \in [\gamma]^{<\omega}$ and $\bar{p} = \langle p_t : t \in T \rangle$ is a standard tree of conditions in $\bar{Q}$,
- $\tau$ is a $\mathbb{P}_\gamma$-name for an element of $\langle w \rangle$ such that $\Vdash_{\mathbb{P}_\gamma} (\forall \alpha < \gamma)(\tau \in V^\mathbb{P}_\alpha)$.

Then there are a tree of conditions $\bar{q} = \langle q_t : t \in T \rangle$ and $N \in \omega$ such that

- $\bar{p} \leq \bar{q}$,
- if $t \in T$, $\text{rk}(t) = \gamma$, then the condition $q_t$ decides $\tau | N$, say $q_t \Vdash_{\mathbb{P}_\gamma} \tau | N = \sigma_t$,
- if $t_0, t_1 \in T$, $\text{rk}(t_0) = \text{rk}(t_1) = \gamma$ and $t_0 \neq t_1$, then $\sigma_{t_0} \neq \sigma_{t_1}$.

**Proof.** For $\alpha < \beta \leq \gamma$, $\bar{P}_{\alpha, \beta}$ is a $\mathbb{P}_\alpha$-name for a forcing notion with universe $P_{\alpha, \beta} = \{p | (\alpha, \beta) : p \in \bar{P}_\beta \}$ such that

if $G_{\alpha} \subseteq \bar{P}_{\alpha}$ is generic over $V$ and $f, g \in P_{\alpha, \beta}$,

then $\Vdash_{\bar{P}_{\alpha} | G_{\alpha}} f \leq_{\bar{P}_{\alpha, \beta}} g$ if and only if $\langle \exists p \in G_{\alpha} \langle p \Vdash f \leq_{\mathbb{P}_{\beta}} p \Vdash g \rangle \rangle$.

Note that $P_{\alpha, \beta}$ is from $\mathbb{V}$, it is only the relation $\leq_{\bar{P}_{\alpha, \beta}}$, which is defined in $\mathbb{V}[G_{\alpha}]$.

Also, $\mathbb{P}_\beta$ is isomorphic with a dense subset of $\bar{P}_{\alpha} * \bar{P}_{\alpha, \beta}$. 

Let \( w = \{ \xi_0, \ldots, \xi_i \} \) be the decreasing enumeration. We may assume that \( 0 \in w \) and thus \( \xi_0 = \max(w) \) and \( \xi_0 = 0 \). Let \( M = |T| + 7, \) \( T^* = \{ t \in T : \text{rk}(t) = \gamma \} \). By induction on \( j \leq i \) we will define \( \mathbb{P}_{\xi_j} \)-names \( \check{q}^j_t, \check{\sigma}^j_{t,k} \) and \( \check{g}^j_{t,k} \) for \( t \in T^* \) and \( k < M \) so that

\[
\begin{align*}
(a)_j & \mid \mathbb{P}_{\varphi_j} \check{q}^j_t \models \check{q}^j_t \in \omega \land \check{\sigma}^j_{t,k} : \check{q}^j_t \rightarrow \omega ; \\
(b)_j & \mid \mathbb{P}_{\varphi_j} \check{g}^j_{t,k} \in \mathbb{P}_{\varphi_j, \gamma} \land \check{p}_t \models |_{\xi_j, \gamma} \check{g}^j_{t,k} \& \text{rk} \check{g}^j_{t,k} \models \check{\sigma}^j_{t,k} ; \\
(c)_j & \text{if } t \in T^* \land k < \ell < M, \text{ then} \\
|_{\mathbb{P}_{\varphi_j}} \check{q}^j_t & \models \check{\sigma}^j_{t,k} \neq \check{\sigma}^j_{\ell,t} ; \\
(d)_j & \text{if } t_0, t_1 \in T^*, \ t = t_0 \cap t_1 \text{ and } \text{rk}(t) > \xi_j, \text{ then} \\
|_{\mathbb{P}_{\varphi_j}} \check{q}^j_{t_0} & \models \text{rk}(t) = \check{g}^j_{t_0} \mid \text{rk}(t) \text{.}
\end{align*}
\]

To start the inductive process suppose that \( G_{\xi_0} \subseteq \mathbb{P}_{\xi_0} \) is generic over \( V \) and work in \( V[G_{\xi_0}] \) for a moment. Note that \( \tau \) may be thought of as a \( \mathbb{P}_{\varphi_0} \)-name for an element of \( \omega \omega \) such that \( |_{\mathbb{P}_{\varphi_0}} \check{\pi} \mid G_{\xi_0} \notin V[G_{\xi_0}] \). Therefore we may find \( n_t \in \omega, \sigma_{t,k} : n_t \rightarrow \omega \) and \( q_{t,k} \in \mathbb{P}_{\varphi_0, \gamma}[G_{\xi_0}] \) (for \( t \in T^* \) and \( k < M \)) such that for each \( t \in T^* \) and \( k, \ell < M, \ell \neq k \):

- \( p_t \models |_{[\xi_0, \gamma]} G_{\xi_0} q_{t,k} \),
- \( q_{t,k} \models |_{[\varphi_0, \gamma]} G_{\xi_0} \tau |_{n_t} = \sigma_{t,k} \),
- \( \sigma_{t,k} \neq \sigma_{t,\ell} \).

Now, let \( \mathbb{P}_{\varphi_j} \check{q}^0_{t,k}, \check{q}^0_t, \check{g}^0_{t,k} \) (for \( t \in T^*, \ k < M \)) be \( \mathbb{P}_{\xi_0} \)-names for objects with properties as those of \( n_t, \sigma_{t,k}, q_{t,k} \) above.

Suppose that \( j < i \) and we have defined \( \mathbb{P}_{\varphi_j} \)-names \( \check{q}^j_t, \check{\sigma}^j_{t,k}, \check{g}^j_{t,k} \) so that \( (a)_j \)–\( (d)_j \) are satisfied. Let \( G_{\xi_{i+1}} \subseteq \mathbb{P}_{\xi_i, +1} \) be generic over \( V \) and work in \( V[G_{\xi_{i+1}}] \) for a moment. For each \( s \in T \) of rank \( \text{rk}(s) = \xi_j \) we may pick a condition \( p_s \models |_{\xi_{i+1}} G_{\xi_{i+1}} \check{q}_s \) stronger than \( p_t \models |_{\xi_{i+1}} G_{\xi_j} \check{q}_t \) and also we may choose \( n_t \in \omega, \sigma_{t,k} : n_t \rightarrow \omega \) and \( q_{t,k} \) (for \( k < M, t \in T^* \)) such that

\[
q_s \models |_{\mathbb{P}_{\varphi_{i+1}, \xi_{i+1}}[G_{\xi_{i+1}}]} (\forall k < M)(\forall t \in T^*) (s < t \Rightarrow [\check{q}^j_t \models n_t \land \check{g}^j_{t,k} = q_{t,k} \& \check{\sigma}^j_{t,k} = \sigma_{t,k}]).
\]

Now let \( \check{q}^{i+1}_{t,k}, \check{\sigma}^{i+1}_{t,k} \) be \( \mathbb{P}_{\varphi_j} \)-names for \( n_t, \sigma_{t,k} \) as above, and let \( \check{g}^{i+1}_{t,k} \) be a \( \mathbb{P}_{\xi_{i+1}} \)-name for \( q_{t,k} \). One easily verifies that demands \( (a)_{j+1} \)–\( (d)_{j+1} \) are satisfied.

Finally note that \( (\xi_i = 0) \ check{q}^i_t, \check{\sigma}^i_{t,k} \) and \( \check{g}^i_{t,k} \) are actually objects in \( V \), not names.

Let \( T^+ = (T^*, \text{rk}^+ \) be a standard \( (w, m + 1)^\gamma \)-tree such that \( \text{proj}_{w, m+1}^w(T^+) = \mathcal{T} \) and

\[
\text{if } t = ((t)_\xi : \xi \in w) \in T^+, \text{rk}^+(t) = \gamma, \text{ then} (t)_\xi(m) < M \text{ and } (\forall \xi \in w \land \xi_0)((t)_\xi(m) = -1).
\]

It should be clear that \( (\check{q}^{i+1}_{t,k} : k < M \land t \in T^*) \) determines a tree of conditions \( \check{q}^* : s \in T^+ \) such that \( \check{p} \leq \check{q}^* \) and \( \check{q}^* = \check{q}^{i+1}_{t,k} \) whenever \( t' \in T^+ \), \( \text{rk}^+(t') = \gamma \), \( t = \text{proj}_{w, m+1}^w(t') \) and \( k = (t')_{\xi_0}(m) \). Let \( N = \max\{|\check{q}^*_t : t \in T^*\} \). Carrying out a procedure similar to that described above we may find a tree of conditions \( \check{q}^* : s \in T^+ \) such that \( \check{q}^* \geq \check{q}^* \) and for some \( \rho_s \in w^\gamma \) (for \( s \in T^+, \text{rk}^+(s) = \gamma \) we have

- \( \rho_s \models |_{\mathbb{P}_s} \tau |_{\mathcal{T}} = \check{N} = \rho_{s_0} \), and
- if \( s_0, s_1 \in T^+, \text{rk}^+(s_0) = \text{rk}^+(s_1) = \gamma, \text{proj}_{w, m+1}^w(s_0) = \text{proj}_{w, m+1}^w(s_1), (s_0)_{\xi_0}(m) \neq (t_1)_{\xi_0}(m) \), then \( \rho_{s_0} \neq \rho_{s_1} \).
Then for each \( t \in T^* \) we may choose \( s_t \in T^+ \) such that \( \proj^{w_m+1}(s_t) = t \) and \( \rho_{s_{t_0}} \neq \rho_{s_{t_1}} \) for distinct \( t_0, t_1 \in T^* \). The choice of \( q \) should be clear now. \( \Box \)

2. \( \oplus_n \)-property and CS iterations

Here we show that CS iterations of forcing notions with \( \oplus_n \)-property result in forcings with the \( n \)-localization property. This result covers examples like the \( n \)-Sacks forcing notion \( \mathbb{D}_n \), or the suitable Geschke forcings \( \mathbb{P}_G \). However, we do not know if the uniform \( n \)-Sacks forcing fits here, so in the next section we will prove a result applicable to a larger family of forcing notions. Still we believe that the proof of 2.1 below is a nice preparation for the arguments in the following section.

**Theorem 2.1.** Let \( \bar{Q} = \langle \mathbb{P}_\xi,\bar{Q}_\xi : \xi < \gamma \rangle \) be a CS iteration such that for every \( \xi < \gamma \),

\[ \forces_{\mathbb{P}_\xi} \text{"} \bar{Q}_\xi \text{ has the } \oplus_n \text{ -property "}. \]

Then

1. \( \mathbb{P}_\gamma = \lim(\bar{Q}) \) has the \( \oplus_n \)-property.
2. \( \mathbb{P}_\gamma = \lim(\bar{Q}) \) has the \( n \)-localization property.

**Proof.** (1) Let \( p \in \mathbb{P}_\gamma \). We are going to describe a strategy \( \text{st} \) for Generic in the game \( \mathcal{D}_n^\gamma(p,\mathbb{P}_\omega) \). This strategy will give Generic, at a stage \( i < \omega \), a standard \( (w_i, i+1)^\gamma \)-tree \( T_i = (T_i, \rk_i) \). These standard trees will satisfy \( T_i = \proj^{(w_i+1,i+2)}(T_{i+1}) \) and \( \{ t \in T_i : \rk_i(t) = \gamma \} \) will correspond to max(\( s_i \)) in the rules of the game. If only we make sure that

\[ (\oplus)_0 \] for each \( t' \in T_i \) with \( \rk_i(t) = \gamma \) we have

\[ 0 < | \{ t \in T_{i+1} : \proj^{(w_i+1,i+2)}_i(t) = t' \} | \leq n, \]

then Generic may easily build trees \( s_i \) and mappings \( \pi_i : \{ t \in T_i : \rk_i(t) = \gamma \} \longrightarrow s_i \) such that

\[ (\oplus)_0^{(a)} \] \( \text{Rng}(\pi_i) = \text{max}(s_i) \subseteq (i+1)n_i \),
\[ (\oplus)_0^{(b)} \] \( (\forall t_0 \in T_i)(\forall t_1 \in T_{i+1})(\pi_i(t_0) \subseteq \pi_i+1(t_1) \iff t_0 = \proj^{(w_i+1,i+2)}(t_1)) \),
\[ (\oplus)_0^{(c)} \] the demands of 1(a) hold.

Later we will even not mention the trees \( s_i \) but we will work directly with \( T_i \).

As we said, in the course of the play the strategy \( \text{st} \) will instruct Generic to choose finite sets \( w_i \subseteq \gamma \) and standard \( (w_i, i+1)^\gamma \)-trees \( T_i \). She will also pick sets \( K_\xi \in [\omega]^{\omega_i} \), conditions \( r_i \in \mathbb{P}_\gamma \) and \( p^\xi, p^\xi, q^\xi, k_i^\xi, s_i^\xi, t_i^\xi, \bar{s}_i^\xi \). All these objects will be constructed so that, assuming \( (T_i, p^\xi, \bar{p}^\xi, \bar{q}^\xi) : i < \omega \) is the result of a play of \( \mathcal{D}_n^\omega(p,\mathbb{P}_\omega) \) in which Generic used \( \text{st} \) and she determined the corresponding side objects, the following conditions are satisfied.

\[ (\oplus)_1 \] \( r_0(0) = p(0), w_i \in [\gamma]^{i+1}, w_0 = \{0\}, w_i \subseteq w_{i+1} \text{ and } \bigcup_{i < \omega} \text{Dom}(r_i) = \bigcup_{i < \omega} w_i. \)
\[ (\oplus)_2 \] If \( j < i < \omega \), then \( (\forall \xi \in w_j+1)(r_j(\xi) = r_i(\xi)) \) and \( p \leq r_j \leq r_i \).
\[ (\oplus)_3 \] If \( \xi \in w_i \), then \( K_\xi \in [\omega]^{\omega_i} \) is known at stage \( i \) of the play and if \( \xi, \zeta \in \bigcup_{i \in w_i} \) are distinct, then \( K_\xi \cap K_\zeta = \emptyset \).
\[ (\oplus)_4 \] For \( \xi \in \bigcup_{i < \omega} w_i \) we have \( i^\xi = \min(\{i : \xi \in w_i\}) \leq \min(K_\xi) \), and \( \text{st}_\xi \) is a \( \mathbb{P}_\xi \)-name for a winning strategy of Generic in \( \mathcal{D}_n^\omega(r_i^\xi(\xi),\bar{Q}_\xi) \) which is nice.
for \( \{ k \in \omega : k + i^*_\xi \in K_\xi \} \) (see (1.2), (1.3)). (So \( st_0 \) is a \( K_0 \)-nice winning strategy of Generic in \( D_0^{n}(r_0(0), \mathcal{Q}_0) \).)

(\( \oplus \))\( _5 \) \( T_i = (T_i, r_k) \) is a standard \((w_i, i + 1)^{\gamma} \)-tree, \( T_i = \text{proj}_{(w_i, i+1)}(T_{i+1}) \).

(\( \oplus \))\( _6 \) \( p^*_{i,t} = (p^*_{i,t}, t \in T_i) \) and \( q^*_{i,t} = (q^*_{i,t}, t \in T_i) \) are standard trees of conditions, \( p^*_{i,t} \leq q^*_{i,t} \leq p^{i+1}_{i,t} \).

(\( \oplus \))\( _7 \) \( \text{Dom}(p^*_{i,t}) = \{(0) \cup \text{Dom}(p) \} \cap \text{rk}(t) \) for each \( t \in T_0 \) and \( p^*_{0,t}(\xi) = p(\xi) \) for \( \xi \in \text{Dom}(p^*_{0,t}) \setminus \{0\}, t \in T_0 \).

(\( \oplus \))\( _8 \) For \( t \in T_{i+1} \) we have \( \text{Dom}(p^*_{i+1})(t) = \text{Dom}(r_k(t)) \cap \text{rk}_{i+1}(t) \) and \( p^*_{i+1}(\xi) = r_k(\xi) \) for \( \xi \in \text{Dom}(p^*_{i+1}) \setminus \text{rk}_{i+1}(t) \).

(\( \oplus \))\( _9 \) If \( \xi \in w_i \), then \( s_{i,\xi} \subseteq \bigcup_{j \leq i+1} \mathcal{I}(n+1) \) is an \( n \)-tree and \( p_{i,\xi} = (p^*_{i,\xi} : \eta \in \text{max}(s_{i,\xi})) \) are \( \mathbb{P}_\xi \)-names for systems of conditions in \( \mathbb{Q}_\xi \) (indexed by \( \text{max}(s_{i,\xi}) \)).

(\( \oplus \))\( _{10} \) For each \( \xi \in \bigcup_{i<\omega} w_i \),

\[ \|p_\xi \|^i \{ s_{i,\xi}, p_{i,\xi}, q_{i,\xi}, i^*_\xi : i^*_\xi \leq i \leq \omega \} \] is a legal play of \( D_0^n(r_\xi(\xi)), \mathbb{Q}_\xi \) in which Generic uses \( st_\xi \).

(\( \oplus \))\( _{11} \) If \( t \in T_i \), \( r_k(t) = \xi < \gamma \) (so \( \xi \in w_i \) and \( i \geq i^*_\xi \)), then

\[ \{ s_\xi : t < s \in T_i \} = \{ \eta : \eta \upharpoonright i^*_\xi \in \mathcal{I}^i \{ \ast \} \ \& \ \exists \nu \in \text{max}(s_{i,\xi}) \} = \{ \eta(\eta \upharpoonright i^*_\xi \upharpoonright \nu) \}

(\( \oplus \))\( _{12} \) If \( t \in T_i \), \( \xi < \text{rk}(t) \), \( \xi \in w_i \) and \( (t_\xi = \langle t^*_\xi \rangle \upharpoonright \nu \) (so \( \nu \in \text{max}(s_{i,\xi}) \)), then

\[ p_{i,t}(\xi) = p^*_{i,t}(\xi) \] and \( q_{i,t}(\xi) = q^*_{i,t}(\xi) \).

(\( \oplus \))\( _{13} \) If \( t_0, t_1 \in T_i \), \( \text{rk}(t_0) = \text{rk}(t_1) \) and \( \xi \in w_i \cap \text{rk}(t_0) \), \( t_0 \upharpoonright \xi = t_1 \upharpoonright \xi \) but \( (t_0) \xi \neq (t_1) \xi \), then

\[ q^*_{i,t_0}(\xi) \] is incompatible.

(\( \oplus \))\( _{14} \) \( \text{Dom}(r_\xi) = \bigcup_{t \in T_i} \text{Dom}(q^*_{i,t}) \cup \text{Dom}(p) \) and if \( t \in T_i \), \( \xi \in \text{Dom}(r_\xi) \cap \text{rk}(t) \setminus w_i \),

\[ q^*_{i,t}(\xi) \] is determined by \( \oplus \))\( _{14} \) (remember that \( r^*_\xi(\xi) \) is determined by \( \oplus \))\( _2 \).

To describe the instructions given by \( st \) at stage \( i < \omega \) of a play of \( D_0^n(p, \mathbb{P}_\gamma) \) let us assume that \( \langle (T_j, \bar{p}_j, \bar{q}_j) : j < i \rangle \) is the result of the play so far and that Generic constructed aside the objects appearing in \( \oplus \))\( _1 \)–\( \oplus \))\( _{14} \) (and they have the respective properties).

For definiteness of our definitions, whenever we say “Generic chooses/picks \( X \) such that” we really mean “Generic takes the \( <^\gamma \)–first \( X \) such that”.

First, Generic uses her favourite bookkeeping device to determine \( w_i \) such that the demands in \( \oplus \))\( _1 \) are satisfied (and that at the end we will have \( \bigcup_{j<\omega} \text{Dom}(r_j) = \bigcup w_j \) and then again she uses the bookkeeping device to determine \( K_\xi \) so that \( \oplus \))\( _3 \)–\( \oplus \))\( _4 \) hold. Note that \( i^*_\xi \) for \( \xi \in w_i \) is defined by \( \oplus \))\( _4 \), also the choice of \( st_\xi \) is determined by \( \oplus \))\( _4 \) (remember that \( r^*_\xi(\xi) \) is determined by \( \oplus \))\( _2 \).
Now $(\oplus)_9 + (\oplus)_{10}$ decide $s_{i,\xi}$ (for $\xi \in w_i$) and since $st_{\xi}$ is (a name for) a nice for $K_{\xi} - i^*_\xi$ strategy, we know that $s_{i,\xi}$ can be easily read from the truth value of “$i + i^*_\xi \in K_{\xi}$.” Plainly $\max(s_{i,\xi}) \leq i^*_\xi n + 1$ and the clauses mentioned before determine $\bar{p}_{i,\xi} = \langle p^0_{i,\xi} : \eta \in \max(s_{i,\xi}) \rangle$. Now the choice of the standard tree $T_i$ is fully described by $(\oplus)_5 + (\oplus)_{11}$ and clearly $(\oplus)_0$ holds then too. For each $t \in T_i$ Generic picks a condition $p^i_{t,t} \in P_{rk_i(t)}$ so that the demands of $(\oplus)_7 + (\oplus)_{12}$ are satisfied. (One may use $(\oplus)_{16}$ to argue that the last demand in $(\oplus)_6$ is satisfied.)

After the above choices are made, Generic (in the play of $\mathcal{D}_n^\omega(p, \mathbb{P}_\gamma)$) puts $T_i$ as her inning and Antigeneric chooses an enumeration $\bar{t} = \{t^i : \ell < k_i\}$ with $\{(t \in T_i : rk_i(t) = \gamma\}$. Now the two players start a subgame of length $k_i = |\{t \in T_i : rk_i(t) = \gamma\}|$. During the subgame Generic will also pick (for temporary use) trees of conditions $q^{\text{tmp}}_{t,\ell} = \langle q^{\text{tmp}}_{t,\ell} : t \in T_i\rangle$ for $\ell \leq k_i$. So, she lets $q^i_{t,0} = p^i_{t,0}$ and she plays $p^i_{t,0} = q^i_{t,0}$ as her first inning in the subgame. Antigeneric answers with $q^i_{t,0} > p^i_{t,0}$ after which Generic picks a tree of conditions $q^{\text{tmp}}_{t,0}$ so that $q^{\text{tmp}}_{t,0} = q^i_{t,0}$ and for each $0 < \ell < k_i$

- if $t^i < t^i_0$, $t^i < t^i_{\ell}$ and $rk_i(t)$ is the largest possible, then
  \[
  q^{\text{tmp}}_{t,0} = q^i_{t,0} \uparrow rk_i(t) \upharpoonright q^{\text{tmp}}_{t,0} \upharpoonright [rk_i(t), \gamma).
  \]

Now, if the players arrived to level $\ell^* < k_i$ of the subgame and $q^{\text{tmp}}_{t,\ell}$ was chosen, then Generic plays $p^i_{t,\ell} = q^{\text{tmp}}_{t,\ell}$. After Antigeneric answered with $q^i_{t,\ell + 1} \geq p^i_{t,\ell}$, Generic builds a tree of conditions $q^{\text{tmp}}_{t,\ell + 1}$ so that $q^{\text{tmp}}_{t,\ell + 1} = q^i_{t,\ell + 1}$ and for each $\ell < k_i$, $\ell \neq \ell^*$

- if $t^i < t^i_{\ell^*}$, $t < t^i_{\ell^*}$ and $rk_i(t)$ is the largest possible, then
  \[
  q^{\text{tmp}}_{t,\ell + 1} = q^i_{t,\ell^*} \uparrow rk_i(t) \upharpoonright q^{\text{tmp}}_{t,\ell + 1} \upharpoonright [rk_i(t), \gamma).
  \]

When the subgame is over Generic lets $q^i_{\xi} = q^{\text{tmp}}_{\xi}$.

Note that the demand of $(\oplus)_{13}$ is satisfied because the strategies $st_{\xi}$ are nice, also the relevant parts of $(\oplus)_{16} + (\oplus)_{18}$ hold. The names $q^i_{\xi,\xi}$ (for $\xi \in w_i$) are chosen so that $q^i_{\xi,\xi} \models \langle \bar{p}_{\xi,\xi} : (\xi, \eta) \in w_i \rangle$ and $\models P_\gamma " q^i_{\xi,\xi} \leq q^i_{\xi,\xi} "$ whenever $t \in T_i$, $\xi \in w_i \cap rk_i(t)$, $\langle (t\xi, i^*_\xi) \upharpoonright \nu = (t\xi, \nu) : \nu \in \max(s_{i,\xi}) \rangle$. Then $(\oplus)_9 + (\oplus)_{12}$ are satisfied. Finally Generic chooses $r_i \in P_\gamma$ essentially by conditions $(\oplus)_{2} + (\oplus)_{14}$ (and our rule of picking “the $<^*\xi$-first such that”).

This completes the description of the side objects constructed by Generic and her innings at stage $i$. We also verified that clauses $(\oplus)_{0} -(\oplus)_{14}$ hold and thus the description of the strategy is complete.

We are going to argue that $st$ is a winning strategy for Generic in $\mathcal{D}_n^\omega(p, \mathbb{P}_\gamma)$. To this end suppose that $\langle \langle T_i, \bar{p}^i, q^i \rangle : i < \omega \rangle$ is the result of a play of $\mathcal{D}_n^\omega(p, \mathbb{P}_\gamma)$ in which Generic used $st$, and the objects constructed at each stage $i < \omega$ are

\[
(\oplus) \ w_i, T_i, \bar{p}^i, q^i, i^*_\xi, r_i, \bar{p}_i, q^i_{\xi,\xi}, K_i, i^*_\xi, st_{\xi}, s_{i,\xi}, p_{i,\xi}, q_{i,\xi} \text{ for } \xi \in w_i,
\]

and they satisfy the requirements $(\oplus)_{0} -(\oplus)_{14}$.

We define a condition $q \in \mathbb{P}_\gamma$ as follows. Let $\text{Dom}(q) = \bigcup_{i < \omega} w_i = \bigcup_{i < \omega} \text{Dom}(r_i)$ and for $\xi \in \text{Dom}(q)$ let $q(\xi)$ be a $\mathbb{P}_\gamma$-name for a condition in $Q_{\xi,\xi}$ such that

\[
\models P_\gamma " q(\xi) \geq r^i_{\xi,\xi} (\xi) \text{ and } q(\xi) \models Q_{\xi,\xi} \forall i \geq i^*_\xi (\exists \nu \in \max(s_{i,\xi})) (q^i_{\xi,\xi} \in \Gamma q_{\xi,\xi}) ".
\]

Clearly $q$ is well defined (remember $(\oplus)_{10}$) and $q \geq p$ (remember $(\oplus)_{1} + (\oplus)_{2}$). Also $q \geq r_i$ for all $i < \omega$. 


We will show that for each $i < \omega$ the family  \( \{ q^+_j : t \in T_i \text{ & } \text{rk}_i(t) = \gamma \} \) is predense above $q$ (and this clearly will imply that Generic won the play). So suppose $q^+ \geq q$, $i < \omega$ and $\omega \cup \{ \gamma \} = \{ \xi_0, \xi_1, \ldots, \xi_i, \xi_{i+1} \}$ (the increasing enumeration, so $\xi_0 = 0$). By induction on $j \leq i$ we choose an increasing sequence  \( \{ q^+_j : j \leq i \} \subseteq P_\gamma \) and we will also define $t^{i+1}_j$.

First, by the choice of $q(0)$ there is $\nu \in \text{max}(s_i, \xi_0)$ such that the conditions $q^+ (0)$ and $q^+_\nu, \xi_0$ are compatible. Let $(t)_0 = \nu$ (this defines $t^{i+1}_0$). Let $q^+_0 \in P_{\xi_i}$ be such that $q^+_0 (0)$ is stronger than both $q^+(0)$ and $q^+_{\nu, \xi_0}$, and let $q^+_0 | (\xi_0, \xi_1) = q^+ | (\xi_0, \xi_1)$. It follows from $(\oplus)_{12}$ and $(\oplus)_{14}$ that $q^+_0$ is stronger than $q^+_\nu, t^{i+1}_0$ and, of course, it is stronger than $q^+ | \xi_0$. Now suppose that $j < i$ and we have defined $t^{i+1}_j \in T_i$ and a condition $q^+_j \in P_{\xi_{i+1}}$, stronger than both $q^+ | \xi_{i+1}$ and $q^+_\nu, t^{i+1}_j$. Necessarily \( q^+_j \models_{\nu, \xi_{i+1}} \langle \exists \nu \in \max(s_i, \xi_{i+1}) \rangle (q^+_\nu, \xi_{i+1}, q^+ | \xi_{i+1}) \) are compatible.

so we may choose $\nu \in \max(s_i, \xi_{i+1})$ and a condition $q_{i+1} \in P_{\xi_{i+1}}$ stronger than $q^+_j$ such that

\[ q^+_j \models_{\nu, \xi_{i+1}} \langle \exists \nu \in \max(s_i, \xi_{i+1}) \rangle (q^+_\nu, \xi_{i+1}, q^+ | \xi_{i+1}) \] are compatible.

Let $(t)_{\xi_{i+1}} = (\ldots) \nu$ (thus $t^{i+1}_j \in T_i$ has been defined) and let $q^+_j \in P_{\xi_{i+2}}$ be such that $q^+_j \models_{t^{i+1}_j} \xi_{i+1} = q^+_j$

\[ q^+_j, t^{i+1}_j \models_{\nu, \xi_{i+1}} q^+_\nu, \xi_{i+1}, q^+ | \xi_{i+1} \] are compatible

and $q^+_j, t^{i+1}_j \models_{\nu, \xi_{i+1}} q^+ | \xi_{i+1}$ and $q^+ | \xi_{i+1}$. Then by $(\oplus)_{12}$ and $(\oplus)_{14}$ the condition $q^+_j$ is stronger than $q^+_\nu, t^{i+1}_j$ and $q^+_j \models_{t^{i+1}_j} \xi_{i+1}$.

Finally look at $t = t^{i+1}_j$ and $q^+_j$.

(2) Since we do not know if the $\circ_n$-property implies the $n$-localization property, we cannot just say that the statement in (2) follows from (1). However, the reason for the weaker $\circ_n$ in the conclusion of Lemma 2.4 (and not $\circ_n$) is that in our description of the strategy $\text{st}$, we have to make sure that the conditions played by Antigeneric form a tree of conditions.

So to show that $P_\gamma$ has the $n$-localization property we use [1,8] and the procedure described in the proof of Lemma 2.4. Suppose that $\tau$ is a $P_\gamma$-name for an element of $\omega^\omega$; we may assume that $\models_{P_\gamma} (\forall \alpha < \gamma)(\tau \not\in V^{P_\gamma})$. Let $p \in P_\gamma$. Construct a sequence

\[ \langle w_i, T_i, \bar{q}_i, \bar{r}_i, (i^*_\xi, \rho, K_\xi : \xi \in \omega), \bar{o}, \bar{m} : i < \omega \rangle \]

such that conditions $(\oplus)_{0} - (\oplus)_{17}$ and $(\oplus)_{0} - (\oplus)_{14}$ are satisfied and

\[ (\oplus)_{15} \bar{m} = \langle m_i : i < \omega \rangle \subseteq \omega, 0 \leq m_0 < m_i < m_{i+1} \text{ for } i < \omega, \]

\[ (\oplus)_{16} \bar{o}^s = \langle \sigma_i^s : t \in T_i \text{ & } \text{rk}_i(t) = \gamma \rangle \subseteq \{ m_i, m_{i+1} \} \omega \text{ and if } t, t' \in T_i, t \neq t', \text{ rk}_i(t) = \text{rk}_i(t') = \gamma, \text{ then } \sigma_i^s \neq \sigma_i^{s'}, \text{ and} \]

\[ (\oplus)_{17} \bar{q}_{i,t} \models_{P_\gamma} \sigma_i^s < \tau \text{ for } t \in T_i, \text{rk}_i(t) = \gamma. \]

Then pick $q \in P_\gamma$ stronger than $p$ and such that for each $i < \omega$ the family $\{ q_{i,t}^i : t \in T_i \text{ & } \text{rk}_i(t) = \gamma \}$ is predense above $q$ (this is done exactly as in part (1)). Let $S \subseteq \omega^\omega$ be a tree such that

\[ [S] = \{ f \in \omega^\omega : (\forall i < \omega)(\exists t \in T_i)(\text{rk}_i(t) = \gamma \& f(m_i, m_{i+1}) = \sigma_i^s) \}. \]

Then $S$ is an $n$-ary tree and $q \models_{P_\gamma} \tau \in [S]$. \hfill \Box
3. $\Diamond_n$-property and CS iterations

The result of the previous section is not applicable to $Q_n, S_n$ as these forcing have nice $\Diamond_n$-property only. An iteration theorem suitable for that property is presented below. It is not sufficient for claiming the $n$–localization property, so later we formulate yet another property and we argue that it implies the $n$–localization of the limits of CS iterations.

**Theorem 3.1.** If $\bar{Q} = \langle P_\xi, Q_\xi : \xi < \gamma \rangle$ is a CS iteration such that for every $\xi < \gamma$,

$$\models_{P_\xi} " Q_\xi has the nice $\Diamond_n$-property ".

then $P_\gamma = \text{lim}(\bar{Q})$ has the $\Diamond_n$-property.

**Proof.** Let $p \in P_\gamma$. We are going to describe a strategy $st$ for Generic in the game $G_n^\gamma(p, P_\gamma)$. As in the proof of Theorem 2.1 the strategy $st$ will give Generic, at a stage $i < \omega$, a standard $(w_i, (i + 1))\gamma$–tree $T_i = (T_i, r_i)$ such that $T_i = \text{proj}_{(w_i, i+1)}(T_{i+1})$ and

(\circ) for each $t' \in T_i$ with $r_i(t) = \gamma$ we have

$$0 < |\{ t \in T_{i+1} : \text{proj}_{(w_i, i+1)}(t) = t' \}| \leq n.$$

Generic will also pick sets $K_\xi \in [\omega]^\omega$, conditions $r_i \in P_\gamma$ and $k_i, i^*_\xi, st_\xi, s_i, \bar{p}_i, \bar{q}_i, \bar{g}_i, \tilde{p}_i$. All these objects will be constructed so that, assuming $\langle (T_i, p^i, q^i) : i < \omega \rangle$ is the result of a play in which Generic used $st$ and she determined the corresponding side objects, the following demands are satisfied.

(\circ)_1 $r_0(0) = p(0)$, $w_1 \in [\gamma]^{i+1}$, $w_0 = \{0\}$, $w_i \subseteq w_{i+1}$ and $\bigcup_{i<\omega} \text{Dom}(r_i) = \bigcup_{i<\omega} w_i$.

(\circ)_2 If $j < i < \omega$, then $\forall \xi \in w_{j+1} \langle r_j(\xi) = r_i(\xi) \rangle$ and $p \leq r_j \leq r_i$.

(\circ)_3 If $\xi \in w_i$, then $K_\xi \in [\omega]^\omega$ is known at stage $i$ of the play and if $\xi, \zeta \in \bigcup_{i<\omega} w_i$ are distinct, then $K_\xi \cap K_\zeta = \emptyset$.

(\circ)_4 For $\xi \in \bigcup_{i<\omega} w_i$ we have $i^*_\xi = \min(\{i : \xi \in w_i\}) \leq \min(K_\xi)$, and $st_\xi$ is a $P_\xi$-name for a winning strategy of Generic in $G_n^\gamma(r_i^\zeta, Q_\xi)$ which is nice for $\{k \in \omega : k + i^*_\xi \in K_\xi\}$ (see 1.2). (So $st_0$ is a $K_0$-nice winning strategy of Generic in $G_n^\gamma(r_0(0), Q_0)$.)

(\circ)_5 If $\xi \in w_i$, then $s_i, \xi \subseteq \bigcup_{j \leq i+1-i^*_\xi} j(n + 1)$ is an $n$–tree and $\bar{p}_i, \xi = \langle p_i^n, \xi : n \in \text{max}(s_i, \xi) \rangle$ are $P_\xi$-names for systems of conditions in $Q_\xi$ (indexed by $\text{max}(s_i, \xi)$).

(\circ)_6 For each $\xi \in \bigcup_{i<\omega} w_i$,

$$\models_{P_\xi} " (s_i, \xi, \bar{p}_i, \xi, \bar{q}_i, \xi : i^*_\xi \leq i < \omega) is a legal play of $G_n^\gamma(r_i^\zeta, Q_\xi)$ in which Generic uses $st_\xi$ and the orders of $\max(s_i, \xi)$ chosen by Antigeneric are given by $<^\gamma$."

(\circ)_7 $T_i = (T_i, r_i)$ is a standard $(w_i, i + 1)\gamma$–tree, $T_i = \text{proj}_{(w_i, i+1)}(T_{i+1})$.

(\circ)_8 If $t \in T_i$, $r_i(t) = \xi < \gamma$ (so $\xi \in w_i$ and $i \geq i^*_\xi$), then

$$\{ (s) : t < s \in T_i \} = \{ \eta : \eta i^*_\xi \in i^*_\xi \& (\exists \nu \in \text{max}(s_i, \xi))(\eta = (\eta i^*_\xi) \wedge \nu) \}.$$

(\circ)_9 $k_i = |\{ t \in T_i : r_i(t) = \gamma \}|.$
such that” we really mean “Generic takes the conditions \( p_{t_i}^j(\xi), p_{t_m}^j(\xi) \) are incompatible”.

\((\circ)_{11}\) If \( t \in T_i \), \( \text{rk}_i(t) = \gamma, \xi \in w_i \) and \((t)_{\xi} = (t)_{\xi}[i_{\xi}^*, \eta, \eta \in \text{max}(s_i, \xi)]\), then

\[ p_{t_i}^j(\xi) \leq p_{t_m}^j(\xi) \quad \text{and} \quad q_{t_i}^j(\xi) \leq q_{t_m}^j(\xi). \]

\((\circ)_{12}\) Dom\((r_i) = \bigcup_{t \in T_i} \text{Dom}(q_i) \cup \text{Dom}(p)\) and if \( t \in T_i \), \( \text{rk}_i(t) = \gamma, \xi \in \text{Dom}(r_i) \setminus w_i \),

then \( q_{t_i}^j(\xi) \leq r_i(\xi) \).

To describe the instructions given by \( ST \) at stage \( i < \omega \) of a play of \( \mathcal{O}_n^\omega(p, \mathbb{P}_n) \), let us assume that \( \langle (T_i, \overline{p}, \overline{q}) : j < i \rangle \) is the result of the play so far and that Generic constructed aside the objects appearing in \((\circ)_{11} - (\circ)_{12}\) (and they have the respective properties).

For definiteness of our definitions, whenever we say “Generic chooses/picks \( X \) such that” we really mean “Generic takes the <\_\_\_ first \( X \) such that”.

First, Generic uses her favourite bookkeeping device to determine \( w_i \) and \( K_\xi \) so that \((\circ)_1 + (\circ)_3 + (\circ)_4\) hold. Note that \((\circ)_4\) determines \( i_{\xi}^* \) and \( ST_\xi \) for \( \xi \in w_i \) (remember that \( r_{i_i}(\xi) \) is given by \((\circ)_2\)). Now \((\circ)_5 + (\circ)_6\) and the truth value of “\( i + i_{\xi}^* \in K_\xi \)” decide \( s_i, \xi \) (for \( \xi \in w_i \)), remember that \( ST_\xi \) is (a name for) a strategy which is nice for \( K_\xi - i_{\xi}^* \). Plainly \( \text{max}(s_i, \xi) \subseteq i + i_{\xi}^* + 1 \). The choice of the standard tree \( T_i \) is fully described by \((\circ)_7 + (\circ)_8\) and clearly \((\circ)_9\) holds then too. Also \( k_i \) is given by \((\circ)_9\).

Let \( \{ \zeta_j : j < j^* \} \) be the increasing enumeration of \( \{ \xi \in w_i : |\text{max}(s_i, \xi)| > 1 \} \) (so \( j^* \leq i + 1 \) and we may assume that \( j^* \neq 0 \)). We will think of \( \text{max}(s_i, \xi) \) (for \( \xi \in w_i \)) as linearly ordered by <\_\_\_ (restricted suitably). This linear order determines a list of \( \text{max}(s_i, \xi) \) which will be considered as an inning of Antigeneric in answer to the choice of \( s_i, \xi \) by Generic in \( \mathcal{O}_n^\omega(r_{i_i}(\xi), Q_\xi) \). We may identify \( \{ (t)_{\xi} : t \in T_i \text{ and } \text{rk}_i(t) = \xi \} \) with \( \text{max}(s_i, \xi) \) by the mapping \( (t)_{\xi} \to (t)_{\xi}[i_{\xi}^*, i] \), so in particular the linear order of \( \text{max}(s_i, \xi) \) determines a linear ordering of \( \{ (t)_{\xi} : t \in T_i \text{ and } \text{rk}_i(t) > \xi \} \) (for \( \xi \in w_i \)). Generic takes the lexicographic product of these orderings for all coordinates \( \xi \in w_i \) and she lets \( t_{m}^i : m < k_i \) be the increasing (in this lexicographic order) enumeration of \( \{ t \in T_i : \text{rk}_i(t) = \gamma \} \). Then for each \( t \in T_i \) with \( \text{rk}_i(t) = \gamma \) and \( \xi \in w_i \) we have

\[(\circ)_{13}\] for some \( m_0 < m_1 \leq k_i \),

\[ \{ t' \in T_i : \text{rk}_i(t') = \gamma \& t'_{|\xi} = t_{|\xi} \} = \{ t_{m}^i : m_0 \leq m < m_1 \}. \]

The interval \( \langle m_0, m_1 \rangle \) as in \((\circ)_{13}\) will be called the neighbourhood of \( t \) at \( \xi \). Note that if \( \xi = \zeta_j \) for some \( j < j^* \), then \( m_1 > m_0 + 1 \), otherwise \( m_1 = m_0 + 1 \). For \( j < j^* \) let \( m_j \) be such that \( \langle 0, m_j \rangle \) is the neighbourhood of \( t_{m_0}^i \) at \( \zeta \) (so \( m_{j-1} < m_{j-2} < \ldots < m_0 = k_i \)).

Now the two players start a subgame of length \( k_i \).

For \( j < m_{j-1} \) Generic proceeds in the subgame as follows. First \( p_{t_i}^j \in \mathbb{P}_\gamma \) is such that

\[ \text{Dom}(p_{t_i}^j) = \begin{cases} w_0 \cup \text{Dom}(p) & \text{if } i = 0, \\ w_i \cup \text{Dom}(r_{i-1}) & \text{if } i > 0, \end{cases} \]
and if $\xi \in \text{Dom}(p_{t_0}^i) \setminus w_i$ then

$$p_{t_0}^i(\xi) = \begin{cases} 
p(\xi) & \text{if } i = 0, \\
 r_{i-1}(\xi) & \text{if } i > 0.
\end{cases}$$

For $\xi \in w_i$, $p_{t_0}^i(\xi)$ is a $\mathbb{P}_\xi$–name for an element of $\mathbb{Q}_\xi$ such that

$$\Vdash_{\mathbb{P}_\xi} \text{“} p_{t_0}^i(\xi) \text{ is the condition given to Generic by } s_{t\xi} \text{ in the subgame of }$$

$$\mathcal{O}_n(r_{i\xi}^*(\xi), \mathbb{Q}_\xi) \text{ after } \langle s_{j\xi}, b_{j\xi}, \bar{g}_{j\xi} : i_j^* \leq j < i \rangle \text{ was played and }$$

Antigeneric picked the $\prec_{\lambda^*}$–increasing enumeration of max$(s_{i\xi})$”

A straightforward induction (using $(\circ)_1^2 + (\circ)_2^1$) shows that

$$(\circ)^{0}_{i} \text{ if } i > 0, \quad t' = \text{proj}^{(w_{i-1}, i)}(t_{i-1}^0), \text{ then } p_{t_{i-1}^0}^i \geq q_{i+1}^0 \text{ (and } p_{t_{i-1}^0}^0 \geq p).$$

The condition $p_{t_{i-1}^0}^i$ is the first inning of Generic in the subgame, after which Antigeneric answers with $q_{i+1}^0 \geq p_{t_{i-1}^0}^i$.

Now suppose that the two players arrived to a step $0 < m < m_{j-1}$ of the subgame. The inning of Generic now is defined similarly to that at stage 0: $p_{t_{m}^i}^i \in \mathbb{P}_\gamma$ is such that

$$\text{Dom}(p_{t_{m}^i}^i) = \begin{cases} 
w_0 \cup \text{Dom}(p) & \text{if } i = 0, \\
w_i \cup \text{Dom}(r_{i-1}) \cup (\text{Dom}(q_{t_{m-1}^i}^i) \cap \zeta_{j-1}) & \text{if } i > 0,
\end{cases}$$

and $p_{t_{m}^i}^i \mid \zeta_{j-1} = q_{t_{m-1}^i}^i \mid \zeta_{j-1}$, and $p_{t_{m}^i}^i(\xi) = p_{t_{0}^i}^i(\xi)$ for $\xi \in \text{Dom}(p_{t_{m-1}^i}^i) \setminus (\zeta_{j-1} + 1)$, and finally $p_{t_{m-1}^i}^i(\zeta_{j-1})$ is a $\mathbb{P}_{\zeta_{j-1}}$–name for an element of $\mathbb{Q}_{\zeta_{j-1}}$ such that

$$p_{t_{m-1}^i}^i(\zeta_{j-1}) \mid \mathbb{P}_{\zeta_{j-1}} \text{ “} p_{t_{m}^i}^i(\zeta_{j-1}) \text{ is the inning of Generic given by } s_{t\zeta_{j-1}} \text{ in the subgame of level } i \text{ of } \mathcal{O}_n(r_{i\zeta_{j-1}}^*(\zeta_{j-1}), \mathbb{Q}_{\zeta_{j-1}}) \text{ after the two players played }$$

$$\langle p_{t_{1}^0}^i(\zeta_{j-1}), q_{t_{0}^i}^i(\zeta_{j-1}) \rangle, \ldots, \langle p_{t_{m-1}^i}^i(\zeta_{j-1}), q_{t_{m-2}^i}^i(\zeta_{j-1}) \rangle \text{ as the conditions attached to } (t_{0}^i)_{\zeta_{j-1}}, \ldots, (t_{m-1}^i)_{\zeta_{j-1}}, \text{ respectively”}.$$

Again, one may verify by induction on $\xi \in \text{Dom}(p_{t_{m}^i}^i)$ that

$$(\circ)^{1}_{i} \text{ if } i > 0, \quad t' = \text{proj}^{(w_{i-1}, i)}(t_{m}^i), \text{ then } p_{t_{m}^i}^i \geq q_{i+1}^0 \text{ (and } p_{t_{0}^i}^0 \geq p).$$

Now, $p_{t_{m}^i}^i$ is Generic’s inning at this stage of the subgame, and after this Antigeneric answers with $q_{i+1}^0 \geq p_{t_{m}^i}^i$.

Thus we have described how Generic plays in the first $m_{j-1}$ steps of the subgame — let us call this procedure $\text{proc}_{j-1}(0, m_{j-1}, t_{0}^i, p_{t_{0}^i}^i)$.

Suppose that $m_{j-1} < k_i$ and let $m' > m_{j-1}$ be such that $|m_{j-1}, m'|$ is the neighbourhood of $t_{m_{j-1}}^i$ at $\zeta_{j-1}$ (so $j' > 1$ and so $i > 0$). Let $p_{t_{m_{j-1}}^i}^i \in \mathbb{P}_\gamma$ be such that

$$\text{Dom}(p_{t_{m_{j-1}}^i}^i) = w_i \cup \text{Dom}(r_{i-1}) \cup (\text{Dom}(q_{t_{m_{j-1}}^i}^i) \cap \zeta_{j-2})$$

and $p_{t_{m_{j-1}}^i}^i \mid \zeta_{j-2} = q_{t_{m_{j-1}}^i}^i \mid \zeta_{j-2}$, and $p_{t_{m_{j-1}}^i}^i(\zeta_{j-2})$ is a $\mathbb{P}_{\zeta_{j-2}}$–name for an element of $\mathbb{Q}_{\zeta_{j-2}}$ such that

$$p_{t_{m_{j-1}}^i}^i(\zeta_{j-2}) \mid \mathbb{P}_{\zeta_{j-2}} \text{ “} p_{t_{m_{j-1}}^i}^i(\zeta_{j-2}) \text{ is the inning of Generic given by } s_{t\zeta_{j-2}} \text{ in the subgame of level } i \text{ of } \mathcal{O}_n(r_{i\zeta_{j-2}}^*(\zeta_{j-2}), \mathbb{Q}_{\zeta_{j-2}}) \text{ after the two players played }$$

$$\langle p_{t_{1}^0}^i(\zeta_{j-2}), q_{t_{0}^i}^i(\zeta_{j-2}) \rangle, \ldots, \langle p_{t_{m_{j-1}}^i}^i(\zeta_{j-2}), q_{t_{m_{j-1}-1}^i}^i(\zeta_{j-2}) \rangle \text{ as the conditions attached to } (t_{0}^i)_{\zeta_{j-2}}, \ldots, (t_{m_{j-1}-1}^i)_{\zeta_{j-2}}, \text{ respectively”}.$$
and finally $p^i_{m_{j',-1}}(\xi) = p^i_{t_0}(\xi)$ for $\xi \in \text{Dom}(p^i_{m_{j',-1}}) \setminus (\zeta_{j'-2} + 1)$. Again, a straightforward induction shows that $(\circ)^{m_{j'-1}}_{t'}$ if $t' = \text{proj}^{(w_i, i+1)}(t^i_{m_{j,-1}})$, then $p^i_{m_{j,-1}} \geq q_{i'}^{i-1}$.

The condition $p^i_{m_{j,-1}}$ is played by Generic at stage $m_{j,-1}$ and from now till step $m'$ she plays applying procedure $p_{\zeta_{j-2}}(m_{j,-1}, m', t^i_{m_{j,-1}}, p^i_{m_{j,-1}})$.

Then, if only $m' < m_{j,-2}$, Generic takes $m''$ such that $[m', m'']$ is the neighbourhood of $t^i_{m'}$ at $\zeta_{j-1}$. She defines $p^i_{m'} \in P_\gamma$ like $p^i_{m_{j,-1}}$, so

$$\text{Dom}(p^i_{m'}) = w_i \cup \text{Dom}(r_{i-1}) \cup \left( \text{Dom}(q^i_{m_{j,-1}}) \cap \zeta_{j-2} \right)$$

and $p^i_{m'}(\zeta_{j-2}) = q^i_{m_{j,-1}}(\zeta_{j-2})$, and $p^i_{m'}(\zeta_{j-2})$ is a $P_{\zeta_{j-2}}$-name for an element of $Q_{\zeta_{j-2}}$ such that $p^i_{m'}(\zeta_{j-2}) \not\in P_{\zeta_{j-2}}^{\zeta_{j-2}}$.

Then, if $t' = \text{proj}^{(w_i, i+1)}(t^i_{m'})$, then $p^i_{m'} \geq q^i_{i'-1}$.

The condition $p^i_{m'}$ is played by Generic at stage $m'$ and from now till step $m''$ she plays applying procedure $p_{\zeta_{j-2}}(m', m'', t^i_{m'}, p^i_{m'})$. Following this pattern until the players get to level $m_{j,-2}$ (i.e., at the steps from 0 to $m_{j,-2} - 1$) results in defining the procedure $p_{\zeta_{j-2}}(0, m_{j,-2}, t^i_0, p^i_{t_0})$.

Suppose we have defined the procedure $p_{\zeta_{j-2}}$, $0 < j < j^*$, and we are going to define $p_{\zeta_{j-2}}(0, m_{j-1}, t^i_0, p^i_{t_0})$ following the pattern presented above. We pick $m' > m_j$ such that $[m_j, m']$ is the neighbourhood of $t^i_{m_j}$ at $\zeta_j$ and we define $p^i_{m_j} \in P_\gamma$ so that

- $\text{Dom}(p^i_{m_j}) = w_i \cup \text{Dom}(r_{i-1}) \cup \left( \text{Dom}(q^i_{m_{j,-1}}) \cap \zeta_{j-1} \right)$,
- $p^i_{m_j}(\zeta_{j-1}) = q^i_{m_{j,-1}}(\zeta_{j-1})$,
- $p^i_{m_j}(\zeta_{j-1})$ forces in $P_{\zeta_{j-1}}$ that $p^i_{m_j}(\zeta_{j-1})$ is the inning of Generic given by $s_{\zeta_{j-1}}$ in the subgame of level $i$ of $\text{Dom}(r^i_{m_{j-1}}(\zeta_{j-1}))$ after the two players played $(p^i_{t_0}(\zeta_{j-1}), q^i_{m_{j,-1}}(\zeta_{j-1}))$ as the conditions attached to $(t^i_0, (\zeta_{j-1}))$.

Plainly, $(\ast)^{m_{j}}_{t_0}$ holds and the condition $p^i_{m_j}$ is played by Generic as her inning associated with $t^i_{m_j}$. Antigeneric answers with $q^i_{m_j} \geq p^i_{m_j}$, and then Generic follows procedure $p_{\zeta_{j-1}}(m_j, m', t^i_{m_j}, p^i_{m_j})$ to determine her innings for $m \in (m_j, m')$. If
m' < m_{j-1}, Generic picks m'' > m' such that [m', m''] is the neighbourhood of t'_{m''} at ζ_i and she defines p^i_{t'_{m''}} ∈ P_γ like before, and then she follows the procedure proc_j(m', m'', t'_{m''}, p^i_{t'_{m''}}), and so on. After arriving to m_{j-1} + 1, Generic defined the procedure proc_{j-1}(0, m_{j-1}, t'_0, p^i_{t'_0}).

Finally, the procedure proc_0(0, m_0, t'_0, p^i_{t'_0}) describes the instructions given to Generic by our strategy st in the subgame of level i of D\(\subset_n^\infty\)(p, P_γ). Now, for ξ ∈ w_i, Generic defines \(\tilde{p}_{i,ξ} = (q_{i,ξ}^0 : η ∈ max(s_i,ξ))\) and \(\tilde{q}_{i,ξ} = (q_{i,ξ}^0 : η ∈ max(s_i,ξ))\) so that \(\tilde{p}_{i,ξ}, \tilde{q}_{i,ξ}\) are P_γ names for elements of Q_ξ and the demands in \((\odot)_6 + (\odot)_1\) are satisfied. It should be clear that the choice of \(\tilde{p}_{i,ξ}, \tilde{q}_{i,ξ}\) is possible — note that by niceness of st_ξ (and the last requirement in \([m_2 + 1]\)) we easily justify that \((\odot)_10\) holds. To finish stage i, Generic picks r_i ∈ P_γ essentially by conditions \((\odot)_2\) and \((\odot)_12\).

We are going to argue that st is a winning strategy for Generic in D\(\subset_n^\infty\)(p, P_γ). To this end suppose that \((T_i, \tilde{p}^i, \tilde{q}^i) : i < ω\) is the result of a play of D\(\subset_n^\infty\)(p, P_γ) in which Generic used st, and the side objects constructed at each stage i < ω are

\(\Box\) \(w_i, T_i, r_i, \tilde{p}^i, \tilde{q}^i, k_i, K_i, i^*_i, x_j, s_i, ξ, \tilde{p}_{i,ξ}, \tilde{q}_{i,ξ}\) for ξ ∈ w_i,

and they satisfy the requirements \((\odot)_0 - (\odot)_12\).

We define a condition q ∈ P_γ as follows. Let \(Dom(q) = \bigcup_{i<ω} w_i = \bigcup_{i<ω} Dom(r_i)\) and for ξ ∈ Dom(q) let q(ξ) be a P_γ name for a condition in Q_ξ such that

\((\odot)_{10}\) \(\vdash_{P_γ} "q(ξ) \models r_i^+|ξ(ξ)\) and \(q(ξ) \vdash_{Q_ξ} \langle (\forall η \in max(s_i,ξ)\rangle (\exists ν \in max(s_i,ξ)) \langle ψ_{ξ,ν} | r_i^+|ξ(ξ)\) • • • .

Clearly q is well defined (remember \((\odot)_6\) and q ≥ p (remember \((\odot)_1 + (\odot)_2\)). Also q ≥ r_i for all i < ω.

We will show that for each i < ω the family \(q^i_j : t ∈ T_i & rk_i(t) = γ\) is predence above q (and this clearly implies that Generic won the play). So suppose \(q^+ ≥ q, i < ω\) and \(w_i ∩ \{γ\} = \{ξ_0, ξ_1, ..., ξ_i, ξ_{i+1}\}\) (the increasing enumeration, so ξ_0 = 0, ξ_{i+1} = γ). By induction on j ≤ i we choose an increasing sequence \(q^i_j : j ≤ i \in P_γ\) and we also define t|ξ_j + 1. First, by the choice of q(0) there is ν ∈ max{(s_0, ξ_0)} such that the conditions \(q^+|0\) and \(q^∗_{ξ,ν} = q^∗_{ξ,ν} | 0\) are compatible. Let \((t|0) = ν\) (this defines t|ξ_1). Let \(q^0_{ξ,ν} ∈ P_ξ\) be such that \(q^0_{ξ,ν}|0\) is stronger than both \(q^+|0\) and \(q^∗_{ξ,ν} | 0\). Let \(q^0_ξ | 0\) be such that \(q^0_ξ | 0\) is stronger than both \(q^+|0\) and \(q^∗_{ξ,ν} | 0\).

Suppose that j < i and we have defined t|ξ_{j+1} ∈ T_i and a condition \(q^+_{ξ,j+1}\) such that \(q^+_{ξ,j+1} ≥ q^+|ξ_{j+1}\)

\((\odot)_{16}\) \(\langle \forall t' ∈ T_i \rangle (rk_i(t') = γ & t'|ξ_{j+1} = t|ξ_{j+1}) ⇒ q^+_{ξ,j+1}\) ≥ \(q^+_{ξ,j+1}\).

Necesarily, by \((\odot)_{15}\) \(\vdash_{P_{ξ,j+1}} "(\exists ν ∈ max(s_i,ξ_{j+1})) (q^∗_{ξ,ν,ξ_{j+1}} , q^+|ξ_{j+1}) are compatible \) • • •

so we may choose ν ∈ max(s_i,ξ_{j+1}) and a condition \(q_{j+1} ∈ P_{ξ,j+1}\) stronger than \(q^+_{ξ,j+1}\) such that

\(q^+_{ξ,j+1} | 0\) \(\vdash_{P_{ξ,j+1}} "q^∗_{ξ,ν,ξ_{j+1} , q^+|ξ_{j+1}} are compatible \) • • •

Let \((t|ξ_{j+1} = (s ... s) \vdash_ν (thus t|ξ_{j+1} + 1\) has been defined) and let \(q^+_{ξ,j+1} ∈ P_{ξ,j+2}\) be such that \(q^+_{ξ,j+1}|ξ_{j+1} = q_{j+1}\).
Definition 3.2. Suppose that $\mathbb{P}$ is a forcing notion with $\odot_n$-property and $\mathbf{st} = \langle \mathbf{st}_p : p \in \mathbb{P} \rangle$, where $\mathbf{st}_p$ is a winning strategy for Generic in $\mathcal{D}_n^\odot (p, \mathbb{P})$. Such $\mathbf{st}$ will be called an $\odot_n$-strategy system for $\mathbb{P}$.

1. We say that a finite set $Q$ of conditions is an $\mathbf{st}$-front above $p$ provided that there is a partial play $\langle s_j, \bar{p}^j, \bar{q}^j : j \leq i \rangle$ of $\mathcal{D}_n^\odot (p, \mathbb{P})$ in which Generic uses $\mathbf{st}_p$ and
   - if $\max(s_i) = \langle \eta^j_k : k < K \rangle$ is the enumeration played by Antigeneric at stage $i$ after Generic put $s_i$, then $Q = \{ \eta^j_k : k < K \}$.

2. For a condition $p \in \mathbb{P}$ we define a game $\mathcal{D}_{\odot_n}^\mathbf{st} (p, \mathbb{P})$ as follows. A play of $\mathcal{D}_{\odot_n}^\mathbf{st} (p, \mathbb{P})$ lasts $\omega$ moves and in the course of the play a sequence
   \[(\overset{2}{\mathcal{E}}) \langle s_i, \bar{\eta}^i, \bar{p}^i, \bar{q}^i : i < \omega \rangle\]
   is constructed. At a stage $i < \omega$ of the play,
   - first Generic chooses a finite $n$-ary tree $s_i$ such that the demand (a) of $\mathbb{L}_1(1)$ holds, and then
   - Antigeneric picks an enumeration $\bar{\eta}^i = \langle \eta^j_\ell : \ell < k_i \rangle$ of $\max(s_i)$.

   Now the two players start a subgame of length $k_i$ and they choose successive terms of a sequence $\langle p^{i, j}_{\eta^j_\ell}, Q^{i, j}_{\eta^j_\ell} : \ell < k_i \rangle$. At a stage $\ell < k_i$ of the subgame,
   - first Generic picks a condition $p^{i, j}_{\eta^j_\ell} \in \mathbb{P}$ such that
     \[j < i, \nu \in \max(s_j) \text{ and } \nu < \eta^j_\ell, \text{ then } q^{i, j}_{\eta^j_\ell} \leq p^{i, j}_{\eta^j_\ell}, \text{ and } p \leq p^{i, j}_{\eta^j_\ell}, \]
   - and then Antigeneric picks an $\mathbf{st}$-front $Q^{i, j}_{\eta^j_\ell}$ above $p^{i, j}_{\eta^j_\ell}$.

   After the subgame is completed,
   - Antigeneric chooses $\bar{q}^i = \langle q^{i, \eta} : \eta \in \max(s_i) \rangle$ so that $q^{i, \eta} \in Q^{i, \eta}_\eta$ for $\eta \in \max(s_i)$.

   Finally, Generic wins a play $(\overset{2}{\mathcal{E}})$ if and only if
   - (a) there is a condition $q \geq p$ such that for every $i < \omega$ the family $\{ q^{i, \eta} : \eta \in \max(s_i) \}$ is predense above $q$.

3. Similarly to $\mathbb{L}_2(1)$ we define when a strategy $\mathbf{st}$ of Generic in $\mathcal{D}_{\odot_n}^\mathbf{st} (p, \mathbb{P})$ is nice for an infinite set $K \subseteq \omega$.

4. We say that the forcing notion $\mathbb{P}$ has the uniformly nice $\odot_n^\mathbf{st}$-property if for every $p \in \mathbb{P}$ and an infinite set $K \subseteq \omega$ Generic has a nice for $K$ winning strategy in $\mathcal{D}_{\odot_n}^\mathbf{st} (p, \mathbb{P})$.

Observation 3.3. (1) If $\mathbb{P}$ has the $\odot_n$-property, then it has the uniformly nice $\odot_n^\mathbf{st}$-property for some $\odot_n$-strategy system $\mathbf{st}$.

(2) The uniform $n$–Sacks forcing notion $\mathbb{Q}_n$ has the uniformly nice $\odot_n^\mathbf{st}$-property for some $\odot_n$-strategy system $\mathbf{st}$.

We do not know if the $n$–Silver forcing is equivalent to a forcing with the uniformly nice $\odot_n^\mathbf{st}$-property (for some $\mathbf{st}$).
Theorem 3.4. Assume that $\mathcal{Q} = (\mathcal{P}_\xi, Q_\xi : \xi < \gamma)$ is a CS iteration and $\mathfrak{s}^\xi$ are $\mathcal{P}_\xi$-names such that for every $\xi < \gamma$,

$$\models_{\mathcal{P}_\xi} \text{"$Q_\xi$ has the $\odot_n$-property and $\mathfrak{s}^\xi$ is a $\odot_n$-strategy system for $Q_\xi$ and $Q_\xi$ has the uniformly nice ($\odot_n^\xi$)-property."}$$

Then $\mathbb{P}_\gamma = \lim(\mathcal{Q})$ has the $n$-localization property.

Proof. The following combinatorial observation can be shown by an easy induction.

Claim 3.4.1. Let $M < \omega$. Suppose that for each $m < M$ we are given $k_m < \omega$ and a set $A_m \subseteq k_m\omega$ of size $M$. Then there is a sequence $\langle \sigma_m : m < M \rangle \in \prod_{m < M} A_m$ such that

$$(\forall m < m' < M)(\sigma_m, \sigma_{m'} \text{ are incompatible}).$$

Let $\tau$ be a $\mathbb{P}_\gamma$-name for an element of $\omega$· Without loss of generality we may assume that

$$(\odot)0 \models_{\mathbb{P}_\gamma} (\forall \alpha < \gamma)(\tau \notin V^\mathbb{P}_\gamma).$$

Claim 3.4.2. Let $K, M < \omega, \xi < \gamma, p \in \mathbb{P}_\gamma$. Then there are $N > M, q^* \in \mathcal{P}_\xi$ and a sequence $\langle \sigma_\ell, q_\ell : \ell < L \rangle$ such that

(a) $\sigma_\ell \in N \omega$ and $|\{\sigma_\ell[M,N] : \ell < L\}| > K,$
(b) $p \in \mathbb{P}_\gamma, p \leq q_\ell, q_\ell[\xi] = q^*$ and $q_\ell \Vdash_{\mathbb{P}_\gamma} \tau[M] = \sigma_\ell,$
(c) $q^* \Vdash_{\mathbb{P}_\gamma} \{q_\ell[\xi] : \ell < k\}$ is an $\mathfrak{s}^\xi$-front above $p(\xi)".$

Proof of the Claim. Let $\mathbb{P}_\gamma$ be defined as at the beginning of the proof of Lemma 3.8. Suppose that $G_\xi \subseteq \mathcal{P}_\xi$ is generic over $V$, $p[\xi] \in G_\xi,$ and let us work in $V[G_\xi]$ for a while. Then $\mathbb{P}_\gamma[G_\xi]$ is a dense subset of the limit $\mathbb{P}^\mathbb{P}_\gamma$ of a CS iteration of forcing notions with $\odot_n$-property, so we may use the proof of 3.4. Consider a play $\langle (T_\iota, p^\iota, q^\iota) : i < \omega \rangle$ of $\mathcal{O}^\mathbb{P}_\mathbb{P}_\gamma(p)[\xi, \gamma], \mathbb{P}^\mathbb{P}_\gamma)$ in which

- Generic follows exactly the strategy described in the proof of 3.3 where on the coordinate $\xi$ the strategy $\mathfrak{s}^\xi_{p(\xi)}[G_\xi]$ is used and
- each condition $q^\iota_\ell$ played by Antigeneric (for $t \in T_\iota, \text{rk}_t(t) = \gamma$) is from $\mathbb{P}_\gamma[G_\xi]$ and decides the value of $\tau[M + i]$ in $\mathcal{O}_n^\mathbb{P}_\mathbb{P}_\gamma(p)[\xi, \gamma, \mathbb{P}^\mathbb{P}_\gamma]$.

Let $q \in \mathbb{P}^\mathbb{P}_\mathbb{P}_\gamma$ be the condition defined by $\langle \odot \rangle_{\mathbb{P}_\gamma}(\text{at the end of the proof of 3.4}),$ so it witnesses that Generic won the play, and $\{q^\iota_\ell : t \in T_\iota \& \text{rk}_t(t) = \gamma\}$ is predense above $q$ (for each $i < \omega$). It follows from $(\odot)0$ that for some $i < \omega$ we have

$$K < \{\{\sigma^*_{\iota}[M,M + i] : t \in T_\iota \& \text{rk}_t(t) = \gamma\}\}.$$ 

Also it follows from the description of Generic’s strategy in $\mathcal{O}_n^\mathbb{P}_\mathbb{P}_\gamma(p)[\xi, \gamma, \mathbb{P}^\mathbb{P}_\gamma]$ that the family $\{q^\iota_\ell[\xi] : t \in T_\iota \& \text{rk}_t(t) = \gamma\}$ is an $\mathfrak{s}^\xi_{p(\xi)}[G_\xi]$-front above $p(\xi).$

Now, $T_\iota, \sigma^*_{\iota}, q^\iota_\ell : t \in T_\iota \& \text{rk}_t(t) = \gamma \in V,$ so we may pick a condition $q^* \in \mathcal{P}_\xi$ stronger than $p[\xi]$ which forces that these objects have the properties described above. Let $(t_\ell : \ell < L)$ be an enumeration of $\{t \in T_\iota : \text{rk}_t(t) = \gamma\}$ and $\sigma_\ell = \sigma^*_{\iota}, q_\ell = q^* \cdot q^\iota_{t_\ell}$ (for $\ell < L$).

Let $p \in \mathbb{P}_\gamma$. Following the procedure described in the proof of Theorem 3.4 construct a sequence

$$\langle w_i, T_i, p^i, q^i_{t_i} : q^*_{t_i}, \bar{r}_i, k_i, \alpha_i, M_i, N_i, i_\xi, i_\bar{\xi}, i_\bar{\eta}, i_{\bar{r}_i}, i_{\bar{r}_i}, \bar{p}_i, \bar{s}_i, G_i, \bar{G}_i, g_i : i < \omega, \xi \in w_i \rangle$$

such that the following conditions are satisfied.
We let $q$ as in 3.1 one argues that $T$ is nice. To guarantee demands $(\otimes_1, \otimes_3)$, it is a $P$-name for a winning strategy of Generic in $\mathcal{O}_{\otimes n}(r_{i,\xi}(\xi), Q_{i,\xi})$ which is nice for each $\{k \in \omega : k + i^*_\xi \in K_\xi\}$. If $\xi \in w_i$, then $s_{i,\xi} \subseteq \bigcup_{j \leq i+1-t^*_i} j(n + 1)$ is an $n$-tree, $\tilde{p}_{i,\xi}$ is the enumeration of $\max(s_{i,\xi})$ in the $\prec_{\chi}$-increasing order, and $\tilde{p}_{i,\xi} = (q^i_{\eta,i,:} : \eta \in \max(s_{i,\xi}))$, $Q_{i,\xi} = (Q^i_{\eta,i,:} : \eta \in \max(s_{i,\xi}))$, $\tilde{q}_{i,\xi} = (q^i_{\eta,i,:} : \eta \in \max(s_{i,\xi}))$ are $P_\xi$-names for systems indexed by $\max(s_{i,\xi})$.

For each $\xi \in \bigcup_{i < \omega}$,

\[
\models_{\mathcal{P}_\xi} \langle s_{i,\xi}, \tilde{p}_{i,\xi}, Q_{i,\xi}, \tilde{q}_{i,\xi} : i^*_\xi \leq i < \omega \rangle \text{ is a legal play of } \mathcal{O}_{\otimes n}(r_{i,\xi}(\xi), Q_{i,\xi}) \text{ in which Generic uses } st_{i,\xi}. 
\]

$(\otimes_5)$ As $(N^i_t : t \in T_i \& \rk(t) = \gamma)$, $M_i < N^i_t < M_{i+1} < \omega$, $M_0 = 0$, and $\tilde{q}_i^* = (s^i_{t,\ell} : t \in T_i \& \rk(t) = \gamma \& \ell < k_i!)$, $s^i_{t,\ell} \in N^i_t$, and if $\ell < \ell' < k_i!$, $t \in T_i$, $\rk(t_\ell) = \gamma$, then $s^i_{t,\ell}[1|M_i, N^i_t] \neq s^i_{t,\ell'}[1|M_i, N^i_t]$. $(\otimes_6)$ $\alpha_i = \max(w_i)$, $\tilde{q}_i^* = (q^i_{\eta,i,:} : t \in T_i \& \rk(t) = \gamma \& \ell < k_i!)$, and for $t \in T_i$, $\rk(t) = \gamma$, $\ell < k_i!$ we have $q^i_{\eta,i,:}[1|\alpha_i] = q^i_{\eta,i,:}[1|\alpha_i]$, $q^i_{\eta,i,:}[1|\alpha_i] \models_{P_\xi} q^i_{\eta,i,:}[1|\alpha_i] \in Q^ii_i\alpha_i$, where $\eta \in \max(s_{i,\alpha_i})$ is the end segment of $q^i_{\eta,i,:}$ of length $i + 1 - i^*_\xi$, and $q^i_{\eta,i,:} \models_{P_\xi, \tau} N^i_t$. $(\otimes_7)$ If $t_0, t_1 \in T_i$, $\rk(t_0) = \rk(t_1) = \gamma$, $t_0 \neq t_1$, then for some $\ell_0, \ell_1 < k_i$, we have that $q^i_{t_0} = q^i_{t_1}$, $q^i_{t_1} = q^i_{t_1 \ell_1}$ and the sequences $\sigma^i_{t_0,\ell_0}[1|M_i, N^i_t]$ and $\sigma^i_{t_1,\ell_1}[1|M_i, N^i_t]$ are incomparable.

To guarantee demands $(\otimes_1) - (\otimes_4)$ we follow exactly the lines of the proof of 3.1 to get $(\otimes_5) + (\otimes_6)$ we use Claim 3.4.2 and we ensure $(\otimes_7)$ by Claim 3.4.1.

After the construction is carried out define a condition $q \in \mathcal{P}_\gamma$ in a manner similar to that in the proof of 3.4. Dom$(q) = \bigcup_{i < \omega} w_i = \bigcup_{i < \omega} \text{ Dom}(r_i)$ and for $\xi \in \text{ Dom}(q)$ we let $q(\xi)$ be a $P_\xi$-name for a condition in $\mathcal{Q}_\xi$ such that

\[
(\otimes)_{\xi} \models_{P_\xi} q(\xi) \geq r_{i,\xi}(\xi) \text{ and } q(\xi) \models_{\mathcal{Q}_\xi} (\forall x \in \max(s_{i,\xi})) \langle q^x_{i,\xi} : i^*_\xi \leq i < \omega \rangle = q^x_{i,\xi} \in \Gamma q_{\xi}. 
\]

As in 3.1 one argues that

$(\otimes)_{\xi}$ for each $i \in \omega$ the family $\{q^i_{t} : t \in T_i \& \rk(t) = \gamma\}$ is predense above $q$.

Now we choose a tree $T \subseteq \omega > \omega$ such that

\[
(\forall f \in [T])(\forall i < \omega)(\exists x \in T_i)(\exists k \in k_i)(q^i_{t} = q^i_{t \ell < f} \& s^i_{t,\ell} \prec f).
\]

Plainly, $T$ is an $n$-ary tree and $q \models_{P_\gamma, \tau} T \in [T]$. $\Box$

Remark 3.5. After analyzing the proof of Theorem 3.4 one may notice that the following can be shown by the same proof.

Assume that $\gamma$ is a limit ordinal and $\mathcal{Q} = \{P_\xi, \mathcal{Q}_\xi : \xi < \gamma\}$ is a CS iteration such that for every $\xi < \gamma$

- $\models_{P_\xi} "\mathcal{Q}_\xi \text{ has the nice } \diamond_{n-}\text{property }"$
- $P_\xi \text{ has the } n\text{-localization property.}$

Then $P_\gamma = \lim(\mathcal{Q})$ has the $n$-localization property.
(The assumption that $\gamma$ is limit allows us to make sure in the construction that $i < \min(K_\alpha)$ for all $i < \omega$.)

**Problem 3.6.** (1) Can the implications in Observation 1.4 be reversed? What if we restrict ourselves to (s)nep forcing notions or even Suslin$^+$?

(2) Assume that $\mathbb{P}$ has the $\circ_n$–property. Is it equivalent to a forcing notion with the uniformly nice $\circ_n^\ast$–property (for some $\circ_n$–strategy system $\text{st}$)? Again, we may allow restrictions to nice forcing notions.

**References**


Department of Mathematics, University of Nebraska at Omaha, Omaha, NE 68182-0243, USA

E-mail address: roslanow@member.ams.org

URL: http://www.unomaha.edu/logic