

9-2006

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Recommended Citation

Roslanowski, Andrzej, "n-localization property" (2006). *Mathematics Faculty Publications*. 63.
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n -LOCALIZATION PROPERTY

ANDRZEJ ROSLANOWSKI

0. INTRODUCTION

The present paper is concerned with the n -localization property and its preservation in countable support (CS) iterations. This property was first introduced in Newelski and Roslanowski [10, p. 826].

Definition 0.1. Let n be an integer greater than 1.

- (1) A tree T is an n -ary tree provided that $(\forall s \in T)(|\text{succ}_T(s)| \leq n)$.
- (2) A forcing notion \mathbb{P} has the n -localization property if

$$\Vdash_{\mathbb{P}} \text{ “ } (\forall f \in {}^\omega\omega)(\exists T \in \mathbf{V})(T \text{ is an } n\text{-ary tree and } f \in [T]) \text{ ”.}$$

In [10, Theorem 2.3] we showed that countable support products of the n -Sacks forcing notion \mathbb{D}_n (see Definition 1.5(1) here) have the n -localization property. That theorem was used to obtain some consistency results concerning cardinal characteristics of the ideal determined by unsymmetric games. Soon after this, the uniform n -Sacks forcing notion \mathbb{Q}_n (see Definition 1.5(2)) was introduced in [11, §4] and applied in the proof of [11, Theorem 5.13]. The crucial property of \mathbb{Q}_n which was used there is that the CS iterations of \mathbb{Q}_n have the n -localization property, but in [11] we only stated that the proof is similar to that of [10, Theorem 2.3].

One of the difficulties with the n -localization property was that there was no “preservation theorem” for it. Geschke and Quickert [5] give full and detailed proofs of the 2-localization property for both CS products and CS iterations of the Sacks forcing \mathbb{D}_2 (and those proofs can be easily rewritten for n -localization property and \mathbb{D}_n). And the same proof can be repeated for \mathbb{Q}_n , but a more general theorem has been missing.

Recently, the n -localization property, the σ -ideal generated by n -ary trees and n -Sacks forcing notion \mathbb{D}_n have been found applicable to some questions concerning convexity numbers of closed subsets of \mathbb{R}^n , see Geschke, Kojman, Kubiś and Schipperus [4], Geschke and Kojman [3] and most recently Geschke [2]. The latter paper is *raison d’être* for this note — when I read [2] to write a review for *Mathematical Reviews* I wanted to check as many technical details as I could. In [2, §2] an interesting forcing notion¹ \mathbb{P}_G was introduced and a proof was given that it has the n -localization property. However, the proof that the CS iteration of this forcing has the n -localization property was left to the reader as “similar to that for Sacks”.

Date: July 2005.

1991 Mathematics Subject Classification. 03E40, 03E35.

Key words and phrases. n -localization property, forcing, CS iterations.

The author would like to thank his wife, Małgorzata Jankowiak-Roslanowska for partial support of this research. He also acknowledges partial support from the United States-Israel Binational Science Foundation (Grant no. 2002323).

¹we call it *the Geschke forcing* here, see Definition 1.5(4)

At first I was not sure about technical details of that proof, so I decided to look at \mathbb{P}_G and \mathbb{D}_n together. Soon I have become convinced that a unifying theorem is needed and this note presents a result which has such character.

It was stated in [10, Theorem 2.3] that the same proof as for \mathbb{D}_n works also for CS iterations and products of the n -Silver forcing notions \mathbb{S}_n (see Definition 1.5(3)). Maybe some old wisdom got lost, but it does not look like that *the same arguments work for the n -Silver forcing \mathbb{S}_n* . As a matter of fact, we believe that it is an open question if \mathbb{S}_n and its CS iterations have the n -localization property. Also, motivated by [1, Questions 3.3, 3.4] we asked if the iteration of two 2-Silver forcing notions may add a 4-Silver real, but because of the claim in [10, Corollary 2.4] we did not state the question explicitly in the final version of [1]. In the light of what we said above, it is only proper to pose this problem again.

Problem 0.2. (1) Can a finite iteration of 2-Silver forcings \mathbb{S}_2 add a generic real for the 4-Silver forcing notion \mathbb{S}_4 ?

(2) Does the n -Silver forcing \mathbb{S}_n have the n -localization property? The same about CS iterations of n -Silver forcings.

The author offers “all you can drink in 3 days” coffee/espresso in a place similar to Caffeine Dreams in Omaha for full solution to this problem. Partial solutions may be eligible for partial awards.

It may occur that the answer to the above problem is hidden in Shelah and Steprans [16]. Let us note that Remark 3.5 suggests that if we can show that finite iterations of \mathbb{S}_n have the n -localization property, then we will be able to handle all CS iterations.

The following general question remains still unsolved.

Problem 0.3. Do CS iterations of proper forcing notions with n -localization property have n -localization property? What if we restrict ourselves to (s)nep forcing notions (see Shelah [15]) or even Suslin⁺ (see Goldstern [6] or Kellner [8], [9])?

Content of the paper: In the first section we introduce several properties related to the n -localization property. The strongest one, \oplus_n -property, does imply the n -localization. However not all forcing notions around have the \oplus_n -property so this is why we have weaker relatives. We also remind definitions of the forcing notions that we are interested in and the basic facts on trees of conditions.

The following section shows that CS iterations of forcing notions with the \oplus_n -property have the n -localization (Theorem 2.1). Since we do not know if \mathbb{Q}_n has the \oplus_n -property, in the third section we somewhat weaken that property to cover more forcing notions. From the point of view of applications Theorems 3.1, 3.4 are strongest and they include the result of the second section. Still, we think that the proof of 2.1 is somewhat easier and it is a good preparation for Section 3.

One should note that the proofs of our iteration theorems are very “not proper” in their form. We work with trees of conditions which were used in pre-proper era and our arguments resemble those of Rosłanowski and Shelah [12, §2] and to some extend also [13, §A.2].

Notation: Our notation is rather standard and compatible with that of classical textbooks (like Jech [7]). In forcing we keep the older convention that *a stronger condition is the larger one*.

- (1) n is our fixed integer, $n \geq 2$. Ordinal numbers will be denoted by the lower case initial letters of the Greek alphabet ($\alpha, \beta, \gamma, \delta \dots$) with possible sub- and superscripts. Natural number will be labeled by i, j, k, ℓ, m (also upper cases).

By χ we will denote a *sufficiently large* regular cardinal; $\mathcal{H}(\chi)$ is the family of all sets hereditarily of size less than χ . Moreover, we fix a well ordering $<_\chi^*$ of $\mathcal{H}(\chi)$.

- (2) For two sequences η, ν we write $\nu \triangleleft \eta$ whenever ν is a proper initial segment of η , and $\nu \sqsubseteq \eta$ when either $\nu \triangleleft \eta$ or $\nu = \eta$. The length of a sequence η is denoted by $\text{lh}(\eta)$.
- (3) A *tree* is a family of finite sequences closed under initial segments. For a tree T and $\eta \in T$ we define *the successors of η in T* and *maximal points of T* by:

$$\begin{aligned} \text{succ}_T(\eta) &= \{\nu \in T : \eta \triangleleft \nu \ \& \ \neg(\exists \rho \in T)(\eta \triangleleft \rho \triangleleft \nu)\}, \\ \text{max}(T) &= \{\nu \in T : \text{there is no } \rho \in T \text{ such that } \nu \triangleleft \rho\}. \end{aligned}$$

For a tree T the family of all ω -branches through T is denoted by $[T]$.

- (4) We will consider some games of two players. One player will be called *Generic*, and we will refer to this player as “she”. Her opponent will be called *Antigeneric* and will be referred to as “he”.
- (5) For a forcing notion \mathbb{P} , $\Gamma_{\mathbb{P}}$ stands for the canonical \mathbb{P} -name for the generic filter in \mathbb{P} . With this one exception, all \mathbb{P} -names for objects in the extension via \mathbb{P} will be denoted with a tilde below (e.g., $\tilde{\tau}$, \tilde{X}). The weakest element of \mathbb{P} will be denoted by $\emptyset_{\mathbb{P}}$ (and we will always assume that there is one, and that there is no other condition equivalent to it). We will also assume that all forcing notions under considerations are atomless.

By “CS iterations” we mean iterations in which domains of conditions are countable. However, we will pretend that conditions in a CS iteration $\mathbb{Q} = \langle \mathbb{P}_\zeta, \mathbb{Q}_\zeta : \zeta < \gamma \rangle$ are total functions on γ and for $p \in \text{lim}(\mathbb{Q})$ and $\alpha < \gamma$ we have $\Vdash_{\mathbb{P}_\alpha} p(\alpha) \in \mathbb{Q}_\alpha$, and if $\alpha \in \gamma \setminus \text{Dom}(p)$ then $p(\alpha) = \emptyset_{\mathbb{Q}_\alpha}$.

1. TOOLS

In this section we introduce the main concepts and properties and we show how they are related to various forcing notions. We also introduce the main tool for our forcing arguments: trees of conditions.

Definition 1.1. Let \mathbb{P} be a forcing notion.

- (1) For a condition $p \in \mathbb{P}$ we define a game $\mathfrak{D}_n^\oplus(p, \mathbb{P})$ of two players, *Generic* and *Antigeneric*. A play of $\mathfrak{D}_n^\oplus(p, \mathbb{P})$ lasts ω moves and during it the players construct a sequence $\langle (s_i, \bar{p}^i, \bar{q}^i) : i < \omega \rangle$ as follows. At a stage $i < \omega$ of the play, first *Generic* chooses a finite n -ary tree s_i and a system $\bar{p}^i = \langle p_\eta^i : \eta \in \text{max}(s_i) \rangle$ such that:

- (α) $|\text{max}(s_0)| \leq n$ and if $i = j + 1$ then s_j is a subtree of s_i such that

$$(\forall \eta \in \text{max}(s_i)) (\exists \ell < \text{lh}(\eta)) (\eta \upharpoonright \ell \in \text{max}(s_j)),$$

and

$$(\forall \nu \in \text{max}(s_j)) (0 < |\{\eta \in \text{max}(s_i) : \nu \triangleleft \eta\}| \leq n),$$

- (β) $p_\eta^i \in \mathbb{P}$ for all $\eta \in \text{max}(s_i)$,

(γ) if $j < i$, $\nu \in \max(s_j)$ and $\nu \triangleleft \eta \in \max(s_i)$, then $q_\nu^j \leq p_\eta^i$ and $p \leq p_\eta^i$. Then Antigeneric answers choosing a system $\bar{q}^i = \langle q_\eta^i : \eta \in \max(s_i) \rangle$ of conditions in \mathbb{P} such that $p_\eta^i \leq q_\eta^i$ for each $\eta \in \max(s_i)$.

Finally, Generic wins the play $\langle (s_i, \bar{p}^i, \bar{q}^i) : i < \omega \rangle$ if and only if

(\otimes) there is a condition $q \geq p$ such that for every $i < \omega$ the family $\{q_\eta^i : \eta \in \max(s_i)\}$ is predense above q .

- (2) Let $p \in \mathbb{P}$. We define a game $\mathfrak{D}_n^\odot(p, \mathbb{P})$ of two players, *Generic* and *Antigeneric*. A play of $\mathfrak{D}_n^\odot(p, \mathbb{P})$ lasts ω moves and during it the players construct a sequence $\langle (s_i, \bar{p}^i, \bar{q}^i) : i < \omega \rangle$ as follows. At a stage $i < \omega$ of the play, first Generic chooses a finite n -ary tree s_i such that the demand (α) of (1) above holds. Next

(\odot) Antigeneric picks an enumeration $\langle \eta_\ell^i : \ell < k_i \rangle$ of $\max(s_i)$ (so $k_i < \omega$) and then the two players play a subgame of length k_i alternatively choosing successive terms of a sequence $\langle p_{\eta_\ell^i}^i, q_{\eta_\ell^i}^i : \ell < k_i \rangle$. At a stage $\ell < k_i$ of the subgame, first Generic picks a condition $p_{\eta_\ell^i}^i \in \mathbb{P}$ such that

(γ) $_\ell^i$ if $j < i$, $\nu \in \max(s_j)$ and $\nu \triangleleft \eta_\ell^i$, then $q_\nu^j \leq p_{\eta_\ell^i}^i$ and $p \leq p_{\eta_\ell^i}^i$,

and then Antigeneric answers with a condition $q_{\eta_\ell^i}^i$ stronger than $p_{\eta_\ell^i}^i$.

The winning criterion for the game \mathfrak{D}_n^\odot is the same as the one for \mathfrak{D}_n^\oplus (i.e., (\otimes)).

- (3) A game $\mathfrak{D}_n^\ominus(p, \mathbb{P})$ for $p \in \mathbb{P}$ is defined like $\mathfrak{D}_n^\odot(p, \mathbb{P})$ above, but (\odot) is replaced by

(\ominus) Generic picks an enumeration $\langle \eta_\ell^i : \ell < k_i \rangle$ of $\max(s_i)$.

- (4) We say that \mathbb{P} has the \oplus_n -property whenever Generic has a winning strategy in the game $\mathfrak{D}_n^\oplus(p, \mathbb{P})$ for any $p \in \mathbb{P}$. In a similar manner we define when \mathbb{P} has the \odot_n -property (\ominus_n -property, respectively) replacing the game \mathfrak{D}_n^\oplus by \mathfrak{D}_n^\odot (\mathfrak{D}_n^\ominus , respectively).

Definition 1.2. Let \mathbb{P} be a forcing notion.

- (1) Assume that $K \subseteq \omega$ is infinite, $p \in \mathbb{P}$. A strategy \mathbf{st} for Generic in $\mathfrak{D}_n^\odot(p, \mathbb{P})$ is said to be *nice for K* (or just K -nice) whenever

($\boxtimes_{\text{nice}}^K$) if so far Generic used \mathbf{st} and s_i is given to her as a move at a stage $i < \omega$, then

- $s_i \subseteq \bigcup_{j \leq i+1} j(n+1)$, $\max(s_i) \subseteq {}^{(i+1)}(n+1)$ and
- if $\eta \in \max(s_i)$ and $i \notin K$, then $\eta(i) = n$, and
- if $\eta \in \max(s_i)$ and $i \in K$, then $\text{succ}_{s_i}(\eta \upharpoonright i) = n$,
- if $i \in K$, $\langle \eta_\ell^i : \ell < k \rangle$ is an enumeration of $\max(s_i)$ and $\langle p_{\eta_\ell^i}^i, q_{\eta_\ell^i}^i : \ell < k \rangle$ is the result of the subgame of level i in which Generic uses \mathbf{st} , then the conditions $p_{\eta_\ell^i}^i$ (for $\ell < k$) are pairwise incompatible.

In a similar way we define when a strategy \mathbf{st} for Generic in $\mathfrak{D}_n^\oplus(p, \mathbb{P})$ or $\mathfrak{D}_n^\ominus(p, \mathbb{P})$ is *nice for K* .

- (2) We say that \mathbb{P} has the *nice \odot_n -property* if for every $K \in [\omega]^\omega$ and $p \in \mathbb{P}$, Generic has a K -nice winning strategy in $\mathfrak{D}_n^\odot(p, \mathbb{P})$.

Remark 1.3. (1) At a stage $i < \lambda$ of a play of $\mathfrak{D}_n^\oplus(p, \mathbb{P})$, Antigeneric may play stronger conditions, and we may require that if $\bar{q}^i = \langle q_\eta^i : \eta \in \max(s_i) \rangle$ is his move, then the conditions q_η^i are pairwise incompatible. Thus the winning criterion (\otimes) could be replaced by

- (\otimes)* there are a condition $q \geq p$ and a \mathbb{P} -name ρ such that
 $q \Vdash_{\mathbb{P}} \text{“ } \rho \in [\bigcup_{i < \omega} s_i] \ \& \ (\forall i < \omega) (\exists \ell < \omega) (\rho \upharpoonright \ell \in \max(s_i) \ \& \ q_{\rho \upharpoonright \ell}^i \in \Gamma_{\mathbb{P}}) \text{”}$.

- This would make the game \mathfrak{D}_n^{\oplus} more like the game of [13, Definition A.2.1].
 (2) If Generic has a winning strategy in $\mathfrak{D}_n^{\oplus}(p, \mathbb{P})$ and $K \subseteq \omega$ is infinite, then Generic has a K -nice winning strategy in $\mathfrak{D}_n^{\oplus}(p, \mathbb{P})$.

Observation 1.4. For a forcing notion \mathbb{P} the following implications hold:

$$\begin{array}{ccccccc} \oplus_n\text{-property} & \Rightarrow & \text{nice } \odot_n\text{-property} & \Rightarrow & \odot_n\text{-property} & \Rightarrow & \ominus_n\text{-property} \\ & & \Downarrow & & & & \Downarrow \\ n\text{-localization property} & & & & & & \text{proper.} \end{array}$$

Let us recall definitions of forcing notions that are main examples for the properties introduced in 1.1.

Definition 1.5. (1) **The n -Sacks forcing notion \mathbb{D}_n** consists of perfect trees $p \subseteq \omega^{>n}$ such that

$$(\forall \eta \in p) (\exists \nu \in p) (\eta \triangleleft \nu \ \& \ \text{succ}_p(\nu) = n).$$

The order of \mathbb{D}_n is the reverse inclusion, i.e., $p \leq_{\mathbb{D}_n} q$ if and only if $q \subseteq p$.

(2) **The uniform n -Sacks forcing notion \mathbb{Q}_n** consists of perfect trees $p \subseteq \omega^{>n}$ such that

$$(\exists X \in [\omega]^\omega) (\forall \eta \in p) (\text{lh}(\eta) \in X \Rightarrow \text{succ}_p(\nu) = n).$$

The order of \mathbb{Q}_n is the reverse inclusion, i.e., $p \leq_{\mathbb{Q}_n} q$ if and only if $q \subseteq p$.

(3) **The n -Silver forcing notion \mathbb{S}_n** consists of partial functions p such that $\text{Dom}(p) \subseteq \omega$, $\text{Rng}(p) \subseteq n$ and $\omega \setminus \text{Dom}(p)$ is infinite. The order of \mathbb{S}_n is the inclusion, i.e., $p \leq_{\mathbb{S}_n} q$ if and only if $p \subseteq q$.

(4) Let us assume that $G = (V, E)$ is a hypergraph on a Polish space V which is

- $(n+1)$ -regular open, that is $E \subseteq [V]^{n+1}$ is open in the topology inherited from V^{n+1} , and
- transitive, that is $(\forall e \in E) (\forall v \in V \setminus e) (\exists w \in e) ((e \setminus \{w\}) \cup \{v\} \in E)$,
- uncountably chromatic on every open set, that is for every non-empty open subset U of V and every countable family \mathcal{F} of subsets of U , either $\bigcup \mathcal{F} \neq U$ or $[F]^{n+1} \cap E \neq \emptyset$ for some $F \in \mathcal{F}$.

The Geschke forcing notion \mathbb{P}_G for G consists of all closed sets $C \subseteq V$ such that the hypergraph $(C, E \cap [C]^{n+1})$ is uncountably chromatic on every non-empty open subset of C . The order of \mathbb{P}_G is the inverse inclusion, i.e., $C \leq_{\mathbb{P}_G} D$ if and only if $D \subseteq C$.

- Observation 1.6.** (1) The n -Sacks forcing notion \mathbb{D}_n has the \oplus_n -property.
 (2) The uniform n -Sacks forcing notion \mathbb{Q}_n and the n -Silver forcing notion \mathbb{S}_n have the nice \odot_n -property.
 (3) Assume that $G = (V, E)$ is a transitive $(n+1)$ -regular open hypergraph on a Polish space V which is uncountably chromatic on every open set. Then the corresponding Geschke forcing notion \mathbb{P}_G has the \oplus_n -property.

Proof. (1)–(3) Straightforward.

(4) This is included in the proof of [2, Lemma 2.8]. □

The proofs of our theorems resemble arguments from the pre-proper era of iterated forcing and their crucial ingredients are trees of conditions. Let us first recall the relevant notions — in the definition below we follow the pattern that recently has been used in the context of iterations with uncountable supports.

Definition 1.7 (cf [13, Def. A.1.7], [14, A.3.3, A.3.2]). Let γ be an ordinal and let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \gamma \rangle$ be a CS iteration.

- (1) Let $m < \omega$ and $w \subseteq \gamma$ be finite. A *standard $(w, m)^\gamma$ -tree* is a pair $\mathcal{T} = (T, \text{rk})$ such that
- (T, \triangleleft) is a tree with root $\langle \rangle$, $\text{rk} : T \rightarrow w \cup \{\gamma\}$, and
 - if $t \in T$ and $\text{rk}(t) = \varepsilon$, then t is a sequence $\langle (t)_\zeta : \zeta \in w \cap \varepsilon \rangle$, where each $(t)_\zeta$ is a sequence of length m .

We will keep the convention that \mathcal{T}_y^x is (T_y^x, rk_y^x) .

- (2) Suppose that $w_0 \subseteq w_1$ are finite subsets of γ , $m_0 \leq m_1$, and $\mathcal{T}_1 = (T_1, \text{rk}_1)$ is a standard $(w_1, m_1)^\gamma$ -tree. The *projection* $\text{proj}_{(w_0, m_0)}^{(w_1, m_1)}(\mathcal{T}_1)$ of \mathcal{T}_1 onto (w_0, m_0) is defined as a standard $(w_0, m_0)^\gamma$ -tree $\mathcal{T}_0 = (T_0, \text{rk}_0)$ such that $T_0 = \{ \langle (t)_\zeta \upharpoonright m_0 : \zeta \in w_0 \cap \text{rk}_1(t) \rangle : t = \langle (t)_\zeta : \zeta \in w_1 \cap \text{rk}_1(t) \rangle \in T_1 \}$.

The mapping

$$T_1 \ni \langle (t)_\zeta : \zeta \in w_1 \cap \text{rk}_1(t) \rangle \mapsto \langle (t)_\zeta \upharpoonright m_0 : \zeta \in w_0 \cap \text{rk}_1(t) \rangle \in T_0$$

will be denoted $\text{proj}_{(w_0, m_0)}^{(w_1, m_1)}$ too.

- (3) A *standard tree of conditions in $\bar{\mathbb{Q}}$* is a system $\bar{p} = \langle p_t : t \in T \rangle$ such that
- (T, rk) is a standard $(w, m)^\gamma$ -tree for some finite set $w \subseteq \gamma$ and an integer $m < \omega$,
 - $p_t \in \mathbb{P}_{\text{rk}(t)}$ for $t \in T$, and
 - if $s, t \in T$, $s \triangleleft t$, then $p_s = p_t \upharpoonright \text{rk}(s)$.
- (4) Let \bar{p}^0, \bar{p}^1 be standard trees of conditions in $\bar{\mathbb{Q}}$, $\bar{p}^i = \langle p_t^i : t \in T_i \rangle$, where $\mathcal{T}_0 = \text{proj}_{(w_0, m_0)}^{(w_1, m_1)}(\mathcal{T}_1)$, $w_0 \subseteq w_1 \subseteq \gamma$, $m_0 \leq m_1$. We will write $\bar{p}^0 \leq_{w_0, m_0}^{w_1, m_1} \bar{p}^1$ (or just $\bar{p}^0 \leq \bar{p}^1$) whenever for each $t \in T_1$, letting $t' = \text{proj}_{(w_0, m_0)}^{(w_1, m_1)}(t) \in T_0$, we have $p_{t'}^0 \upharpoonright \text{rk}_1(t) \leq p_t^1$.

Lemma 1.8. *Assume that*

- $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \gamma \rangle$ is a CS iteration,
- (T, rk) is a standard $(w, m)^\gamma$ -tree, $w \in [\gamma]^{< \omega}$ and $\bar{p} = \langle p_t : t \in T \rangle$ is a standard tree of conditions in $\bar{\mathbb{Q}}$, and
- τ is a \mathbb{P}_γ -name for an element of ${}^\omega \omega$ such that $\Vdash_{\mathbb{P}_\gamma} (\forall \alpha < \gamma) (\tau \notin \mathbf{V}^{\mathbb{P}_\alpha})$.

Then there are a tree of conditions $\bar{q} = \langle q_t : t \in T \rangle$ and $N \in \omega$ such that

- $\bar{p} \leq \bar{q}$,
- if $t \in T$, $\text{rk}(t) = \gamma$, then the condition q_t decides $\tau \upharpoonright N$, say $q_t \Vdash_{\mathbb{P}_\gamma} \tau \upharpoonright N = \sigma_t$,
- if $t_0, t_1 \in T$, $\text{rk}(t_0) = \text{rk}(t_1) = \gamma$ and $t_0 \neq t_1$, then $\sigma_{t_0} \neq \sigma_{t_1}$.

Proof. For $\alpha < \beta \leq \gamma$, $\mathbb{P}_{\alpha\beta}$ is a \mathbb{P}_α -name for a forcing notion with universe $P_{\alpha\beta} = \{ p \upharpoonright [\alpha, \beta) : p \in \mathbb{P}_\beta \}$ such that

- if $G_\alpha \subseteq \mathbb{P}_\alpha$ is generic over \mathbf{V} and $f, g \in P_{\alpha\beta}$,
- then $\mathbf{V}[G_\alpha] \models f \leq_{\mathbb{P}_{\alpha\beta}[G_\alpha]} g$ if and only if $(\exists p \in G_\alpha) (p \cup f \leq_{\mathbb{P}_\beta} p \cup g)$.

Note that $P_{\alpha\beta}$ is from \mathbf{V} , it is only the relation $\leq_{\mathbb{P}_{\alpha\beta}}$ which is defined in $\mathbf{V}[G_\alpha]$. Also, \mathbb{P}_β is isomorphic with a dense subset of $\mathbb{P}_\alpha * \mathbb{P}_{\alpha\beta}$.

Let $w = \{\xi_0, \dots, \xi_i\}$ be the decreasing enumeration. We may assume that $0 \in w$ and thus $\xi_0 = \max(w)$ and $\xi_i = 0$. Let $M = |T| + 7$, $T^* = \{t \in T : \text{rk}(t) = \gamma\}$. By induction on $j \leq i$ we will define \mathbb{P}_{ξ_j} -names n_t^j , $\sigma_{t,k}^j$ and $q_{t,k}^j$ for $t \in T^*$ and $k < M$ so that

- (a)_j $\Vdash_{\mathbb{P}_{\xi_j}} \text{“} n_t^j \in \omega \ \& \ \sigma_{t,k}^j : n_t^j \longrightarrow \omega \text{”}$,
- (b)_j $\Vdash_{\mathbb{P}_{\xi_j}} \text{“} q_{t,k}^j \in \mathbb{P}_{\xi_j \gamma} \ \& \ p_t \upharpoonright [\xi_j, \gamma] \leq_{\mathbb{P}_{\xi_j \gamma}} q_{t,k}^j \ \& \ q_{t,k}^j \Vdash_{\mathbb{P}_{\xi_j \gamma}} \mathcal{T} \upharpoonright n_t^j = \sigma_{t,k}^j \text{”}$,
- (c)_j if $t \in T^*$ and $k < \ell < M$, then

$$\Vdash_{\mathbb{P}_{\xi_j}} \text{“} q_{t,k}^j \upharpoonright [\xi_j, \xi_0) = q_{t,\ell}^j \upharpoonright [\xi_j, \xi_0) \ \& \ \sigma_{t,k}^j \neq \sigma_{t,\ell}^j \text{”},$$

- (d)_j if $t_0, t_1 \in T^*$, $t = t_0 \cap t_1$ and $\text{rk}(t) > \xi_j$, then

$$\Vdash_{\mathbb{P}_{\xi_j}} \text{“} q_{t_0,k}^j \upharpoonright \text{rk}(t) = q_{t_1,k}^j \upharpoonright \text{rk}(t) \text{”}.$$

To start the inductive process suppose that $G_{\xi_0} \subseteq \mathbb{P}_{\xi_0}$ is generic over \mathbf{V} and work in $\mathbf{V}[G_{\xi_0}]$ for a moment. Note that \mathcal{T} may be thought of as a $\mathbb{P}_{\xi_0 \gamma}[G_{\xi_0}]$ -name for an element of ${}^\omega \omega$ such that $\Vdash_{\mathbb{P}_{\xi_0 \gamma}[G_{\xi_0}]} \mathcal{T} \notin \mathbf{V}[G_{\xi_0}]$. Therefore we may find $n_t \in \omega$, $\sigma_{t,k} : n_t \longrightarrow \omega$ and $q_{t,k} \in \mathbb{P}_{\xi_0 \gamma}[G_{\xi_0}]$ (for $t \in T^*$ and $k < M$) such that for each $t \in T^*$ and $\ell, k < M$, $\ell \neq k$:

- $p_t \upharpoonright [\xi_0, \gamma] \leq_{\mathbb{P}_{\xi_0 \gamma}[G_{\xi_0}]} q_{t,k}$,
- $q_{t,k} \Vdash_{\mathbb{P}_{\xi_0 \gamma}[G_{\xi_0}]} \mathcal{T} \upharpoonright n_t = \sigma_{t,k}$,
- $\sigma_{t,k} \neq \sigma_{t,\ell}$.

Now, let $n_t^0, \sigma_{t,k}^0, q_{t,k}^0$ (for $t \in T^*$, $k < M$) be \mathbb{P}_{ξ_0} -names for objects with properties as those of $n_t, \sigma_{t,k}, q_{t,k}$ above.

Suppose that $j < i$ and we have defined \mathbb{P}_{ξ_j} -names $n_t^j, \sigma_{t,k}^j, q_{t,k}^j$ so that (a)_j–(d)_j are satisfied. Let $G_{\xi_{j+1}} \subseteq \mathbb{P}_{\xi_{j+1}}$ be generic over \mathbf{V} and work in $\mathbf{V}[G_{\xi_{j+1}}]$ for a moment. For each $s \in T$ of rank $\text{rk}(s) = \xi_j$ we may pick a condition $q_s \in \mathbb{P}_{\xi_{j+1} \xi_j}[G_{\xi_{j+1}}]$ stronger than $p_s \upharpoonright [\xi_{j+1}, \xi_j)$ and also we may choose $n_t \in \omega$, $\sigma_{t,k} : n_t \longrightarrow \omega$ and $q_{t,k}$ (for $k < M$, $t \in T^*$) such that

$$q_s \Vdash_{\mathbb{P}_{\xi_{j+1} \xi_j}[G_{\xi_{j+1}}]} (\forall k < M) (\forall t \in T^*) (s \triangleleft t \Rightarrow [n_t^j = n_t \ \& \ q_{t,k}^j = q_{t,k} \ \& \ \sigma_{t,k}^j = \sigma_{t,k}]).$$

Now let $n_t^{j+1}, \sigma_{t,k}^{j+1}$ be $\mathbb{P}_{\xi_{j+1}}$ -names for $n_t, \sigma_{t,k}$ as above, and let $q_{t,k}^{j+1}$ be a $\mathbb{P}_{\xi_{j+1}}$ -name for $q_t \upharpoonright \xi_j \widehat{\ } q_{t,k}$. One easily verifies that demands (a)_{j+1}–(d)_{j+1} are satisfied.

Finally note that (as $\xi_i = 0$) $n_t^i, \sigma_{t,k}^i$ and $q_{t,k}^i$ are actually objects in \mathbf{V} , not names.

Let $\mathcal{T}^+ = (T^+, \text{rk}^+)$ be a standard $(w, m+1)^\gamma$ -tree such that $\text{proj}_{w,m}^{w,m+1}(\mathcal{T}^+) = \mathcal{T}$ and

$$\begin{aligned} & \text{if } t = \langle (t)_\xi : \xi \in w \rangle \in T^+, \text{rk}^+(t) = \gamma, \\ & \text{then } (t)_{\xi_0}(m) < M \text{ and } (\forall \xi \in w \cap \xi_0) ((t)_\xi(m) = *). \end{aligned}$$

It should be clear that $\langle q_{t,k}^i : k < M \ \& \ t \in T^* \rangle$ determines a tree of conditions $\bar{q}' = \langle q'_t : t' \in T^+ \rangle$ such that $\bar{p} \leq \bar{q}'$ and $q'_t = q_{t,k}^i$ whenever $t' \in T^+$, $\text{rk}^+(t') = \gamma$, $t = \text{proj}_{w,m}^{w,m+1}(t')$ and $k = (t')_{\xi_0}(m)$. Let $N = \max(\{n_t^i : t \in T^*\})$. Carrying out a procedure similar to that described above we may find a tree of conditions $\bar{q}^* = \langle q_s^* : s \in T^+ \rangle$ such that $\bar{q}^* \geq \bar{q}'$ and for some $\rho_s \in {}^n \omega$ (for $s \in T^+$, $\text{rk}^+(s) = \gamma$) we have

- $q_s^* \Vdash_{\mathbb{P}_\gamma} \mathcal{T} \upharpoonright N = \rho_s$, and
- if $s_0, s_1 \in T^+$, $\text{rk}^+(s_0) = \text{rk}^+(s_1) = \gamma$, $\text{proj}_{w,m}^{w,m+1}(s_0) = \text{proj}_{w,m}^{w,m+1}(s_1)$, $(s_0)_{\xi_0}(m) \neq (s_1)_{\xi_0}(m)$, then $\rho_{s_0} \neq \rho_{s_1}$.

Then for each $t \in T^*$ we may choose $s_t \in T^+$ such that $\text{proj}_{w,m}^{w,m+1}(s_t) = t$ and $\rho_{s_{t_0}} \neq \rho_{s_{t_1}}$ for distinct $t_0, t_1 \in T^*$. The choice of \bar{q} should be clear now. \square

2. \oplus_n -PROPERTY AND CS ITERATIONS

Here we show that CS iterations of forcing notions with \oplus_n -property result in forcings with the n -localization property. This result covers examples like the n -Sacks forcing notion \mathbb{D}_n or the suitable Geschke forcings \mathbb{P}_G . However, we do not know if the uniform n -Sacks forcing fits here, so in the next section we will prove a result applicable to a larger family of forcing notions. Still we believe that the proof of 2.1 below is a nice preparation for the arguments in the following section.

Theorem 2.1. *Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underline{\mathbb{Q}}_\xi : \xi < \gamma \rangle$ be a CS iteration such that for every $\xi < \gamma$,*

$$\Vdash_{\mathbb{P}_\xi} \text{“ } \underline{\mathbb{Q}}_\xi \text{ has the } \oplus_n\text{-property ”.}$$

Then

- (1) $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ has the \odot_n -property.
- (2) $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ has the n -localization property.

Proof. (1) Let $p \in \mathbb{P}_\gamma$. We are going to describe a strategy \mathbf{st} for Generic in the game $\mathfrak{D}_n^\odot(p, \mathbb{P}_\gamma)$. This strategy will give Generic, at a stage $i < \omega$, a standard $(w_i, (i+1)^\gamma)$ -tree $\mathcal{T}_i = (T_i, \text{rk}_i)$. These standard trees will satisfy $\mathcal{T}_i = \text{proj}_{(w_i, i+1)}^{(w_{i+1}, i+2)}(\mathcal{T}_{i+1})$ and $\{t \in T_i : \text{rk}_i(t) = \gamma\}$ will correspond to $\max(s_i)$ in the rules of the game. If only we make sure that

$(\oplus)_0$ for each $t' \in T_i$ with $\text{rk}_i(t) = \gamma$ we have

$$0 < |\{t \in T_{i+1} : \text{proj}_{(w_i, i+1)}^{(w_{i+1}, i+2)}(t) = t'\}| \leq n,$$

then Generic may easily build trees s_i and mappings $\pi_i : \{t \in T_i : \text{rk}_i(t) = \gamma\} \rightarrow s_i$ such that

- $(\oplus)_0^{(a)}$ $\text{Rng}(\pi_i) = \max(s_i) \subseteq {}^{(i+1)}n$,
- $(\oplus)_0^{(b)}$ $(\forall t_0 \in T_i)(\forall t_1 \in T_{i+1})(\pi_i(t_0) \triangleleft \pi_{i+1}(t_1) \Leftrightarrow t_0 = \text{proj}_{(w_i, i+1)}^{(w_{i+1}, i+2)}(t_1))$,
- $(\oplus)_0^{(c)}$ the demands of 1.1(1(α)) hold.

Later we will even not mention the trees s_i but we will work directly with T_i .

As we said, in the course of the play the strategy \mathbf{st} will instruct Generic to choose finite sets $w_i \subseteq \gamma$ and standard $(w_i, i+1)^\gamma$ -trees \mathcal{T}_i . She will also pick sets $K_\xi \in [\omega]^\omega$, conditions $r_i \in \mathbb{P}_\gamma$ and $\bar{t}^i, \bar{p}^i, \bar{q}^i, k_i, i_\xi^*, \mathbf{st}_\xi, s_{i,\xi}, \bar{p}_{i,\xi}, \bar{q}_{i,\xi}$. All these objects will be constructed so that, assuming $\langle (\mathcal{T}_i, \bar{t}^i, \bar{p}^i, \bar{q}^i) : i < \omega \rangle$ is the result of a play of $\mathfrak{D}_n^\odot(p, \mathbb{P}_\gamma)$ in which Generic used \mathbf{st} and she determined the corresponding side objects, the following conditions are satisfied.

- $(\oplus)_1$ $r_0(0) = p(0)$, $w_i \in [\gamma]^{i+1}$, $w_0 = \{0\}$, $w_i \subseteq w_{i+1}$ and $\bigcup_{i < \omega} \text{Dom}(r_i) = \bigcup_{i < \omega} w_i$.
- $(\oplus)_2$ If $j < i < \omega$, then $(\forall \xi \in w_{j+1})(r_j(\xi) = r_i(\xi))$ and $p \leq r_j \leq r_i$.
- $(\oplus)_3$ If $\xi \in w_i$, then $K_\xi \in [\omega]^\omega$ is known at stage i of the play and if $\xi, \zeta \in \bigcup_{i \in \omega} w_i$ are distinct, then $K_\zeta \cap K_\xi = \emptyset$.
- $(\oplus)_4$ For $\xi \in \bigcup_{i < \omega} w_i$ we have $i_\xi^* = \min(\{i : \xi \in w_i\}) \leq \min(K_\xi)$, and \mathbf{st}_ξ is a \mathbb{P}_ξ -name for a winning strategy of Generic in $\mathfrak{D}_n^\oplus(r_{i_\xi^*}(\xi), \underline{\mathbb{Q}}_\xi)$ which is nice

for $\{k \in \omega : k + i_\xi^* \in K_\xi\}$ (see 1.2, 1.3(2)). (So \mathbf{st}_0 is a K_0 -nice winning strategy of Generic in $\mathfrak{D}_n^\oplus(r_0(0), \mathbb{Q}_0)$.)

- (\oplus)₅ $\mathcal{T}_i = (T_i, \text{rk}_i)$ is a standard $(w_i, i+1)^\gamma$ -tree, $\mathcal{T}_i = \text{proj}_{(w_i, i+1)}^{(w_{i+1}, i+2)}(\mathcal{T}_{i+1})$.
- (\oplus)₆ $\bar{p}_*^i = \langle p_{*,t}^i : t \in T_i \rangle$ and $\bar{q}_*^i = \langle q_{*,t}^i : t \in T_i \rangle$ are standard trees of conditions, $\bar{p}_*^i \leq \bar{q}_*^i \leq_{w_i, i+1}^{w_{i+1}, i+2} \bar{p}_*^{i+1}$.
- (\oplus)₇⁰ $\text{Dom}(p_{*,t}^0) = (\{0\} \cup \text{Dom}(p)) \cap \text{rk}_0(t)$ for each $t \in T_0$ and $p_{*,t}^0(\xi) = p(\xi)$ for $\xi \in \text{Dom}(p_{*,t}^0) \setminus \{0\}$, $t \in T_0$.
- (\oplus)₇ⁱ⁺¹ For $t \in T_{i+1}$ we have $\text{Dom}(p_{*,t}^{i+1}) = (\text{Dom}(r_i) \cup w_{i+1}) \cap \text{rk}_{i+1}(t)$ and $p_{*,t}^{i+1}(\xi) = r_i(\xi)$ for $\xi \in \text{Dom}(p_{*,t}^{i+1}) \setminus w_{i+1}$.
- (\oplus)₈ $k_i = |\{t \in T_i : \text{rk}_i(t) = \gamma\}|$, $\bar{t}^i = \langle t_\ell^i : \ell < k_i \rangle$ is an enumeration of $\{t \in T_i : \text{rk}_i(t) = \gamma\}$, and for each $t \in T_i$ with $\text{rk}_i(t) = \gamma$ we have

$$p_{*,t}^i \leq p_t^i \leq q_t^i \leq q_{*,t}^i.$$

- (\oplus)₉ If $\xi \in w_i$, then $s_{i,\xi} \subseteq \bigcup_{j \leq i+1-i_\xi^*}^j (n+1)$ is an n -tree and $\bar{p}_{i,\xi} = \langle p_{i,\xi}^\eta : \eta \in \max(s_{i,\xi}) \rangle$, $\bar{q}_{i,\xi} = \langle q_{i,\xi}^\eta : \eta \in \max(s_{i,\xi}) \rangle$ are \mathbb{P}_ξ -names for systems of conditions in \mathbb{Q}_ξ (indexed by $\max(s_{i,\xi})$).

- (\oplus)₁₀ For each $\xi \in \bigcup_{i < \omega} w_i$,

$\Vdash_{\mathbb{P}_\xi}$ “ $\langle s_{i,\xi}, \bar{p}_{i,\xi}, \bar{q}_{i,\xi} : i_\xi^* \leq i < \omega \rangle$ is a legal play of $\mathfrak{D}_n^\oplus(r_{i_\xi^*}(\xi), \mathbb{Q}_\xi)$ in which Generic uses \mathbf{st}_ξ ”.

- (\oplus)₁₁ If $t \in T_i$, $\text{rk}_i(t) = \xi < \gamma$ (so $\xi \in w_i$ and $i \geq i_\xi^*$), then

$$\{(s)_\xi : t \triangleleft s \in T_i\} = \{\eta : \eta \upharpoonright i_\xi^* \in i_\xi^* \{*\} \ \& \ (\exists \nu \in \max(s_{i,\xi})) (\eta = (\eta \upharpoonright i_\xi^*) \frown \nu)\}.$$

- (\oplus)₁₂ If $t \in T_i$, $\xi < \text{rk}_i(t)$, $\xi \in w_i$ and $(t)_\xi = ((t)_\xi \upharpoonright i_\xi^*) \frown \nu$ (so $\nu \in \max(s_{i,\xi})$), then

$$p_{*,t}^i \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} “ p_{*,t}^i(\xi) = p_{i,\xi}^\nu ” \quad \text{and} \quad q_{*,t}^i \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} “ q_{*,t}^i(\xi) = q_{i,\xi}^\nu ”.$$

- (\oplus)₁₃ If $t_0, t_1 \in T_i$, $\text{rk}_i(t_0) = \text{rk}_i(t_1)$ and $\xi \in w_i \cap \text{rk}_i(t_0)$, $t_0 \upharpoonright \xi = t_1 \upharpoonright \xi$ but $(t_0)_\xi \neq (t_1)_\xi$, then

$$q_{*,t_0}^i \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} “ \text{the conditions } q_{*,t_0}^i(\xi), q_{*,t_1}^i(\xi) \text{ are incompatible } ”.$$

- (\oplus)₁₄ $\text{Dom}(r_i) = \bigcup_{t \in T_i} \text{Dom}(q_{*,t}^i) \cup \text{Dom}(p)$ and if $t \in T_i$, $\xi \in \text{Dom}(r_i) \cap \text{rk}_i(t) \setminus w_i$, then $q_{*,t}^i \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} “ r_i(\xi) \geq q_{*,t}^i(\xi) ”$.

To describe the instructions given by \mathbf{st} at stage $i < \omega$ of a play of $\mathfrak{D}_n^\oplus(p, \mathbb{P}_\gamma)$ let us assume that $\langle (\mathcal{T}_j, \bar{t}^j, \bar{p}^j, \bar{q}^j) : j < i \rangle$ is the result of the play so far and that Generic constructed aside the objects appearing in (\oplus)₁–(\oplus)₁₄ (and they have the respective properties).

For definiteness of our definitions, whenever we say “Generic chooses/picks X such that” we really mean “Generic takes the $<_\chi^*$ -first X such that”.

First, Generic uses her favourite bookkeeping device to determine w_i such that the demands in (\oplus)₁ are satisfied (and that at the end we will have $\bigcup_{j < \omega} \text{Dom}(r_j) =$

$\bigcup_{j < \omega} w_j$) and then again she uses the bookkeeping device to determine K_ξ so that

(\oplus)₃ + (\oplus)₄ hold. Note that i_ξ^* for $\xi \in w_i$ is defined by (\oplus)₄, also the choice of \mathbf{st}_ξ is determined by (\oplus)₄ (remember that $r_{i_\xi^*}(\xi)$ is determined by (\oplus)₂).

Now $(\oplus)_9 + (\oplus)_{10}$ decide $s_{i,\xi}$ (for $\xi \in w_i$) and since \mathbf{st}_ξ is (a name for) a nice for $K_\xi - i_\xi^*$ strategy, we know that $s_{i,\xi}$ can be easily read from the truth value of “ $i + i_\xi^* \in K_\xi$ ”. Plainly $\max(s_{i,\xi}) \subseteq {}^{i+1-i_\xi^*}(n+1)$ and the clauses mentioned before determine $\bar{p}_{i,\xi} = \langle p_{i,\xi}^\eta : \eta \in \max(s_{i,\xi}) \rangle$. Now the choice of the standard tree \mathcal{T}_i is fully described by $(\oplus)_5 + (\oplus)_{11}$ and clearly $(\oplus)_0$ holds then too. For each $t \in T_i$ Generic picks a condition $p_{*,t}^i \in \mathbb{P}_{\text{rk}_i(t)}$ so that the demands of $(\oplus)_7 + (\oplus)_{12}$ are satisfied. (One may use $(\oplus)_{14}$ to argue that the last demand in $(\oplus)_6$ is satisfied.)

After the above choices are made, Generic (in the play of $\mathcal{D}_n^\circ(p, \mathbb{P}_\gamma)$) puts T_i as her inning and Antigeneric chooses an enumeration $\bar{t}^i = \langle t_\ell^i : \ell < k_i \rangle$ of $\{t \in T_i : \text{rk}_i(t) = \gamma\}$. Now the two players start a subgame of length $k_i = |\{t \in T_i : \text{rk}_i(t) = \gamma\}|$. During the subgame Generic will also pick (for temporary use) trees of conditions $\bar{q}_\ell^{\text{tmp}} = \langle q_{\ell,t}^{\text{tmp}} : t \in T_i \rangle$ for $\ell \leq k_i$. So, she lets $\bar{q}_0^{\text{tmp}} = \bar{p}_*^i$ and she plays $p_{t_0^i}^i = q_{0,t_0^i}^{\text{tmp}}$ as her first inning in the subgame. Antigeneric answers with $q_{t_0^i}^i \geq p_{t_0^i}^i$ after which Generic picks a tree of conditions \bar{q}_1^{tmp} so that $q_{1,t_0^i}^{\text{tmp}} = q_{t_0^i}^i$ and for each $0 < \ell < k_i$

- if $t \triangleleft t_0^i$, $t \triangleleft t_\ell^i$ and $\text{rk}_i(t)$ is the largest possible, then

$$q_{1,t_\ell^i}^{\text{tmp}} = q_{t_\ell^i}^i \upharpoonright \text{rk}_i(t) \frown q_{0,t_\ell^i}^{\text{tmp}} \upharpoonright [\text{rk}_i(t), \gamma).$$

Now, if the players arrived to level $\ell^* < k_i$ of the subgame and $\bar{q}_{\ell^*}^{\text{tmp}}$ was chosen, then Generic plays $p_{t_{\ell^*}^i}^i = q_{\ell^*,t_{\ell^*}^i}^{\text{tmp}}$. After Antigeneric answered with $q_{t_{\ell^*}^i}^i \geq p_{t_{\ell^*}^i}^i$, Generic builds a tree of conditions $\bar{q}_{\ell^*+1}^{\text{tmp}}$ so that $q_{\ell^*+1,t_{\ell^*}^i}^{\text{tmp}} = q_{t_{\ell^*}^i}^i$ and for each $\ell < k_i$, $\ell \neq \ell^*$

- if $t \triangleleft t_{\ell^*}^i$, $t \triangleleft t_\ell^i$ and $\text{rk}_i(t)$ is the largest possible, then

$$q_{\ell^*+1,t_\ell^i}^{\text{tmp}} = q_{t_\ell^i}^i \upharpoonright \text{rk}_i(t) \frown q_{\ell^*,t_\ell^i}^{\text{tmp}} \upharpoonright [\text{rk}_i(t), \gamma).$$

When the subgame is over Generic lets $\bar{q}_*^i = \bar{q}_{k_i}^{\text{tmp}}$. Note that the demand of $(\oplus)_{13}$ is satisfied because the strategies \mathbf{st}_ξ are nice, also the relevant parts of $(\oplus)_6 + (\oplus)_8$ hold. The names $\bar{q}_{i,\xi}$ (for $\xi \in w_i$) are chosen so that $q_{*,t}^i \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} “q_{i,\xi}^\nu = q_{*,t}^i(\xi)”$ and $\Vdash_{\mathbb{P}_\xi} “p_{i,\xi}^\nu \leq \bar{q}_{i,\xi}^\nu”$ whenever $t \in T_i$, $\xi \in w_i \cap \text{rk}_i(t)$, $((t)_\xi \upharpoonright i_\xi^*) \frown \nu = (t)_\xi$, $\nu \in \max(s_{i,\xi})$. Then $(\oplus)_9 + (\oplus)_{12}$ are satisfied. Finally Generic chooses $r_i \in \mathbb{P}_\gamma$ essentially by conditions $(\oplus)_2 + (\oplus)_{14}$ (and our rule of picking “the $<_\chi^*$ -first such that”).

This completes the description of the side objects constructed by Generic and her innings at stage i . We also verified that clauses $(\oplus)_0 - (\oplus)_{14}$ hold and thus the description of the strategy is complete.

We are going to argue that \mathbf{st} is a winning strategy for Generic in $\mathcal{D}_n^\circ(p, \mathbb{P}_\gamma)$. To this end suppose that $\langle (\mathcal{T}_i, \bar{t}^i, \bar{p}^i, \bar{q}^i) : i < \omega \rangle$ is the result of a play of $\mathcal{D}_n^\circ(p, \mathbb{P}_\gamma)$ in which Generic used \mathbf{st} , and the objects constructed at each stage $i < \omega$ are

$$(\square) w_i, \mathcal{T}_i, \bar{t}^i, \bar{p}^i, \bar{q}^i, r_i, \bar{p}_*^i, \bar{q}_*^i, k_i, K_\xi, i_\xi^*, \mathbf{st}_\xi, s_{i,\xi}, \bar{p}_{i,\xi}, \bar{q}_{i,\xi} \text{ for } \xi \in w_i,$$

and they satisfy the requirements $(\oplus)_0 - (\oplus)_{14}$.

We define a condition $q \in \mathbb{P}_\gamma$ as follows. Let $\text{Dom}(q) = \bigcup_{i < \omega} w_i = \bigcup_{i < \omega} \text{Dom}(r_i)$ and for $\xi \in \text{Dom}(q)$ let $q(\xi)$ be a \mathbb{P}_ξ -name for a condition in \mathbb{Q}_ξ such that

$$\Vdash_{\mathbb{P}_\xi} “q(\xi) \geq r_{i_\xi^*}(\xi) \text{ and } q(\xi) \Vdash_{\mathbb{Q}_\xi} (\forall i \geq i_\xi^*) (\exists \nu \in \max(s_{i,\xi})) (q_{i,\xi}^\nu \in \Gamma_{\mathbb{Q}_\xi})”.$$

Clearly q is well defined (remember $(\oplus)_{10}$) and $q \geq p$ (remember $(\oplus)_1 + (\oplus)_2$). Also $q \geq r_i$ for all $i < \omega$.

We will show that for each $i < \omega$ the family $\{q_{*,t}^i : t \in T_i \text{ \& } \text{rk}_i(t) = \gamma\}$ is predense above q (and this clearly will imply that Generic won the play). So suppose $q^+ \geq q$, $i < \omega$ and $w_i \cup \{\gamma\} = \{\xi_0, \xi_1, \dots, \xi_i, \xi_{i+1}\}$ (the increasing enumeration, so $\xi_0 = 0$). By induction on $j \leq i$ we choose an increasing sequence $\langle q_j^+ : j \leq i \rangle \subseteq \mathbb{P}_\gamma$ and we will also define $t \upharpoonright \xi_j + 1$.

First, by the choice of $q(0)$ there is $\nu \in \max(s_{i,\xi_0})$ such that the conditions $q^+(0)$ and $q_{i,\xi_0}^\nu = q_{i,\xi_0}^\nu$ are compatible. Let $(t)_0 = \nu$ (this defines $t \upharpoonright (\xi_0 + 1)$). Let $q_0^+ \in \mathbb{P}_{\xi_1}$ be such that $q_0^+(0)$ is stronger than both $q^+(0)$ and q_{i,ξ_0}^ν , and let $q_0^+ \upharpoonright (\xi_0, \xi_1) = q^+ \upharpoonright (\xi_0, \xi_1)$. It follows from $(\oplus)_{12} + (\oplus)_{14}$ that q_0^+ is stronger than $q_{*,t \upharpoonright (\xi_0+1)}^i$ (and, of course, it is stronger than $q^+ \upharpoonright \xi_1$). Now suppose that $j < i$ and we have defined $t \upharpoonright (\xi_j + 1) \in T_i$ and a condition $q_j^+ \in \mathbb{P}_{\xi_{j+1}}$ stronger than both $q^+ \upharpoonright \xi_{j+1}$ and $q_{*,t \upharpoonright (\xi_j+1)}^i$. Necessarily

$$q_j^+ \Vdash_{\mathbb{P}_{\xi_{j+1}}} \text{ “ } (\exists \nu \in \max(s_{i,\xi_{j+1}})) (q_{i,\xi_{j+1}}^\nu, q^+(\xi_{j+1}) \text{ are compatible) ”}$$

so we may choose $\nu \in \max(s_{i,\xi_{j+1}})$ and a condition $q_{j+1} \in \mathbb{P}_{\xi_{j+1}}$ stronger than q_j^+ such that

$$q_{j+1} \Vdash_{\mathbb{P}_{\xi_{j+1}}} \text{ “ } q_{i,\xi_{j+1}}^\nu, q^+(\xi_{j+1}) \text{ are compatible ”.}$$

Let $(t)_{\xi_{j+1}} = \langle * \dots * \rangle \smallfrown \nu$ (thus $t \upharpoonright (\xi_{j+1} + 1)$ has been defined) and let $q_{j+1}^+ \in \mathbb{P}_{\xi_{j+2}}$ be such that $q_{j+1}^+ \upharpoonright \xi_{j+1} = q_{j+1}$,

$$q_{j+1}^+ \upharpoonright \xi_{j+1} \Vdash_{\mathbb{P}_{\xi_{j+1}}} \text{ “ } q_{j+1}^+(\xi_{j+1}) \geq q_{i,\xi_{j+1}}^\nu \quad \& \quad q_{j+1}^+(\xi_{j+1}) \geq q^+(\xi_{j+1}) \text{ ”}$$

and $q_{j+1}^+ \upharpoonright (\xi_{j+1}, \xi_{j+2}) = q^+ \upharpoonright (\xi_{j+1}, \xi_{j+2})$. Then by $(\oplus)_{12} + (\oplus)_{14}$ the condition q_{j+1}^+ is stronger than $q_{*,t \upharpoonright (\xi_{j+1}+1)}^i$ and $q^+ \upharpoonright \xi_{j+2}$.

Finally look at $t = t \upharpoonright \xi_{i+1}$ and q_{i+1}^+ .

(2) Since we do not know if “the \odot_n -property” implies “the n -localization property”, we cannot just say that the statement in (2) follows from (1). However, the reason for the weaker “ \odot_n ” in the conclusion of 2.1(1) (and not “ \oplus_n ”) is that in our description of the strategy **st**, we have to make sure that the conditions played by Antigeneric form a tree of conditions.

So to show that \mathbb{P}_γ has the n -localization property we use 1.8 and the procedure described in the proof of 2.1(1). Suppose that \mathcal{T} is a \mathbb{P}_γ -name for an element of ${}^\omega\omega$; we may assume that $\Vdash_{\mathbb{P}_\gamma} (\forall \alpha < \gamma) (\mathcal{T} \notin \mathbf{V}^{\mathbb{P}^\alpha})$. Let $p \in \mathbb{P}_\gamma$. Construct a sequence

$$\langle w_i, \mathcal{T}_i, \vec{p}_*^i, \vec{q}_*^i, r_i, \langle i_\xi^*, \mathbf{st}_\xi, K_\xi : \xi \in w_i \rangle, \vec{\sigma}^i, \vec{m} : i < \omega \rangle$$

such that conditions $(\oplus)_{0-} - (\oplus)_7$ and $(\oplus)_{9-} - (\oplus)_{14}$ are satisfied and

- $(\oplus)_{15}$ $\vec{m} = \langle m_i : i < \omega \rangle \subseteq \omega$, $0 = m_0 \leq m_i < m_{i+1}$ for $i < \omega$,
- $(\oplus)_{16}$ $\vec{\sigma}^i = \langle \sigma_t^i : t \in T_i \text{ \& } \text{rk}_i(t) = \gamma \rangle \subseteq [m_i, m_{i+1})^\omega$ and if $t, t' \in T^i$, $t \neq t'$, $\text{rk}_i(t) = \text{rk}_i(t') = \gamma$, then $\sigma_t^i \neq \sigma_{t'}^i$, and
- $(\oplus)_{17}$ $q_{*,t}^i \Vdash_{\mathbb{P}_\gamma} \sigma_t^i \triangleleft \mathcal{T}$ for $t \in T_i$, $\text{rk}_i(t) = \gamma$.

Then pick $q \in \mathbb{P}_\gamma$ stronger than p and such that for each $i < \omega$ the family $\{q_{*,t}^i : t \in T_i \text{ \& } \text{rk}_i(t) = \gamma\}$ is predense above q (this is done exactly as in part (1)). Let $S \subseteq {}^{>\omega}\omega$ be a tree such that

$$[S] = \{f \in {}^\omega\omega : (\forall i < \omega) (\exists t \in T_i) (\text{rk}_i(t) = \gamma \text{ \& } f \upharpoonright [m_i, m_{i+1}) = \sigma_t^i)\}.$$

Then S is an n -ary tree and $q \Vdash_{\mathbb{P}_\gamma} \mathcal{T} \in [S]$. \square

3. \odot_n -PROPERTY AND CS ITERATIONS

The result of the previous section is not applicable to $\mathbb{Q}_n, \mathbb{S}_n$ as these forcing have nice \odot_n -property only. An iteration theorem suitable for that property is presented below. It is not sufficient for claiming the n -localization property, so later we formulate yet another property and we argue that it implies the n -localization of the limits of CS iterations.

Theorem 3.1. *If $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \gamma \rangle$ is a CS iteration such that for every $\xi < \gamma$,*

$$\Vdash_{\mathbb{P}_\xi} \text{ “ } \mathbb{Q}_\xi \text{ has the nice } \odot_n\text{-property ”,}$$

then $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ has the \odot_n -property.

Proof. Let $p \in \mathbb{P}_\gamma$. We are going to describe a strategy \mathbf{st} for Generic in the game $\mathfrak{D}_n^\odot(p, \mathbb{P}_\gamma)$. As in the proof of Theorem 2.1, the strategy \mathbf{st} will give Generic, at a stage $i < \omega$, a standard $(w_i, (i+1)^\gamma)$ -tree $\mathcal{T}_i = (T_i, \text{rk}_i)$ such that $\mathcal{T}_i = \text{proj}_{(w_i, i+1)}^{(w_{i+1}, i+2)}(\mathcal{T}_{i+1})$ and

(\odot)₀ for each $t' \in T_i$ with $\text{rk}_i(t) = \gamma$ we have

$$0 < |\{t \in T_{i+1} : \text{proj}_{(w_i, i+1)}^{(w_{i+1}, i+2)}(t) = t'\}| \leq n.$$

Generic will also pick sets $K_\xi \in [\omega]^\omega$, conditions $r_i \in \mathbb{P}_\gamma$ and $k_i, i_\xi^*, \mathbf{st}_\xi, s_{i,\xi}, \bar{p}_{i,\xi}, \bar{q}_{i,\xi}$. All these objects will be constructed so that, assuming $\langle (\mathcal{T}_i, \bar{p}^i, \bar{q}^i) : i < \omega \rangle$ is the result of a play in which Generic used \mathbf{st} and she determined the corresponding side objects, the following demands are satisfied.

(\odot)₁ $r_0(0) = p(0)$, $w_i \in [\gamma]^{i+1}$, $w_0 = \{0\}$, $w_i \subseteq w_{i+1}$ and $\bigcup_{i < \omega} \text{Dom}(r_i) = \bigcup_{i < \omega} w_i$.

(\odot)₂ If $j < i < \omega$, then $(\forall \xi \in w_{j+1})(r_j(\xi) = r_i(\xi))$ and $p \leq r_j \leq r_i$.

(\odot)₃ If $\xi \in w_i$, then $K_\xi \in [\omega]^\omega$ is known at stage i of the play and if $\xi, \zeta \in \bigcup_{i < \omega} w_i$

are distinct, then $K_\zeta \cap K_\xi = \emptyset$.

(\odot)₄ For $\xi \in \bigcup_{i < \omega} w_i$ we have $i_\xi^* = \min(\{i : \xi \in w_i\}) \leq \min(K_\xi)$, and \mathbf{st}_ξ is a \mathbb{P}_ξ -name for a winning strategy of Generic in $\mathfrak{D}_n^\odot(r_{i_\xi^*}(\xi), \mathbb{Q}_\xi)$ which is nice for $\{k \in \omega : k + i_\xi^* \in K_\xi\}$ (see 1.2). (So \mathbf{st}_0 is a K_0 -nice winning strategy of Generic in $\mathfrak{D}_n^\odot(r_0(0), \mathbb{Q}_0)$.)

(\odot)₅ If $\xi \in w_i$, then $s_{i,\xi} \subseteq \bigcup_{j \leq i+1-i_\xi^*} j(n+1)$ is an n -tree and $\bar{p}_{i,\xi} = \langle \bar{p}_{i,\xi}^\eta : \eta \in \max(s_{i,\xi}) \rangle$, $\bar{q}_{i,\xi} = \langle \bar{q}_{i,\xi}^\eta : \eta \in \max(s_{i,\xi}) \rangle$ are \mathbb{P}_ξ -names for systems of conditions in \mathbb{Q}_ξ (indexed by $\max(s_{i,\xi})$).

(\odot)₆ For each $\xi \in \bigcup_{i < \omega} w_i$,

$$\Vdash_{\mathbb{P}_\xi} \text{ “ } \langle s_{i,\xi}, \bar{p}_{i,\xi}, \bar{q}_{i,\xi} : i_\xi^* \leq i < \omega \rangle \text{ is a legal play of } \mathfrak{D}_n^\odot(r_{i_\xi^*}(\xi), \mathbb{Q}_\xi) \text{ in which Generic uses } \mathbf{st}_\xi \text{ and the orders of } \max(s_{i,\xi}) \text{ chosen by Antigeneric are given by } <_\chi^* \text{ ”.}$$

(\odot)₇ $\mathcal{T}_i = (T_i, \text{rk}_i)$ is a standard $(w_i, i+1)^\gamma$ -tree, $\mathcal{T}_i = \text{proj}_{(w_i, i+1)}^{(w_{i+1}, i+2)}(\mathcal{T}_{i+1})$.

(\odot)₈ If $t \in T_i$, $\text{rk}_i(t) = \xi < \gamma$ (so $\xi \in w_i$ and $i \geq i_\xi^*$), then

$$\{(s)_\xi : t \triangleleft s \in T_i\} = \{\eta : \eta \upharpoonright i_\xi^* \in i_\xi^* \{*\} \ \& \ (\exists \nu \in \max(s_{i,\xi}))(\eta = (\eta \upharpoonright i_\xi^*) \smallfrown \nu)\}.$$

(\odot)₉ $k_i = |\{t \in T_i : \text{rk}_i(t) = \gamma\}|$.

(\odot)₁₀ If $\langle t_\ell^i : \ell < k_i \rangle$ is the list of $\{t \in T_i : \text{rk}_i(t) = \gamma\}$ chosen by Generic, $\ell < m < k_i$, $\xi \in w_i$ and $t_\ell^i \upharpoonright \xi = t_m^i \upharpoonright \xi$ but $(t_\ell^i)_\xi \neq (t_m^i)_\xi$, then $q_{t_\ell^i}^i \upharpoonright \xi \leq p_{t_m^i}^i \upharpoonright \xi$ and

$$p_{t_m^i}^i \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} \text{“ the conditions } p_{t_\ell^i}^i(\xi), p_{t_m^i}^i(\xi) \text{ are incompatible ”.}$$

(\odot)₁₁ If $t \in T_i$, $\text{rk}_i(t) = \gamma$, $\xi \in w_i$ and $(t)_\xi = (t)_\xi \upharpoonright i_\xi^* \frown \eta$, $\eta \in \max(s_{i,\xi})$, then

$$p_t^i \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} p_{t_\xi}^\eta \leq p_t^i(\xi) \quad \text{and} \quad q_t^i \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} q_t^i(\xi) \leq q_{t_\xi}^\eta.$$

(\odot)₁₂ $\text{Dom}(r_i) = \bigcup_{t \in T_i} \text{Dom}(q_t^i) \cup \text{Dom}(p)$ and if $t \in T_i$, $\text{rk}_i(t) = \gamma$, $\xi \in \text{Dom}(r_i) \setminus w_i$, then $q_t^i \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} \text{“ } q_t^i(\xi) \leq r_i(\xi) \text{ ”.}$

To describe the instructions given by **st** at stage $i < \omega$ of a play of $\mathcal{D}_n^\odot(p, \mathbb{P}_\gamma)$ let us assume that $\langle (\mathcal{T}_j, \bar{p}^j, \bar{q}^j) : j < i \rangle$ is the result of the play so far and that Generic constructed aside the objects appearing in (\odot)₁–(\odot)₁₂ (and they have the respective properties).

For definiteness of our definitions, whenever we say “Generic chooses/picks X such that” we really mean “Generic takes the $<_\chi^*$ –first X such that”.

First, Generic uses her favourite bookkeeping device to determine w_i and K_ξ so that (\odot)₁ + (\odot)₃ + (\odot)₄ hold. Note that (\odot)₄ determines i_ξ^* and \mathbf{st}_ξ for $\xi \in w_i$ (remember that $r_{i_\xi^*}(\xi)$ is given by (\odot)₂). Now (\odot)₅ + (\odot)₆ and the truth value of “ $i + i_\xi^* \in K_\xi$ ” decide $s_{i,\xi}$ (for $\xi \in w_i$), remember that \mathbf{st}_ξ is (a name for) a strategy which is nice for $K_\xi - i_\xi^*$. Plainly $\max(s_{i,\xi}) \subseteq {}^{i+1-i_\xi^*}(n+1)$. The choice of the standard tree \mathcal{T}_i is fully described by (\odot)₇ + (\odot)₈ and clearly (\odot)₀ holds then too. Also k_i is given by (\odot)₉.

Let $\{\zeta_j : j < j^*\}$ be the increasing enumeration of $\{\xi \in w_i : |\max(s_{i,\xi})| > 1\}$ (so $j^* \leq i+1$ and we may assume that $j^* \neq 0$). We will think of $\max(s_{i,\xi})$ (for $\xi \in w_i$) as linearly ordered by $<_\chi^*$ (restricted suitably). This linear order determines a list of $\max(s_{i,\xi})$ which will be considered as an inning of Antigeneric in answer to the choice of $s_{i,\xi}$ by Generic in $\mathcal{D}_n^\odot(r_{i_\xi^*}(\xi), \mathbb{Q}_\xi)$. We may identify $\{(t)_\xi : t \in T_i \ \& \ \text{rk}_i(t) = \xi\}$ with $\max(s_{i,\xi})$ by the mapping $(t)_\xi \mapsto (t)_\xi \upharpoonright [i_\xi^*, i)$, so in particular the linear order of $\max(s_{i,\xi})$ determines a linear ordering of $\{(t)_\xi : t \in T_i \ \& \ \text{rk}_i(t) > \xi\}$ (for $\xi \in w_i$). Generic takes the lexicographic product of these orderings for all coordinates $\xi \in w_i$ and she lets $\langle t_m^i : m < k_i \rangle$ be the increasing (in this lexicographic order) enumeration of $\{t \in T_i : \text{rk}_i(t) = \gamma\}$. Then for each $t \in T_i$ with $\text{rk}_i(t) = \gamma$ and $\xi \in w_i$ we have

(\odot)₁₃ for some $m_0 < m_1 \leq k_i$,

$$\{t' \in T_i : \text{rk}_i(t') = \gamma \ \& \ t' \upharpoonright \xi = t \upharpoonright \xi\} = \{t_m^i : m_0 \leq m < m_1\}.$$

The interval $[m_0, m_1)$ as in (\odot)₁₃ will be called *the neighbourhood of t at ξ* . Note that if $\xi = \zeta_j$ for some $j < j^*$, then $m_1 > m_0 + 1$, otherwise $m_1 = m_0 + 1$. For $j < j^*$ let m_j be such that $[0, m_j)$ is the neighbourhood of t_0^i at ζ_j (so $m_{j^*-1} < m_{j^*-2} < \dots < m_0 = k_i$).

Now the two players start a subgame of length k_i .

For $j < m_{j^*-1}$ Generic proceeds in the subgame as follows. First $p_{t_0^i}^i \in \mathbb{P}_\gamma$ is such that

$$\text{Dom}(p_{t_0^i}^i) = \begin{cases} w_0 \cup \text{Dom}(p) & \text{if } i = 0, \\ w_i \cup \text{Dom}(r_{i-1}) & \text{if } i > 0, \end{cases}$$

and if $\xi \in \text{Dom}(p_{t_0}^i) \setminus w_i$ then

$$p_{t_0}^i(\xi) = \begin{cases} p(\xi) & \text{if } i = 0, \\ r_{i-1}(\xi) & \text{if } i > 0. \end{cases}$$

For $\xi \in w_i$, $p_{t_0}^i(\xi)$ is a \mathbb{P}_ξ -name for an element of \mathbb{Q}_ξ such that

$$\begin{aligned} \Vdash_{\mathbb{P}_\xi} \quad & \text{“ } p_{t_0}^i(\xi) \text{ is the condition given to Generic by } \mathbf{st}_\xi \text{ in the subgame of} \\ & \mathcal{D}_n^\odot(r_{i^*}(\xi), \mathbb{Q}_\xi) \text{ after } \langle s_{j,\xi}, \bar{p}_{j,\xi}, \bar{q}_{j,\xi} : i_\xi^* \leq j < i \rangle \text{ was played and} \\ & \text{Antigeneric picked the } <^*_\chi \text{-increasing enumeration of } \max(s_{i,\xi}) \text{”} \end{aligned}$$

A straightforward induction (using $(\odot)_{12} + (\odot)_{11}$) shows that

$$(\odot)_{14}^0 \text{ if } i > 0, t' = \text{proj}_{(w_{i-1}, i)}^{(w_i, i+1)}(t_0^i), \text{ then } p_{t_0}^i \geq q_{t'}^{i-1} \text{ (and } p_{t_0}^0 \geq p).$$

The condition $p_{t_0}^i$ is the first inning of Generic in the subgame, after which Antigeneric answers with $q_{t_0}^i \geq p_{t_0}^i$.

Now suppose that the two players arrived to a step $0 < m < m_{j^*-1}$ of the subgame. The inning of Generic now is defined similarly to that at stage 0: $p_{t_m}^i \in \mathbb{P}_\gamma$ is such that

$$\text{Dom}(p_{t_m}^i) = \begin{cases} w_0 \cup \text{Dom}(p) & \text{if } i = 0, \\ w_i \cup \text{Dom}(r_{i-1}) \cup (\text{Dom}(q_{t_{m-1}}^i) \cap \zeta_{j^*-1}) & \text{if } i > 0, \end{cases}$$

and $p_{t_m}^i \upharpoonright \zeta_{j^*-1} = q_{t_{m-1}}^i \upharpoonright \zeta_{j^*-1}$, and $p_{t_m}^i(\xi) = p_{t_0}^i(\xi)$ for $\xi \in \text{Dom}(p_{t_m}^i) \setminus (\zeta_{j^*-1} + 1)$,

and finally $p_{t_m}^i(\zeta_{j^*-1})$ is a $\mathbb{P}_{\zeta_{j^*-1}}$ -name for an element of $\mathbb{Q}_{\zeta_{j^*-1}}$ such that

$$\begin{aligned} p_{t_m}^i \upharpoonright \zeta_{j^*-1} \Vdash_{\mathbb{P}_{\zeta_{j^*-1}}} \quad & \text{“ } p_{t_m}^i(\zeta_{j^*-1}) \text{ is the inning of Generic given by } \mathbf{st}_{\zeta_{j^*-1}} \text{ in} \\ & \text{the subgame of level } i \text{ of } \mathcal{D}_n^\odot(r_{i^*}(\zeta_{j^*-1}), \mathbb{Q}_{\zeta_{j^*-1}}) \text{ after the two} \\ & \text{players played } \langle p_{t_0}^i(\zeta_{j^*-1}), q_{t_0}^i(\zeta_{j^*-1}) \rangle, \dots, \langle p_{t_{m-1}}^i(\zeta_{j^*-1}), q_{t_{m-1}}^i(\zeta_{j^*-1}) \rangle \\ & \text{as the conditions attached to } (t_0^i)_{\zeta_{j^*-1}}, \dots, (t_{m-1}^i)_{\zeta_{j^*-1}}, \text{ respectively”}. \end{aligned}$$

Again, one may verify by induction on $\xi \in \text{Dom}(p_{t_m}^i)$ that

$$(\odot)_{14}^m \text{ if } i > 0, t' = \text{proj}_{(w_{i-1}, i)}^{(w_i, i+1)}(t_m^i), \text{ then } p_{t_m}^i \geq q_{t'}^{i-1} \text{ (and } p_{t_m}^0 \geq p).$$

Now, $p_{t_m}^i$ is Generic's inning at this stage of the subgame, and after this Antigeneric answers with $q_{t_m}^i \geq p_{t_m}^i$.

Thus we have described how Generic plays in the first m_{j^*-1} steps of the subgame — let us call this procedure $\text{proc}_{j^*-1}(0, m_{j^*-1}, t_0^i, p_{t_0}^i)$.

Suppose that $m_{j^*-1} < k_i$ and let $m' > m_{j^*-1}$ be such that $[m_{j^*-1}, m')$ is the neighbourhood of $t_{m_{j^*-1}}^i$ at ζ_{j^*-1} (so $j^* > 1$ and so $i > 0$). Let $p_{t_{m_{j^*-1}}}^i \in \mathbb{P}_\gamma$ be such that

$$\text{Dom}(p_{t_{m_{j^*-1}}}^i) = w_i \cup \text{Dom}(r_{i-1}) \cup (\text{Dom}(q_{t_{m_{j^*-1}-1}}^i) \cap \zeta_{j^*-2})$$

and $p_{t_{m_{j^*-1}}}^i \upharpoonright \zeta_{j^*-2} = q_{t_{m_{j^*-1}-1}}^i \upharpoonright \zeta_{j^*-2}$, and $p_{t_{m_{j^*-1}}}^i(\zeta_{j^*-2})$ is a $\mathbb{P}_{\zeta_{j^*-2}}$ -name for an element of $\mathbb{Q}_{\zeta_{j^*-2}}$ such that

$$\begin{aligned} p_{t_{m_{j^*-1}}}^i \upharpoonright \zeta_{j^*-2} \Vdash_{\mathbb{P}_{\zeta_{j^*-2}}} \quad & \text{“ } p_{t_{m_{j^*-1}}}^i(\zeta_{j^*-2}) \text{ is the inning of Generic given by } \mathbf{st}_{\zeta_{j^*-2}} \\ & \text{in the subgame of level } i \text{ of } \mathcal{D}_n^\odot(r_{i^*}(\zeta_{j^*-2}), \mathbb{Q}_{\zeta_{j^*-2}}) \text{ after the two players} \\ & \text{played } \langle p_{t_0}^i(\zeta_{j^*-2}), q_{t_{m_{j^*-1}-1}}^i(\zeta_{j^*-2}) \rangle \text{ as the conditions attached to } (t_0^i)_{\zeta_{j^*-2}} \text{”}, \end{aligned}$$

and finally $p_{t_{m_{j^*-1}}^i}^i(\xi) = p_{t_0^i}^i(\xi)$ for $\xi \in \text{Dom}(p_{t_{m_{j^*-1}}^i}^i) \setminus (\zeta_{j^*-2} + 1)$. Again, a straightforward induction shows that

$$(\odot)_{14}^{m_{j^*-1}} \text{ if } t' = \text{proj}_{(w_{i-1}, i)}^{(w_i, i+1)}(t_{m_{j^*-1}}^i), \text{ then } p_{t_{m_{j^*-1}}^i}^i \geq q_{t'}^{i-1}.$$

The condition $p_{t_{m_{j^*-1}}^i}^i$ is played by Generic at stage m_{j^*-1} and from now till step m' she plays applying procedure $\text{proc}_{j^*-1}(m_{j^*-1}, m', t_{m_{j^*-1}}^i, p_{t_{m_{j^*-1}}^i}^i)$.

Then, if only $m' < m_{j^*-2}$, Generic takes m'' such that $[m', m'']$ is the neighbourhood of $t_{m'}^i$ at ζ_{j^*-1} . She defines $p_{t_{m'}^i}^i \in \mathbb{P}_\gamma$ like $p_{t_{m_{j^*-1}}^i}^i$, so

$$\text{Dom}(p_{t_{m'}^i}^i) = w_i \cup \text{Dom}(r_{i-1}) \cup (\text{Dom}(q_{t_{m'-1}}^i) \cap \zeta_{j^*-2})$$

and $p_{t_{m'}^i}^i \upharpoonright \zeta_{j^*-2} = q_{t_{m'-1}}^i \upharpoonright \zeta_{j^*-2}$, and $p_{t_{m'}^i}^i(\zeta_{j^*-2})$ is a $\mathbb{P}_{\zeta_{j^*-2}}$ -name for an element of $\mathbb{Q}_{\zeta_{j^*-2}}$ such that

$p_{t_{m'}^i}^i \upharpoonright \zeta_{j^*-2} \Vdash_{\mathbb{P}_{\zeta_{j^*-2}}} \text{“ } p_{t_{m'}^i}^i(\zeta_{j^*-2}) \text{ is the inning of Generic given by } \mathbf{st}_{\zeta_{j^*-2}} \text{ in the subgame of level } i \text{ of } \mathcal{D}_n^\odot(r_{\zeta_{j^*-2}}^i(\zeta_{j^*-2}), \mathbb{Q}_{\zeta_{j^*-2}}) \text{ after the two players played } \langle p_{t_0^i}^i(\zeta_{j^*-2}), q_{t_{m_{j^*-1}-1}^i(\zeta_{j^*-2})}^i \rangle, \langle p_{t_{m_{j^*-1}}^i}^i(\zeta_{j^*-2}), q_{t_{m'-1}^i}^i(\zeta_{j^*-2}) \rangle \text{ as the conditions attached to } (t_0^i)_{\zeta_{j^*-2}}, (t_{m_{j^*-1}}^i)_{\zeta_{j^*-2}}, \text{ respectively ”}$,

and $p_{t_{m'}^i}^i(\xi) = p_{t_0^i}^i(\xi)$ for $\xi \in \text{Dom}(p_{t_{m'}^i}^i) \setminus (\zeta_{j^*-2} + 1)$. Like before,

$$(\odot)_{14}^{m'} \text{ if } t' = \text{proj}_{(w_{i-1}, i)}^{(w_i, i+1)}(t_{m'}^i), \text{ then } p_{t_{m'}^i}^i \geq q_{t'}^{i-1}.$$

The condition $p_{t_{m'}^i}^i$ is played by Generic at stage m' and from now till step m'' she plays applying procedure $\text{proc}_{j^*-1}(m', m'', t_{m'}^i, p_{t_{m'}^i}^i)$. Following this pattern until the players get to level m_{j^*-2} (i.e., at the steps from 0 to $m_{j^*-2} - 1$) results in defining the procedure $\text{proc}_{j^*-2}(0, m_{j^*-2}, t_0^i, p_{t_0^i}^i)$.

Suppose we have defined the procedure proc_j , $0 < j < j^*$, and we are going to define $\text{proc}_{j-1}(0, m_{j-1}, t_0^i, p_{t_0^i}^i)$ following the pattern presented above. We pick $m' > m_j$ such that $[m_j, m']$ is the neighbourhood of $t_{m_j}^i$ at ζ_j and we define $p_{t_{m_j}^i}^i \in \mathbb{P}_\gamma$ so that

- $\text{Dom}(p_{t_{m_j}^i}^i) = w_i \cup \text{Dom}(r_{i-1}) \cup (\text{Dom}(q_{t_{m_{j-1}}^i}^i) \cap \zeta_{j-1})$,
- $p_{t_{m_j}^i}^i \upharpoonright \zeta_{j-1} = q_{t_{m_{j-1}}^i}^i \upharpoonright \zeta_{j-1}$,
- $p_{t_{m_j}^i}^i \upharpoonright \zeta_{j-1}$ forces in $\mathbb{P}_{\zeta_{j-1}}$ that
 - “ $p_{t_{m_j}^i}^i(\zeta_{j-1})$ is the inning of Generic given to her by $\mathbf{st}_{\zeta_{j-1}}$ in the subgame of level i of $\mathcal{D}_n^\odot(r_{\zeta_{j-1}}^i(\zeta_{j-1}), \mathbb{Q}_{\zeta_{j-1}})$ after the two players played $\langle p_{t_0^i}^i(\zeta_{j-1}), q_{t_{m_{j-1}}^i}^i(\zeta_{j-1}) \rangle$ as the conditions attached to $(t_0^i)_{\zeta_{j-1}}$ ”,
- $p_{t_{m_j}^i}^i(\xi) = p_{t_0^i}^i(\xi)$ for $\xi \in \text{Dom}(p_{t_{m_j}^i}^i) \setminus (\zeta_{j-1} + 1)$.

Plainly, $(*)_{14}^{m_j}$ holds and the condition $p_{t_{m_j}^i}^i$ is played by Generic as her inning associated with $t_{m_j}^i$. Antigeneric answers with $q_{t_{m_j}^i}^i \geq p_{t_{m_j}^i}^i$, and then Generic follows procedure $\text{proc}_j(m_j, m', t_{m_j}^i, p_{t_{m_j}^i}^i)$ to determine her innings for $m \in (m_j, m')$. If

$m' < m_{j-1}$, Generic picks $m'' > m'$ such that $[m', m'')$ is the neighbourhood of $t_{m'}^i$ at ζ_j and she defines $p_{t_{m''}^i} \in \mathbb{P}_\gamma$ like before, and then she follows the procedure $\text{proc}_j(m', m'', t_{m'}^i, p_{t_{m''}^i}^i)$, and so on. After arriving to $m_{j-1} - 1$ Generic defined the procedure $\text{proc}_{j-1}(0, m_{j-1}, t_0^i, p_{t_0^i}^i)$.

Finally, the procedure $\text{proc}_0(0, m_0, t_0^i, p_{t_0^i}^i)$ describes the instructions given to Generic by our strategy \mathbf{st} in the subgame of level i of $\mathcal{D}_n^\ominus(p, \mathbb{P}_\gamma)$. Now, for $\xi \in w_i$, Generic defines $\bar{p}_{i,\xi} = \langle p_{i,\xi}^\eta : \eta \in \max(s_{i,\xi}) \rangle$ and $\bar{q}_{i,\xi} = \langle q_{i,\xi}^\eta : \eta \in \max(s_{i,\xi}) \rangle$ so that $\bar{p}_{i,\xi}^\eta, \bar{q}_{i,\xi}^\eta$ are \mathbb{P}_ξ -names for elements of \mathbb{Q}_ξ and the demands in $(\odot)_6 + (\odot)_{11}$ are satisfied. It should be clear that the choice of $\bar{p}_{i,\xi}, \bar{q}_{i,\xi}$ is possible — note that by niceness of \mathbf{st}_ξ (and the last requirement in 1.2(1)) we easily justify that $(\odot)_{10}$ holds. To finish stage i , Generic picks $r_i \in \mathbb{P}_\gamma$ essentially by conditions $(\odot)_2 + (\odot)_{12}$.

We are going to argue that \mathbf{st} is a winning strategy for Generic in $\mathcal{D}_n^\ominus(p, \mathbb{P}_\gamma)$. To this end suppose that $\langle (T_i, \bar{p}^i, \bar{q}^i) : i < \omega \rangle$ is the result of a play of $\mathcal{D}_n^\ominus(p, \mathbb{P}_\gamma)$ in which Generic used \mathbf{st} , and the side objects constructed at each stage $i < \omega$ are

$$(\square) \quad w_i, T_i, r_i, \bar{p}^i, \bar{q}^i, k_i, K_\xi, i_\xi^*, \mathbf{st}_\xi, s_{i,\xi}, \bar{p}_{i,\xi}, \bar{q}_{i,\xi} \text{ for } \xi \in w_i,$$

and they satisfy the requirements $(\odot)_0 - (\odot)_{12}$.

We define a condition $q \in \mathbb{P}_\gamma$ as follows. Let $\text{Dom}(q) = \bigcup_{i < \omega} w_i = \bigcup_{i < \omega} \text{Dom}(r_i)$ and for $\xi \in \text{Dom}(q)$ let $q(\xi)$ be a \mathbb{P}_ξ -name for a condition in \mathbb{Q}_ξ such that

$$(\odot)_{15}^{\xi} \Vdash_{\mathbb{P}_\xi} \text{ “ } q(\xi) \geq r_{i_\xi^*}(\xi) \text{ and } q(\xi) \Vdash_{\mathbb{Q}_\xi} (\forall i \geq i_\xi^*) (\exists \nu \in \max(s_{i,\xi})) (q_{i,\xi}^\nu \in \Gamma_{\mathbb{Q}_\xi}) \text{ ”}.$$

Clearly q is well defined (remember $(\odot)_6$) and $q \geq p$ (remember $(\odot)_1 + (\odot)_2$). Also $q \geq r_i$ for all $i < \omega$.

We will show that for each $i < \omega$ the family $\{q_t^i : t \in T_i \text{ \& } \text{rk}_i(t) = \gamma\}$ is predense above q (and this clearly implies that Generic won the play). So suppose $q^+ \geq q$, $i < \omega$ and $w_i \cup \{\gamma\} = \{\xi_0, \xi_1, \dots, \xi_i, \xi_{i+1}\}$ (the increasing enumeration, so $\xi_0 = 0$, $\xi_{i+1} = \gamma$). By induction on $j \leq i$ we choose an increasing sequence $\langle q_j^+ : j \leq i \rangle \subseteq \mathbb{P}_\gamma$ and we also define $t \upharpoonright \xi_j + 1$. First, by the choice of $q(0)$ there is $\nu \in \max(s_{0,\xi_0})$ such that the conditions $q^+(0)$ and $q_{i,\xi_0}^\nu = q_{i,\xi_0}^\nu$ are compatible. Let $(t)_0 = \nu$ (this defines $t \upharpoonright \xi_1$). Let $q_0^+ \in \mathbb{P}_{\xi_1}$ be such that $q_0^+(0)$ is stronger than both $q^+(0)$ and q_{i,ξ_0}^ν , and $q_0^+ \upharpoonright (\xi_0, \xi_1) = q^+ \upharpoonright (\xi_0, \xi_1)$. It follows from $(\odot)_{10} + (\odot)_{12}$ that

$$(\odot)_{16}^0 \quad (\forall t' \in T_i) ([\text{rk}_i(t') = \gamma \text{ \& } t' \upharpoonright \xi_1 = t \upharpoonright \xi_1] \Rightarrow q_{t'}^i \upharpoonright \xi_1 \leq q_0^+).$$

Suppose that $j < i$ and we have defined $t \upharpoonright \xi_{j+1} \in T_i$ and a condition $q_j^+ \in \mathbb{P}_{\xi_{j+1}}$ such that $q_j^+ \geq q^+ \upharpoonright \xi_{j+1}$ and

$$(\odot)_{16}^j \quad (\forall t' \in T_i) ([\text{rk}_i(t') = \gamma \text{ \& } t' \upharpoonright \xi_{j+1} = t \upharpoonright \xi_{j+1}] \Rightarrow q_{t'}^i \upharpoonright \xi_{j+1} \leq q_j^+).$$

Necessarily, by $(\odot)_{15}^{\xi_{j+1}}$,

$$q_j^+ \Vdash_{\mathbb{P}_{\xi_{j+1}}} \text{ “ } (\exists \nu \in \max(s_{i,\xi_{j+1}})) (q_{i,\xi_{j+1}}^\nu, q^+(\xi_{j+1}) \text{ are compatible) ”}$$

so we may choose $\nu \in \max(s_{i,\xi_{j+1}})$ and a condition $q_{j+1} \in \mathbb{P}_{\xi_{j+1}}$ stronger than q_j^+ such that

$$q_{j+1} \Vdash_{\mathbb{P}_{\xi_{j+1}}} \text{ “ } q_{i,\xi_{j+1}}^\nu, q^+(\xi_{j+1}) \text{ are compatible ”}.$$

Let $(t)_{\xi_{j+1}} = \langle * \dots * \rangle \frown \nu$ (thus $t \upharpoonright (\xi_{j+1} + 1)$ has been defined) and let $q_{j+1}^+ \in \mathbb{P}_{\xi_{j+2}}$ be such that $q_{j+1}^+ \upharpoonright \xi_{j+1} = q_{j+1}$,

(\odot)₁₇ $q_{j+1}^+ \upharpoonright \xi_{j+1} \Vdash_{\mathbb{P}_{\xi_{j+1}}} "q_{j+1}^+(\xi_{j+1}) \geq q_{i, \xi_{j+1}}^\nu \quad \& \quad q_{j+1}^+(\xi_{j+1}) \geq q^+(\xi_{j+1})"$,

and $q_{j+1}^+ \upharpoonright (\xi_{j+1}, \xi_{j+2}) = q^+ \upharpoonright (\xi_{j+1}, \xi_{j+2})$. It follows from (\odot)₁₁ + (\odot)₁₆ that

(\odot)₁₈ if $t' \in T_i$, $\text{rk}_i(t') = \gamma$, $t' \upharpoonright \xi_{j+2} = t \upharpoonright \xi_{j+2}$, then $q_{j+1}^+ \upharpoonright \xi_{j+1} \Vdash_{\mathbb{P}_{\xi_{j+1}}} q_{t'}^i(\xi_{j+1}) \leq q_{i, \xi_{j+1}}^\nu$.

We may use (\odot)₁₇ + (\odot)₁₈ and then (\odot)₁₂ to argue that (\odot)₁₆ ^{$j+1$} holds true.

Finally look at $t = t \upharpoonright \xi_{i+1}$ and q_{i+1}^+ . \square

Definition 3.2. Suppose that \mathbb{P} is a forcing notion with \odot_n -property and $\bar{\mathbf{st}} = \langle \mathbf{st}_p : p \in \mathbb{P} \rangle$, where \mathbf{st}_p is a winning strategy for Generic in $\mathcal{D}_n^\odot(p, \mathbb{P})$. Such $\bar{\mathbf{st}}$ will be called an \odot_n -strategy system for \mathbb{P} .

- (1) We say that a finite set Q of conditions is an $\bar{\mathbf{st}}$ -front above p provided that there is a partial play $\langle s_j, \bar{p}^j, \bar{q}^j : j \leq i \rangle$ of $\mathcal{D}_n^\odot(p, \mathbb{P})$ in which Generic uses \mathbf{st}_p and
 - if $\max(s_i) = \langle \eta_k^i : k < K \rangle$ is the enumeration played by Antigeneric at stage i after Generic put s_i , then $Q = \{q_{\eta_k^i}^i : k < K\}$.

- (2) For a condition $p \in \mathbb{P}$ we define a game $\mathcal{D}_{\odot, n}^{\bar{\mathbf{st}}}(p, \mathbb{P})$ as follows. A play of $\mathcal{D}_{\odot, n}^{\bar{\mathbf{st}}}(p, \mathbb{P})$ lasts ω moves and in the course of the play a sequence

$$(\boxtimes) \quad \langle s_i, \bar{\eta}^i, \bar{p}^i, \bar{Q}^i, \bar{q}^i : i < \omega \rangle$$

is constructed. At a stage $i < \omega$ of the play,

- first Generic chooses a finite n -ary tree s_i such that the demand (α) of 1.1(1) holds, and then
- Antigeneric picks an enumeration $\bar{\eta}^i = \langle \eta_\ell^i : \ell < k_i \rangle$ of $\max(s_i)$.

Now the two players start a subgame of length k_i and they choose successive terms of a sequence $\langle p_{\eta_\ell^i}^i, Q_{\eta_\ell^i}^i : \ell < k_i \rangle$. At a stage $\ell < k_i$ of the subgame,

- first Generic picks a condition $p_{\eta_\ell^i}^i \in \mathbb{P}$ such that
 - if $j < i$, $\nu \in \max(s_j)$ and $\nu \triangleleft \eta_\ell^i$, then $q_\nu^j \leq p_{\eta_\ell^i}^i$ and $p \leq p_{\eta_\ell^i}^i$,
- and then Antigeneric picks an $\bar{\mathbf{st}}$ -front $Q_{\eta_\ell^i}^i$ above $p_{\eta_\ell^i}^i$.

After the subgame is completed,

- Antigeneric chooses $\bar{q}^i = \langle q_\eta^i : \eta \in \max(s_i) \rangle$ so that $q_\eta^i \in Q_\eta^i$ for $\eta \in \max(s_i)$.

Finally, Generic wins a play (\boxtimes) if and only if

- (\otimes) there is a condition $q \geq p$ such that for every $i < \omega$ the family $\{q_\eta^i : \eta \in \max(s_i)\}$ is predense above q .

- (3) Similarly to 1.2(1) we define when a strategy \mathbf{st} of Generic in $\mathcal{D}_{\odot, n}^{\bar{\mathbf{st}}}(p, \mathbb{P})$ is nice for an infinite set $K \subseteq \omega$.
- (4) We say that the forcing notion \mathbb{P} has the uniformly nice (\odot) _{n} ^{$\bar{\mathbf{st}}$} -property if for every $p \in \mathbb{P}$ and an infinite set $K \subseteq \omega$ Generic has a nice for K winning strategy in $\mathcal{D}_{\odot, n}^{\bar{\mathbf{st}}}(p, \mathbb{P})$.

Observation 3.3. (1) If \mathbb{P} has the \oplus_n -property, then it has the uniformly nice (\odot) _{n} ^{$\bar{\mathbf{st}}$} -property for some \odot_n -strategy system $\bar{\mathbf{st}}$.

- (2) The uniform n -Sacks forcing notion \mathbb{Q}_n has the uniformly nice (\odot) _{n} ^{$\bar{\mathbf{st}}$} -property for some \odot_n -strategy system $\bar{\mathbf{st}}$.

We do not know if the n -Silver forcing is equivalent to a forcing with the uniformly nice (\odot) _{n} ^{$\bar{\mathbf{st}}$} -property (for some $\bar{\mathbf{st}}$).

Theorem 3.4. *Assume that $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \gamma \rangle$ is a CS iteration and $\bar{\mathbf{st}}^\xi$ are \mathbb{P}_ξ -names such that for every $\xi < \gamma$,*

$$\Vdash_{\mathbb{P}_\xi} \text{ “ } \mathbb{Q}_\xi \text{ has the } \odot_n\text{-property and } \bar{\mathbf{st}}^\xi \text{ is a } \odot_n\text{-strategy system for } \mathbb{Q}_\xi \text{ and } \mathbb{Q}_\xi \text{ has the uniformly nice } (\odot)_n^{\bar{\mathbf{st}}^\xi}\text{-property ”.}$$

Then $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ has the n -localization property.

Proof. The following combinatorial observation can be shown by an easy induction.

Claim 3.4.1. *Let $M < \omega$. Suppose that for each $m < M$ we are given $k_m < \omega$ and a set $A_m \subseteq {}^{k_m}\omega$ of size $M!$. Then there is a sequence $\langle \sigma_m : m < M \rangle \in \prod_{m < M} A_m$ such that*

$$(\forall m < m' < M) (\sigma_m, \sigma_{m'} \text{ are incompatible}).$$

Let τ be a \mathbb{P}_γ -name for an element of ${}^\omega\omega$. Without loss of generality we may assume that

$$(\otimes)_0 \quad \Vdash_{\mathbb{P}_\gamma} (\forall \alpha < \gamma) (\tau \notin \mathbf{V}^{\mathbb{P}^\alpha}).$$

Claim 3.4.2. *Let $K, M < \omega$, $\xi < \gamma$, $p \in \mathbb{P}_\gamma$. Then there are $N > M$, $q^* \in \mathbb{P}_\xi$ and a sequence $\langle \sigma_\ell, q_\ell : \ell < L \rangle$ such that*

- (a) $\sigma_\ell \in {}^N\omega$ and $|\{\sigma_\ell \upharpoonright [M, N] : \ell < L\}| > K$,
- (b) $q_\ell \in \mathbb{P}_\gamma$, $p \leq q_\ell$, $q_\ell \upharpoonright \xi = q^*$ and $q_\ell \Vdash_{\mathbb{P}_\gamma} \tau \upharpoonright N = \sigma_\ell$,
- (c) $q^* \Vdash_{\mathbb{P}_\xi}$ “ $\{q_\ell(\xi) : \ell < k\}$ is an $\bar{\mathbf{st}}^\xi$ -front above $p(\xi)$ ”.

Proof of the Claim. Let $\mathbb{P}_{\xi\gamma}$ be defined as at the beginning of the proof of Lemma 1.8. Suppose that $G_\xi \subseteq \mathbb{P}_\xi$ is generic over \mathbf{V} , $p \upharpoonright \xi \in G_\xi$, and let us work in $\mathbf{V}[G_\xi]$ for a while. Then $\mathbb{P}_{\xi\gamma}[G_\xi]$ is a dense subset of the limit $\mathbb{P}^{\xi\gamma}$ of a CS iteration of forcing notions with \odot_n -property, so we may use the proof of 3.1. Consider a play $\langle (T_i, \bar{p}^i, \bar{q}^i) : i < \omega \rangle$ of $\mathcal{D}_n^\odot(p \upharpoonright [\xi, \gamma], \mathbb{P}^{\xi\gamma})$ in which

- Generic follows exactly the strategy described in the proof of 3.1, where on the coordinate ξ the strategy $\bar{\mathbf{st}}_{p(\xi)}^\xi[G_\xi]$ is used and
- each condition q_t^i played by Antigeneric (for $t \in T_i$, $\text{rk}_i(t) = \gamma$) is from $\mathbb{P}_{\xi\gamma}[G_\xi]$ and decides the value of $\tau \upharpoonright (M+i)$, say $q_t^i \Vdash \tau \upharpoonright (M+i) = \sigma_t^i$.

Let $q \in \mathbb{P}^{\xi\gamma}$ be the condition defined by $(\odot)_{15}^\xi$ (at the end of the proof of 3.1), so it witnesses that Generic won the play, and $\{q_t^i : t \in T_i \ \& \ \text{rk}_i(t) = \gamma\}$ is predense above q (for each $i < \omega$). It follows from $(\otimes)_0$ that for some $i < \omega$ we have

$$K < |\{\sigma_t^i \upharpoonright [M, M+i] : t \in T_i \ \& \ \text{rk}_i(t) = \gamma\}|.$$

Also it follows from the description of Generic's strategy in $\mathcal{D}_n^\odot(p \upharpoonright [\xi, \gamma], \mathbb{P}^{\xi\gamma})$ that the family $\{q_t^i(\xi) : t \in T_i \ \& \ \text{rk}_i(t) = \gamma\}$ is an $\bar{\mathbf{st}}^\xi[G_\xi]$ -front above $p(\xi)$.

Now, $T_i, \langle \sigma_t^i, q_t^i : t \in T_i \ \& \ \text{rk}_i(t) = \gamma \rangle \in \mathbf{V}$, so we may pick a condition $q^* \in \mathbb{P}_\xi$ stronger than $p \upharpoonright \xi$ which forces that these objects have the properties described above. Let $\langle t_\ell : \ell < L \rangle$ be an enumeration of $\{t \in T_i : \text{rk}_i(t) = \gamma\}$ and $\sigma_\ell = \sigma_{t_\ell}^i$, $q_\ell = q^* \frown q_{t_\ell}^i$ (for $\ell < L$). \square

Let $p \in \mathbb{P}_\gamma$. Following the procedure described in the proof of Theorem 3.1 construct a sequence

$$\langle w_i, T_i, \bar{p}^i, \bar{q}^i, \bar{q}^{i,*}, \bar{\sigma}^i, r_i, k_i, \alpha_i, M_i, \bar{N}^i, i_\xi^*, \bar{\mathbf{st}}_\xi, s_{i,\xi}, \bar{\eta}_{i,\xi}, \bar{p}_{i,\xi}, \bar{Q}_{i,\xi}, \bar{q}_{i,\xi} : i < \omega, \xi \in w_i \rangle$$

such that the following conditions are satisfied.

- (\otimes)₁ The demands formulated in (\odot)₀–(\odot)₃ and (\odot)₇–(\odot)₁₂ of the proof of 3.1.
 (\otimes)₂ For $\xi \in \bigcup_{i < \omega} w_i$ we have $i_\xi^* = \min(\{i : \xi \in w_i\}) \leq \min(K_\xi)$, and \mathfrak{st}_ξ is a \mathbb{P}_ξ -name for a winning strategy of Generic in $\mathfrak{D}_{\odot, n}^{\mathfrak{st}_\xi}(r_{i_\xi^*}(\xi), \mathbb{Q}_\xi)$ which is nice for $\{k \in \omega : k + i_\xi^* \in K_\xi\}$.
 (\otimes)₃ If $\xi \in w_i$, then $s_{i, \xi} \subseteq \bigcup_{j \leq i+1-i_\xi^*}^j (n+1)$ is an n -tree, $\bar{\eta}_{i, \xi}$ is the enumeration of $\max(s_{i, \xi})$ in the $<_{\chi^*}$ -increasing order, and $\bar{p}_{i, \xi} = \langle p_{i, \xi}^\eta : \eta \in \max(s_{i, \xi}) \rangle$, $\bar{Q}_{i, \xi} = \langle Q_{i, \xi}^\eta : \eta \in \max(s_{i, \xi}) \rangle$, $\bar{q}_{i, \xi} = \langle q_{i, \xi}^\eta : \eta \in \max(s_{i, \xi}) \rangle$ are \mathbb{P}_ξ -names for systems indexed by $\max(s_{i, \xi})$.
 (\otimes)₄ For each $\xi \in \bigcup_{i < \omega} w_i$,

$\Vdash_{\mathbb{P}_\xi}$ “ $\langle s_{i, \xi}, \bar{\eta}_{i, \xi}, \bar{p}_{i, \xi}, \bar{Q}_{i, \xi}, \bar{q}_{i, \xi} : i_\xi^* \leq i < \omega \rangle$ is a legal play of $\mathfrak{D}_{\odot, n}^{\mathfrak{st}_\xi}(r_{i_\xi^*}(\xi), \mathbb{Q}_\xi)$ in which Generic uses \mathfrak{st}_ξ ”.

- (\otimes)₅ $\bar{N}^i = \langle N_t^i : t \in T_i \ \& \ \text{rk}_i(t) = \gamma \rangle$, $M_i < N_t^i < M_{i+1} < \omega$, $M_0 = 0$, and $\bar{\sigma}^i = \langle \sigma_{t, \ell}^i : t \in T_i \ \& \ \text{rk}_i(t) = \gamma \ \& \ \ell < k_i! \rangle$, $\sigma_{t, \ell}^i \in N_t^i \omega$, and if $\ell < \ell' < k_i!$, $t \in T_i$, $\text{rk}_i(t) = \gamma$, then $\sigma_{t, \ell}^i \upharpoonright [M_i, N_t^i] \neq \sigma_{t, \ell'}^i \upharpoonright [M_i, N_t^i]$.
 (\otimes)₆ $\alpha_i = \max(w_i)$, $\bar{q}^{i, *} = \langle q_{t, \ell}^{i, *} : t \in T_i \ \& \ \text{rk}_i(t) = \gamma \ \& \ \ell < k_i! \rangle$, and for $t \in T_i$, $\text{rk}_i(t) = \gamma$, $\ell < k_i!$ we have $q_{t, \ell}^{i, *} \upharpoonright \alpha_i = q_t^i \upharpoonright \alpha_i$, $q_{t, \ell}^{i, *} \upharpoonright \alpha_i \Vdash_{\mathbb{P}_{\alpha_i}} q_{t, \ell}^{i, *}(\alpha_i) \in Q_{i, \alpha_i}^\eta$, where $\eta \in \max(s_{i, \alpha_i})$ is the end segment of $(t)_{\alpha_i}$ of length $i+1-i_\xi^*$, and $q_{t, \ell}^{i, *} \Vdash_{\mathbb{P}_\gamma} \mathcal{T} \upharpoonright N_t^i = \sigma_{t, \ell}^i$.
 (\otimes)₇ If $t_0, t_1 \in T_i$, $\text{rk}_i(t_0) = \text{rk}_i(t_1) = \gamma$, $t_0 \neq t_1$, then for some $\ell_0, \ell_1 < k_i!$ we have that $q_{t_0}^i = q_{t_0, \ell_0}^{i, *}$, $q_{t_1}^i = q_{t_1, \ell_1}^{i, *}$ and the sequences $\sigma_{t_0, \ell_0}^i \upharpoonright [M_i, N_{t_0}^i]$ and $\sigma_{t_1, \ell_1}^i \upharpoonright [M_i, N_{t_1}^i]$ are incomparable

To guarantee demands (\otimes)₁–(\otimes)₄ we follow exactly the lines of the proof of 3.1, to get (\otimes)₅ + (\otimes)₆ we use Claim 3.4.2 and we ensure (\otimes)₇ by Claim 3.4.1.

After the construction is carried out define a condition $q \in \mathbb{P}_\gamma$ in a manner similar to that in the proof of 3.1: $\text{Dom}(q) = \bigcup_{i < \omega} w_i = \bigcup_{i < \omega} \text{Dom}(r_i)$ and for $\xi \in \text{Dom}(q)$ we let $q(\xi)$ be a \mathbb{P}_ξ -name for a condition in \mathbb{Q}_ξ such that

- (\otimes)₈ $\Vdash_{\mathbb{P}_\xi}$ “ $q(\xi) \geq r_{i_\xi^*}(\xi)$ and $q(\xi) \Vdash_{\mathbb{Q}_\xi} (\forall i \geq i_\xi^*) (\exists \nu \in \max(s_{i, \xi})) (q_{i, \xi}^\nu \in \Gamma_{\mathbb{Q}_\xi})$ ”.

As in 3.1 one argues that

- (\otimes)₉ for each $i < \omega$ the family $\{q_t^i : t \in T_i \ \& \ \text{rk}_i(t) = \gamma\}$ is predense above q .

Now we choose a tree $T \subseteq \omega^{>\omega}$ such that

$$(\forall f \in [T])(\forall i < \omega)(\exists t \in T_i)(\exists \ell < k_i)(q_t^i = q_{t, \ell}^{i, *} \ \& \ \sigma_{t, \ell}^i \triangleleft f).$$

Plainly, T is an n -ary tree and $q \Vdash_{\mathbb{P}_\gamma} \mathcal{T} \in [T]$. □

Remark 3.5. After analyzing the proof of Theorem 3.4 one may notice that the following can be shown by the same proof.

Assume that γ is a limit ordinal and $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \gamma \rangle$ is a CS iteration such that for every $\xi < \gamma$

- $\Vdash_{\mathbb{P}_\xi}$ “ \mathbb{Q}_ξ has the nice \odot_n -property ”,
- \mathbb{P}_ξ has the n -localization property.

Then $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ has the n -localization property.

(The assumption that γ is limit allows us to make sure in the construction that $i < \min(K_{\alpha_i})$ for all $i < \omega$.)

- Problem 3.6.** (1) Can the implications in Observation 1.4 be reversed? What if we restrict ourselves to (s)nep forcing notions or even Suslin⁺ ?
- (2) Assume that \mathbb{P} has the \odot_n -property. Is it equivalent to a forcing notion with the uniformly nice $(\odot)_n^{\text{st}}$ -property (for some \odot_n -strategy system st) ? Again, we may allow restrictions to nice forcing notions.

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