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Some New Refinements of the Arithmetic, Geometric and Harmonic Mean Inequalities with Applications

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Abstract

Some new refinements of the arithmetic, geometric and harmonic mean inequalities are presented which improve on the inequalities of P. R. Mercer given in [5]. In addition, we present a new method to obtain inequalities. We discuss a few applications to probability theory and obtain bounds for certain central moments of positive random variables in terms of these means.

Keywords: Arithmetic, geometric and harmonic mean inequalities

1 Introduction

Let $X_1, X_2, \ldots, X_n$ be positive real numbers and let $P_1, P_2, \ldots, P_n$ be positive weights with $\sum_{j=1}^{n} P_j = 1$. Then the weighted arithmetic, geometric and harmonic means are respectively given by

$$A = \sum_{j=1}^{n} P_j X_j, \qquad G = \prod_{j=1}^{n} X_j^{P_j}, \qquad H = (\sum_{j=1}^{n} \frac{P_j}{X_j})^{-1}.$$

We are interested in obtaining refinements of some inequalities given by P. R. Mercer in [5] which are concerned with obtaining upper and lower bounds for $A-G, A-H, \log A - \log G$ and $\log G - \log H$. We will also be interested in obtaining bounds for some other functions of the X_j and P_j values, including the third central moment:

$$\mu_3 = \sum_{j=1}^{n} P_j (X_j - A)^3.$$

It will be useful to keep in mind the well-known inequality $H \leq G \leq A$. Then, of course, $\log H \leq \log G \leq \log A$, with equality holding iff all X_j values are equal. First, let's discuss some needed results to prove our later theorems.

Lemma A ([3], p.364-366)

Let $f(t)$ be a real-values function on $[c, d]$ with continuous third derivative $f^{(3)}(t)$ on $[c, d]$. Suppose $f''(t) \geq 0$ and $f^{(3)}(t) \leq 0$ on $[c, d]$. Let $L(t)$ be the equation of the line passing through the points $(c, f(c))$ and $(d, f(d))$. Let $E(t) = f(t) - L(t)$ be the linear interpolation error, $c \leq t \leq d$. Then:

$$\begin{aligned} &[f(d) - f(c) - (d-c)f'(c)]\frac{(t-c)(t-d)}{(d-c)^2} \leq E(t) \\ &\leq [(d-c)f'(d) - f(d) + f(c)]\frac{(t-c)(t-d)}{(d-c)^2} \end{aligned} \tag{1}$$

Lemma B

Suppose $f^{(3)}(t)$ is continuous on $[c, d]$ with $f^{(3)}(t) \geq 0$ on $[c, d]$. Then

$$\frac{\int_c^d f(t)dt}{d-c} \leq f\left(\frac{c+d}{2}\right) + \left(\frac{d-c}{12}\right) \cdot \left(f'(d) - f'\left(\frac{c+d}{2}\right)\right) \tag{2}$$

and

$$\frac{\int_c^d f(t)dt}{d-c} \geq f\left(\frac{c+d}{2}\right) + \left(\frac{d-c}{12}\right) \cdot \left(f'\left(\frac{c+d}{2}\right) - f'(c)\right) \tag{3}$$

Proof. See [2], pages 21-22.

Next, we present Propositions 1-5 given in [5] to keep this paper mostly self-contained. We shall also need the well-known Ky Fan inequality. We shall improve an all five of these propositions. We shall also present some new inequalities in the symmetric P_j's and X_j's case.

Proposition 1.([5], page 1460)

$$L_1 \leq A - G \leq U_1,$$

where

$$L_1 = \sum_{j=1}^{n} \frac{P_j (X_j - G)^2}{X_j + \max(X_j, G)} \tag{4}$$

and

$$U_1 = \sum_{j=1}^{n} \frac{P_j (X_j - G)^2}{X_j + \min(X_j, G)} \tag{5}$$

Proposition 2.([5], page 1460)

$$L_2 \leq \log A - \log G \leq U_2,$$

where

$$L_2 = \frac{1}{A} \sum_{j=1}^{n} \frac{P_j (X_j - G)^2}{X_j + \max(X_j, G)} \tag{6}$$

and

$$U_2 = \frac{1}{A} \sum_{j=1}^{n} \frac{P_j (X_j - G)^2}{X_j + \min(X_j, G)} \tag{7}$$

Proposition 3.([5], page 1461)

$$L_3 \leq \log G - \log H \leq U_3,$$

where

$$L_3 = \sum_{j=1}^{n} \frac{P_j}{X_j} \cdot \frac{(X_j - H)^2}{H + \max(X_j, H)} \tag{8}$$

and

$$U_3 = \sum_{j=1}^{n} \frac{P_j}{X_j} \cdot \frac{(X_j - H)^2}{H + \min(X_j, H)} \tag{9}$$

Proposition 4.([5], page 1462)

$$L_4 \leq A - H \leq U_4,$$

where

$$L_4 = \sum_{j=1}^{n} P_j (X_j - H)^2 \cdot \left[\frac{X_j + 2H + \max(X_j, H)}{(X_j + \max(X_j, H))^2} \right] \tag{10}$$

and

$$U_4 = \sum_{j=1}^{n} P_j (X_j - H)^2 \cdot \left[\frac{X_j + 2H + \min(X_j, H)}{(X_j + \min(X_j, H))^2} \right] \tag{11}$$

Proposition 5.([5], p.1463)

If not all the X_j's are equal, then

$$\frac{A'}{G'} \leq \left(\frac{A}{G} \right)^q, \tag{12}$$

where $q < 1$ is given by

$$q = \left(\frac{A}{1 - A} \right) \frac{\sum_{j=1}^{n} P_j (X_j - A)^2 / (2 - X_j - \max(X_j, A))}{\sum_{j=1}^{n} P_j (X_j - A)^2 / (X_j + \max(X_j, A))} \tag{13}$$

Ky Fan's inequality

Suppose, $0 \leq X_j \leq \frac{1}{2}, j = 1, 2, ..., n$ and let $Y_j = 1 - X_j$. Let A' and B' be the weighted arithmetic and geometric mean of the Y_j's. Then

$$\frac{A'}{G'} \leq \frac{A}{G} \tag{14}$$

In all of the above results equality holds iff all the X_j's are equal.

In Section 2, we shall replace L_i by L_i^* and U_i by U_i^* where $L_i^* > L_i$ and $U_i^* < U_i, i = 1, 2, 3, 4$ in Propositions 1-4 above, thereby improving the bounds given in [5]. We shall also show that we may replace q by q^* in Proposition 5 where $q^* < q$, improving this proposition as well. In Section 3, we present some inequalities for the symmetric case, a case which does not appear to have been previously considered in research papers on A, G and H inequalities.

Proposition 1 above improves upon inequalities given in [1] and [6]. We shall obtain refinements for Propositions 1-5 in this paper.

2 Some New Refinements

First, we need the following lemma.

Lemma C.

a) For $0 < X \le 1$,

$$X - 1 - \log(X) \ge \frac{(X-1)^2}{X+1} + \frac{(X-1)^2}{12} \cdot \left(\frac{3 - 2X - X^2}{(1+X)^2}\right) \ge \frac{(X-1)^2}{(X+1)} \tag{15}$$

b) For $X \ge 1$

$$X - 1 - \log(X) \le \frac{(X-1)^2}{X+1} + \frac{(X-1)^2}{12} \cdot \left(\frac{1 + 2X - 3X^2}{X^2(1+X)^2}\right) \le \frac{(X-1)^2}{(X+1)} \tag{16}$$

c) For $0 < X \le 1$,

$$X - 1 - \log(X) \le \frac{(X-1)^2}{2X} + \frac{(X-1)^3}{6X} \le \frac{(X-1)^2}{2X} \tag{17}$$

d) For $X \ge 1$,

$$X - 1 - \log(X) \ge \frac{(X-1)^2}{2X} + \frac{(X-1)^3}{6X^2} \ge \frac{(X-1)^2}{2X} \tag{18}$$

e) For $X \ge 1$,

$$\frac{(X-1)^2(X+1)}{2X^2} + \frac{(X-1)^3(X+2)}{6X^3} \le X - 2 + \frac{1}{X} \le \frac{(X-1)^2(X+3)}{(X+1)^2} + \frac{(X-1)^2}{12}\left(\frac{-14X^3 + 6X^2 + 6X + 2}{X^3(1+X)^3}\right) \tag{19}$$

f) For $0 < X \le 1$,

$$\frac{(X+3)(X-1)^2}{(X+1)^2} + \frac{(X-1)^2}{12} \cdot \left(\frac{14 - 2X^3 - 6X^2 - 6X}{(X+1)^3}\right) \le X - 2 + \frac{1}{X} \le \frac{(X-1)^2(X+1)}{2X^2} + \frac{1}{6}\left(\frac{1+2X}{X}\right)(X-1)^3 \tag{20}$$

Proof.

To prove part (a), let $f(t) = \frac{(t-1)}{t}, c = X, d = 1$ in part (1) of Lemma B. Then $f^{(3)}(t) = 6t^{-4} > 0$ on $[X, 1]$ and

$$\begin{aligned}\frac{\int_X^1 \left(\frac{t-1}{t}\right)dt}{1-X} = \frac{X-1-\log(X)}{X-1} &\le f\left(\frac{1+X}{2}\right) \\ &+\left(\frac{1-X}{12}\right)\cdot\left(f^{'}(1)-f^{'}\left(\frac{1+X}{2}\right)\right) \\ &= \frac{X-1}{1+X}+\left(\frac{1-X}{12}\right)\cdot\left(\frac{X^2+2X-3}{(1+X)^2}\right),\end{aligned} \tag{21}$$

which gives

$$X-1-\log(X) \ge \frac{(X-1)^2}{1+X}+\frac{(X-1)^2(3-2X-X^2)}{12(1+X)^2}$$

since $X-1<0$.

This proves part (a), since $3-2X-X^2 \ge 0, 0 < X \le 1$.
To prove part (b), let $f(t) = \frac{t-1}{t}, c = 1, d = X$ in part (1) of Lemma B and proceed as done above.
To prove part (c), apply Lemma A, the right half of (21) with $f(t) = \frac{1-t}{t}, c = X, d = 1$. Then

$$\begin{aligned}\int_X^1 (f(t)-L(t))dt &= \int_X^1 \left(\frac{1-t}{t}\right)dt - \left(\frac{f(X)+f(1)}{2}\right)\cdot(1-X) \\ &= X-1-\log(X)-\left(\frac{f(X)+f(1)}{2}\right)\cdot(1-X)\end{aligned} \tag{22}$$

Also,

$$\int_X^1 (f(t)-L(t))dt \le \int_X^1 \left((1-X)f^{'}(1)-f(1)+f(X)\right)\cdot\frac{(t-X)(t-1)}{(1-X)^2}dt$$

$$\begin{aligned}&= \left[(1-X)f^{'}(1)-f(1)+f(X)\right]\cdot\int_X^1 \frac{(t-X)(t-1)}{(1-X)^2}dt \\ &= \frac{1}{6X}(X-1)^3.\end{aligned} \tag{23}$$

(21) - (23) give the desired result, after some algebra, since $1+2X-3X^2 \leq 0$, for $X \geq 1$.

The proofs of parts (d)-(f) are very similar, except, we use other parts of Lemmas A and B. Also, in parts (e) and (f), we use instead $f(t) = \pm(\frac{1-t^2}{t^2})$. We omit the details here.

Now, we are ready to present refinements of Propositions 1-5 above. Theorem 1 is a refinement of Proposition 1.

Theorem 1.

Let L_1 and U_1 be given by (4)-(5). Let

$$\delta_1 = \sum_{X_j \leq G} P_j \frac{(X_j - G)^2}{12G} \cdot \left(\frac{3G^2 - 2GX_j - (X_j)^2}{(X_j + G)^2} \right) + \sum_{X_j > G} \frac{1}{3!} \frac{(X_j - G)^3}{X_j^2} P_j \quad (24)$$

$$\varepsilon_1 = \frac{1}{3!} \sum_{X_j \leq G} \frac{(X_j - G)^3}{GX_j} P_j + \sum_{X_j > G} \frac{(X_j - G)^2}{12G} \frac{(G^2 + 2X_jG - 3X_j^2)}{X_j^2 \left(1 + \frac{X_j}{G}\right)^2} P_j \quad (25)$$

Then

$$L_1 + \delta_1 \leq A - G \leq U_1 - \varepsilon_1, \quad (26)$$

where $\varepsilon_1 \geq 0$ and $\delta_1 \geq 0$, with $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ unless all X_j's are equal.

Proof.

We employ the same method as used by P.R.Mercer in [5], except we use Lemma C instead of the Hermite-Hadamard inequality.

Suppose $X_j \leq G$. Let $X = \frac{X_j}{G}$. Then $0 < X \leq 1$ and Lemma C, part (a) gives, after multiplying by P_j and summing over j with $X_j \leq G_j$,:

$$\begin{aligned} &\sum_{X_j \leq G} \left(\frac{X_j}{G} - 1 - \log\left(\frac{X_j}{G}\right) \right) \cdot P_j \\ \geq &\sum_{X_j \leq G} \left[\frac{\left(\frac{X_j}{G} - 1\right)^2}{\frac{X_j}{G} + 1} + \frac{\left(\frac{X_j}{G} - 1\right)^2}{12} \cdot \frac{3 - \frac{2X_j}{G} - \left(\frac{X_j}{G}\right)^2}{\left(1 + \frac{X_j}{G}\right)^2} \right] \cdot P_j \end{aligned} \quad (27)$$

If $X_j > G$ instead, let $X = \frac{X_j}{G}$. Then $X > 1$ and Lemma C, part (d) gives, after multiplying by P_j and summing over j with $X_j > G$.

$$\sum_{X_j>G}\left(\frac{X_j}{G}-1-\log\left(\frac{X_j}{G}\right)\right)\cdot P_j \geq \sum_{X_j>G}\left[\frac{\left(\frac{X_j}{G}-1\right)^2}{2\left(\frac{X_j}{G}\right)}+\frac{\left(\frac{X_j}{G}-1\right)^3}{3!\left(\frac{X_j}{G}\right)^2}\right]\cdot P_j \quad (28)$$

Addition of (27) and(28) gives, after some algebra and combining terms containing $\left(\frac{X_j}{G}-1\right)^2$:

$$A - G \geq L_1 + \delta_1, \text{as desired} \quad (29)$$

To prove, the upper bound, we use Lemma C, part (c) to those terms with $X_j \leq G$ and use part (b) instead for those terms with $X_j > G$ and proceed as done above. We omit the algebraic details.

The proofs of Theorem 2-5 below are very similar to the proof of Theorem 1, except we use a different choices for X and $f(X)$ and possibly a different part of Lemma C. Hence, we omit the proofs of these theorems and merely indicate what choice of X and $f(X)$ to use and/or what part of Lemma C to use. The rest is straightforward.

Theorem 2.

Let L_2 and U_2 be given by (6)-(7). Then

$$L_2 + \delta_2 \leq \log(A) - \log(G) \leq U_2 - \varepsilon_2,$$

where

$$\delta_2 = \frac{1}{A}\left[\sum_{X_j\leq A}\frac{(X_j-A)^2}{12A}\cdot\frac{(3A^2-2AX_j-X_j^2)}{(X_j+A)^2}P_j+\frac{1}{3!}\sum_{X_j>A}\frac{(X_j-A)^3}{X_j^2}P_j\right], \quad (30)$$

and

$$\varepsilon_2 = \frac{1}{A}\left[\frac{1}{3!}\sum_{X_j\leq A}\frac{1}{X_jA}(X_j-A)^3\cdot P_j+\sum_{X_j>A}\frac{(X_j-A)^2}{12A}\cdot\frac{(A^2+2X_jA-3X_j^2)}{X_j^2\left(1+\frac{X_j}{A}\right)^2}\cdot P_j\right] \quad (31)$$

where $\delta_2 \geq 0, \varepsilon_2 \leq 0$, with $\delta_2 > 0$ and $\varepsilon_2 < 0$, unless all X_j's are equal.

Proof.

Let $X = \frac{X_j}{A}$ instead. Use Lemma C, part (a) for $X_j \le A$ and use part (d) instead for $X_j > A$ and proceed as in proof of Theorem 1. $\delta_2 \ge 0$ follows since in Lemma C, part (a), $3 - 2X - X^2 \ge 0, 0 < X \le 1$, since $3 - 2X - X^2 = 0$ only at $X = 1$ and this must occur for all X_j. Thus $X_j = A$ for all j iff $\delta_2 = 0$. Thus $\delta_2 > 0$, unless all X_j's are equal to A. Similarly for the proof of $\varepsilon_2 \le 0$ and $\varepsilon_2 < 0$. For the upper bound, use parts (b) and (c) of Lemma C instead.

Theorem 3.

Let L_3 and U_3 be given by (8)-(9). Then

$$L_3 + \delta_3 \le \log(G) - \log(H) \le U_3 - \varepsilon_3,$$

where

$$\delta_3 = \sum_{X_j > H} \frac{(X_j - H)^2}{12X_j^2} \frac{(3X_j^2 - 2HX_j - H^2)}{(X_j + H)^2} \cdot P_j + \frac{1}{3!} \sum_{X_j \le H} \frac{(H - X_j)^3}{X_j H^2} \cdot P_j, \quad (32)$$

and

$$\varepsilon_3 = \sum_{X_j \le H} \frac{(X_j - H)^2}{12X_j^2} \frac{(X_j^2 + 2HX_j - 3H^2)}{H^2\left(1 + \frac{H}{X_j}\right)^2} \cdot P_j + \frac{1}{3!} \sum_{X_j > H} \frac{(H - X_j)^3}{HX_j^2} \cdot P_j \quad (33)$$

Also, $\delta_3 \ge 0, \varepsilon_3 \le 0$ with $\delta_3 > 0$ and $\varepsilon_3 < 0$ unless all the X_j's are equal.

Proof.

Let $X = \frac{H}{X_j}$. Use Lemma C, part (a) for $X_j > H$ and use part (d) for $X_j \le H$. This gives the lower bound. For the upper bound, use parts (b) and (c) of Lemma C instead.

Theorem 4.

Let L_4 and U_4 be given by (10)-(11). Then

$$L_4 + \delta_4 \le A - H \le U_4 - \varepsilon_4,$$

where

$$\delta_4 = \sum_{X_j \le H} \frac{(X_j - H)^2}{12H} \cdot \left(\frac{14H^3 - 2X_j^3 - 6X_j^2 H - 6X_j H^2}{(X_j + H)^3} \right) \cdot P_j + \frac{1}{3!} \sum_{X_j > H} \frac{(X_j - H)^3 (X_j + 2H)}{X_j^3} \cdot P_j, \tag{34}$$

and

$$\varepsilon_4 = \sum_{X_j > H} \frac{(X_j - H)^2}{12H} \cdot \frac{(-14X_j^3 + 6X_j^2 H + 6X_j H^2 + 2H^3)}{X_j^3 \left(1 + \frac{X_j}{H}\right)^3} \cdot P_j + \frac{1}{3!} \sum_{X_j \le H} \frac{(H + 2X_j)}{X_j} \cdot \left(\frac{(X_j - H)^3}{H^2} \right) \cdot P_j \tag{35}$$

Also, $\delta_4 \ge 0, \varepsilon_4 \le 0$ with $\delta_4 > 0$ and $\varepsilon_4 < 0$ unless all the X_j's are equal.

Proof.

Use parts (e) and (f) of Lemma C with $X = \frac{X_j}{H}$ and proceed as above. $\delta_4 \ge 0$ and $\varepsilon_4 \le 0$ follow since $-14X^3 + 6X^2 + 6X + 2 \le 0$ for $X \ge 1$ and $14 - 2X^3 - 6X^2 - 6X \ge 0$ for $0 < X \le 1$, by simple calculus. Also, $(X-1)^3 \le 0$ for $0 < X \le 1$ and $(X-1)^3 \ge 0$ for $X > 1$.

Next, we improve on Proposition 5, which itself is an improvement on Ky Fan's inequality.

Theorem 5.

Suppose that not all the X_j values are equal. Let $q^* = \frac{C_1 + C_2}{D_1 + D_2}$, where C_1, C_2, D_2 and D_2 are given in the proof below. Then

$$\frac{A'}{G'} < \left(\frac{A}{G} \right)^{q^*}$$

where $q^* < q < 1$ and q was given in (14).

Proof.

In Theorem 2, replace X_j by $Y_j = 1 - X_j$, A by A'.

Let

$$C_1 = \frac{1}{A'} \sum_{j=1}^{n} \frac{P_j(Y_j - A')^2}{(Y_j + \min(Y_j, A'))},$$

$$C_2 = \frac{1}{A'}\left[\frac{1}{3!}\sum_{Y_j \le A'} (Y_j - A)^3 P_j\right] + \frac{1}{A'}\left[\sum_{Y_j > A'} \frac{(Y_j - A')^2}{12A'} \cdot \left(\frac{(A^{'})^2 + 2Y_j A^{'} - 3Y_j^2}{(Y_j + A')^2}\right) \cdot P_j\right],$$

$$D_1 = \frac{1}{A} \sum_{j=1}^{n} \frac{P_j(X_j - A)^2}{X_j + \max(X_j, A)},$$

$$D_2 = \frac{1}{A}\left[\sum_{X_j \le A} \frac{(X_j - A)^2}{12A} \cdot \frac{(3A^2 - 2X_j A - X_j^2)}{(X_j + A)^2} \cdot P_j\right] + \frac{1}{A}\left[\sum_{X_j > A} \frac{1}{3!} \frac{(X_j - A)^3}{X_j^2} \cdot P_j\right].$$

Then in [5], it is shown in the proof of Proposition 5, that $\log\left(\frac{A'}{G'})\right) \le C_1, \log\left(\frac{A}{G}\right) \ge D_1$, and $q = \frac{C_1}{D_1}$.

Now $C_2 < 0$, since $(Y_j - A)^3 \le 0$ and $1 + 2X - 3X^2 < 0, 0 < X \le 1$. Thus, for at least one j, either $Y_j < A$ or $(A^{'})^2 + 2Y_j A^{'} - 3Y_j^2 < 0$ in C_2 expression. Similarly, $D_2 > 0$, since $(X_j - A)^2 \ge 0$ and $3 - 2X - X^2 > 0$ for $0 < X < 1$ gives $3A^2 - 2X_j A - X_j^2 \le 0$. Thus, there exists a j with either $(X_j - A)^2 > 0$ or $(3A^2 - 2X_j A - X_j^2) > 0$. Theorem 2 applied to Y_j values gives

$$\log(A^{'}) - \log(G^{'}) \le C_1 + C_2 < C_1$$

$$\log A - \log G \ge D_1 + D_2 > D_1$$

Thus, we obtain

$$q^* = \frac{C_1 + C_2}{D_1 + D_2} < \frac{C_1}{D_1} = q.$$

This complete the proof.

3 New Method and Inequalities

In this section, we shall obtain inequalities relating A, G, H and μ_3. In probability and statistics, μ_3 is related to various measure of skewness. We shall relate μ_3 to A, G and H via some inequalities.

In [4], a generalization of the Taylor series expansion is discussed. Power series expansions are given which are usually more accurate than usual Taylor series expansions. Firstly, we need the following lemma to derive new inequalities using this generalized Taylor expansion.

Lemma D([4], p.243-244)

Let $f(X)$ be a real-valued function defined on an interval I. Let m and n be nonnegative integers. Let $f', f'', f^{(3)}, f^{(4)}, ..., f^{(m+n+1)}$ denote the first $(m+n+1)$ derivatives of f, which are assumed to be continuous on I. Let $X, a \epsilon I$. Then

$$f(X) = f(a) + \sum_{k=1}^{L} \frac{(m+n-k)!}{(m+n)!} \left[\binom{m}{k} f^{(k)}(a) - (-1)^k \binom{n}{k} f^{(k)}(X) \right] \cdot (X-a)^k + R,$$

where $L = \max(m, n), f^{(1)} = f', f^{(2)} = f''$, and where the remainder term is

$$R = (-1)^n \frac{m!n!}{(m+n)!(m+n+1)!} f^{(m+n+1)}(\theta),$$

where θ is some real number between a and X, and where $\binom{m}{k}$ and $\binom{n}{k}$ are binomial coefficients with $\binom{a}{b} = 0$, if $a < b$. (If $n = 0$, we obtain the usual m term with remainder Taylor expansion of f about a.)

We are now ready to state and prove some new inequalities relating A, G, H and μ_3.

Theorem 6.

(a)

$$\log(A) - \log(G) \geq \frac{1}{2}\left(\frac{A}{H} - 1\right) + \frac{1}{6} \sum_{j=1}^{n} \left(\frac{(X_j - A)}{X_j}\right)^3 \cdot P_j$$

(b)

$$\log(A) - \log(G) \leq \frac{1}{2}\left(\frac{A}{H} - 1\right) + \frac{\mu_3}{6A^3}$$

Proof (a).

Let $f(X) = -\log(X), a = A$ in Lemma D using $m = 1, n = 1$. Then

$$-\log(X) = -\log(A) + \frac{1}{2}(X-A)\cdot\left(\frac{-1}{A} - \frac{1}{X}\right) + \frac{(X-A)^3}{12}\cdot\left(\frac{2}{\theta^3}\right),$$

where θ is between a and X. If $X > A$, we obtain, substituting X for θ_j:

$$-\log(X) \geq -\log(A) - \frac{1}{2}(X-A)\left(\frac{1}{A} + \frac{1}{X}\right) + \frac{(X-A)^3}{6X^3} \tag{36}$$

and substituting A for θ, we obtain:

$$-\log(X) \leq -\log(A) - \frac{1}{2}(X-A)\left(\frac{1}{A} + \frac{1}{X}\right) + \frac{(X-A)^3}{6A^3}, \tag{37}$$

If $X \leq A$, (36) and (37) hold again, except $\theta\epsilon[X, A]$ instead. Now proceed as done in [5] and in Section 2 of this paper. Replace X by X_j, multiply by P_j, and sum over j, using (36). This proves (a). To prove (b), we use (37) instead of (36) and apply the same procedure.

Corollary 1.

$$\mu_3 \geq 6A^3\left(\log\left(\frac{A}{G}\right) - \frac{1}{2}\left(\frac{A}{H} - 1\right)\right) \geq -3A^3\left(\frac{A}{H} - 1\right)$$

Proof.

The proof is immediate from part (b) of Theorem 6 and since $A \geq G$. Next, we discuss the case of symmetric P_j weights and X_j values symmetric about the arithmetic mean A.

Corollary 2.

Suppose $P_i = P_{n+1-i}, i = 1, 2, ..., N$ and $\frac{X_i + X_{n+1-i}}{2} = A, i = 1, 2, .., N$, where $N = \lfloor\frac{n}{2}\rfloor$, and where $X_{N+1} = A$, if n is odd. Suppose $X_1 < X_2 < ... < X_n$. Then

$$\log(A) - \log(G) \leq \frac{1}{2}\left(\frac{A}{H} - 1\right)$$

Proof.

The conditions on P_j and X_j given in the corollary give $\mu_3 = 0$, by a simple computation. The result follows from part (b) of Theorem 6 above.

Corollary 3.

a)

$$\log(A) - \log(G) \geq \frac{1}{2}\left(\frac{A}{H} - 1\right) + \frac{V_1}{12A^2} - \frac{1}{12}\left(1 - \frac{2A}{H} + A^2 V_2\right) - \frac{1}{30A^5}\sum_{i=1}^{n}(X_i - A)^5 P_i,$$

where $V_1 = \sum_{j=1}^{n}(X_j - A)^2 P_j$ is the variance and $V_2 = \sum_{j=1}^{n} \frac{(P_j}{X_j^2}$. Also,

b)

$$\log(A) - \log(G) \leq \frac{1}{2}\left(\frac{A}{H} - 1\right) + \frac{V_1}{12A^2} - \frac{1}{12}\left(1 - \frac{2A}{H} + A^2 V_2\right) - \frac{1}{30}\sum_{j=1}^{n}\left(\frac{X_j - A}{X_j}\right)^5 P_j.$$

Proof.

Let $m = 2, n = 2, f(X) = -\log(X), a = A$ in Lemma D. Then

$$-\log(X) = -\log(A) - \frac{1}{2}\left(\frac{1}{A} + \frac{1}{X}\right) \cdot (X - A) + \frac{1}{12}\left(\frac{1}{A^2} - \frac{1}{X^2}\right)(X - a)^2 + R,$$

where $R = -\frac{1}{30\theta^5 (X-A)^5}$, θ between a and X.

Proceeding as in the proof of Corollary 2 above, upon replacing X by X_j multiplying by P_j and summing over j and replacing θ by either A or X_j for parts (a) and (b) respectively, we obtain the desired results.

Theorem 7.

(a)

$$\log(A) - \log(G) \leq \frac{2}{3}\left(\frac{A}{H} - 1\right) - \frac{1}{6}\left(1 - \frac{2A}{H} + \sum_{j=1}^{n}\frac{A^2}{X_j^2} \cdot P_j\right) + \frac{1}{12}\left[\sum_{X_j \leq A}\left(\frac{X_j - A}{X_j}\right)^4 \cdot P_j + \sum_{X_j > A}\left(\frac{X_j - A}{A}\right)^4 \cdot P_j\right]$$

(b)

$$\log(A) - \log(G) \geq \frac{2}{3}\left(\frac{A}{H} - 1\right) - \frac{1}{6}\left(1 - \frac{2A}{H} + \sum_{j=1}^{n} \frac{A^2}{X_j^2} \cdot P_j\right)$$
$$+\frac{1}{12}\left[\sum_{X_j \leq A} \left(\frac{X_j - A}{A}\right)^4 \cdot P_j + \sum_{X_j > A} \left(\frac{X_j - A}{X_j}\right)^4 \cdot P_j\right]$$

Proof.

Apply Lemma D with $f(X) = -\log(X), m = 1, n = 2$. Then

$$-\log X = -\log A - \left(\frac{X - A}{3A}\right) - \frac{2}{3}\left(\frac{X - A}{X}\right) - \frac{1}{6}\left(\frac{X - A}{X}\right)^2$$
$$+\frac{1}{12}\left(\frac{X - A}{\theta}\right)^4 \text{where } \theta \text{ is between } a \text{ and } X$$

Since $m+n = 3$ is odd, we must consider separately the cases: $X_j \leq A$ and $X_j > A$, unlike in the proof of Theorem 6. We still utilize the same procedure originally given by P.R.Mercer in [5]. Replacing X by X_j, multiplying by P_j and summing either over $X_j \leq A$ or over $X_j > A$, we obtain parts (a) and (b), upon replacing θ by either A or X_j

The next theorem relates $\log A - \log G$ in terms of the second central moment (variance) and possibly the fourth central moment.

Theorem 8.

(a)

$$\log(A) - \log(G) \leq \frac{1}{3}\left(\frac{A}{H} - 1\right) + \frac{1}{6A^2}\sum_{j=1}^{n}(X_j - A)^2 P_j$$
$$-\frac{1}{12}\left[\sum_{X_j \leq A} \left(\frac{X_j - A}{A}\right)^4 P_j + \sum_{X_j > A} \left(\frac{X_j - A}{X_j}\right)^4 P_j\right]$$

(b)

$$\log(A) - \log(G) \geq \frac{1}{3}\left(\frac{A}{H} - 1\right) + \frac{1}{6A^2}\sum_{j=1}^{n}(X_j - A)^2 P_j$$
$$-\frac{1}{12}\left[\sum_{X_j \leq A} \left(\frac{X_j - A}{X_j}\right)^4 P_j + \sum_{X_j > A} \left(\frac{X_j - A}{A}\right)^4 P_j\right]$$

Proof.

Let $f(X) = -\log X, m = 2, n = 1, a = A$ in Lemma D and proceed as in the proof of Theorem 7. We omit the details since the proof is very similar to the proof of Theorem 7.

It should be mentioned that another proof of Proposition 2 and Theorem 2 can be obtained from Lemma D, so that these are other uses for this lemma. From Lemma D, many more inequalities can be derived by choosing different choices for $f(X), a, m$ and n.

4 Conclusions

In this paper, refinements of various inequalities involving A, G, H and several central moments are discussed which improve on those given in [5]. In addition, a generalized Taylor series method is utilized to obtain more inequalities. In particular, the case of symmetric P_j and X_j values symmetric about A was considered, a case not previously considered in the literature.

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