Mathematics Faculty Publications   Department of Mathematics

1998

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Recommended Citation

Matache, Valentin The eigenfunctions of a certain composition operator. Studies on composition operators (Laramie, WY, 1996), 121–136, Contemp. Math., 213, Amer. Math. Soc., Providence, RI, 1998.

THE EIGENFUNCTIONS OF A
CERTAIN COMPOSITION OPERATOR

VALENTIN MATACHE

ABSTRACT. The composition operator on the classical Hardy space H^2, induced by a hyperbolic disk automorphism is considered. It is investigated when a H^2-function induces under the given operator a minimal invariant cyclic subspace. Theorems where we use the behaviour of this function in the neighbourhood of the fixed points of the hyperbolic automorphism in order to decide if the cyclic subspace mentioned above is minimal invariant or not, are obtained. The inner eigenfunctions of the operator under consideration are characterized.

1. Introduction

We denote by H^2 the classical Hardy space of all functions analytic on the open unit disk $\mathbb{D}$, having square summable Taylor coefficients. ϕ is the hyperbolic Möbius transform

$$(1) \qquad\qquad \phi(z) = \frac{2z+1}{z+2} \qquad z \in \mathbb{D}$$

having fixed points 1 and -1. C_ϕ is the composition operator induced by ϕ on H^2. $\mathrm{Lat}C_\phi$ denotes the invariant subspace lattice of C_ϕ. Subspace means always closed, linear manifold. $\mathcal{M} \in \mathrm{Lat}C_\phi$ is minimal invariant for C_ϕ if $\mathcal{M} \neq 0$ and there is no nonzero $\mathcal{N} \in \mathrm{Lat}C_\phi$, such that $\mathcal{N} \subseteq \mathcal{M}$, and $\mathcal{N} \neq \mathcal{M}$. It is important to see if all the minimal invariant subspaces of C_ϕ are 1- dimensional. Clearly, any minimal invariant subspace of C_ϕ is cyclic. In [8], this author's main idea was to choose some arbitrary $u \in H^2$, consider the cyclic subspace in $\mathrm{Lat}C_\phi$ spanned by u, $\mathcal{K}_u = \vee_{n=0}^\infty C_\phi^n u$ (that is the closure of the linear manifold spanned by the functions $\{u, u \circ \phi, u \circ \phi^{(2)}, ..., u \circ \phi^{(n)}, ...\}$), and deduce by the properties of u if $\mathcal{K}_u$ is minimal invariant or not. By $\phi^{(n)}$ we mean $\phi \circ \phi \circ \cdots \circ \phi$, n times, for each positive integer n. Clearly $\mathcal{K}_u$ is minimal invariant and finite dimensional if and only if u is an eigenfunction of C_ϕ. Related to those $u \in H^2$ such that $\mathcal{K}_u$ is minimal invariant but $\dim \mathcal{K}_u > 1$ we wish to point out that this means $\dim \mathcal{K}_u$ is infinite. Indeed, if $\dim \mathcal{K}_u$ is finite, then the restriction $C_\phi|_{\mathcal{K}_u}$ has nonempty point spectrum. Consequently we can choose f an eigenfunction in $\mathcal{K}_u$, denote by $\mathcal{N}$ the one dimensional subspace spanned by f and observe that $\mathcal{N} \subseteq \mathcal{K}_u$,

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$\mathcal{N} \neq 0$, and $\mathcal{N} \in \mathrm{Lat} C_\phi$. If $\dim \mathcal{K}_u > 1$, then $\mathcal{K}_u$ is not minimal invariant. On the other hand, finding $\mathcal{K}_u$ minimal invariant with infinite dimension is no easy job because, if such $\mathcal{K}_u$ existed then $T = C_\phi|_{\mathcal{K}_u}$ would be an example of a Hilbert space operator, acting on an infinitely-dimensional, complex space with trivial invariant subspace lattice. It is unknown if such an operator exists. This problem is called the invariant subspace problem and stays open for some decades now. The authors of [12] observed that a Hilbert space operator (acting on a complex, separable space of dimension bigger than 1), without proper invariant subspaces exists if and only if $\mathrm{Lat} C_\phi$ contains an infinitely-dimensional minimal invariant subspace. Since finding such a $\mathcal{K}_u$ is not an easy job, our idea in [8], was to discard as many u as possible. Theorem 2 in [8] is a result in this direction. it states that

Theorem 1.1 *If α is 1 or -1, $u(e^{i\theta})$ is the radial limit function of $u \in H^2$, and if we can assign a nonzero value to u at α, $u(\alpha) \neq 0$, such that the extension of $u(e^{i\theta})$ we get in this manner is continuous, then $\mathcal{K}_u$ is nonminimal unless u is constant.*

To see in which way we can discard functions by using the previous result we give the following

Example 1.2 *Suppose $(z_n)_n$ is a sequence in $\mathbb{D}$, such that $\sum_{n=1}^{\infty}(1 - |z_n|) < \infty$ and 1 or -1 is not an accumulation point of the set $\{z_n : n \geq 1\}$. Denote by B the Blaschke product having zeros $(z_n)_n$. $\mathcal{K}_B$ is not minimal invariant.*

Proof. Indeed, since $(z_n)_n$ do not cluster to 1 (or -1), B is continuously extendable at this point (see [6]), and $B(1) \neq 0$ (respectively $B(-1) \neq 0$) because $|B(e^{i\theta})| = 1$ a.e. on the unit circle. By Theorem 1.1 $\mathcal{K}_B$ is nonminimal because B is not constant.   □

Our first concern in this paper will be obtaining an improved version of Theorem 1.1. Before doing that we would like to recall for later reference a theorem obtained by this author in [8].

Theorem 1.3 *If u is an inner function and $\mathcal{K}_u$ is minimal invariant, then the greatest common inner divisor of the following family of inner functions $\{u \circ \phi^{(n)} : n \geq k\}$, must be an eigenfunction of C_ϕ, for each fixed integer k.*

Recently, V. Chkliar answered in [2] a question raised by this author in [8] asking if you can remove the restriction $u(\alpha) \neq 0$ in Theorem 1.1. His result is as follows.

Theorem 1.4 *If $u \in H^2$, $u \neq 0$ satisfies the following conditions,*

(a)
$$\overline{\lim_{z \to -1}} |u(z)| < \infty$$

(b) $\quad |u(z)| \leq c|1 - z|^\varepsilon \quad$ *for some $c, \varepsilon > 0 \quad$ and each z in a neighbourhood of 1*

then $\sum_{n=-\infty}^{\infty} \lambda^n (u \circ \phi^{(n)})$ is an eigenfunction of C_ϕ associated to the eigenvalue λ^{-1} for each λ in the annulus $\{\lambda \in \mathbb{C} : 1 < |\lambda| < 3^{\min(\varepsilon, \frac{1}{2})}\}$ except for some discrete subset of points.

This author observed in [8] that if $\mathcal{K}_u$ is minimal invariant, then $\mathcal{K}_u$ coincides with the subspace $\vee_{n=-\infty}^{\infty} C_\phi^n u$, spanned by all the iterates of C_ϕ and its inverse at u. Under the assumptions on u in Theorem 1.4, this last space is not minimal invariant and hence neither is $\mathcal{K}_u$. Clearly condition (b) tells us u is continuously extendable by 0 at 1.

2. Minimal Invariant Subspaces

By $\phi^{(-n)}$ we will denote $\phi^{-1} \circ \phi^{-1} \circ \cdots \circ \phi^{-1}$, n times. Denote $\gamma(z) = (1+z)/(1-z)$, observe that γ maps the unit disk onto the right half-plane, and $\gamma \circ \phi = 3\gamma$, hence $\gamma \circ \phi^{(n)} = 3^n \gamma$ and $\gamma \circ \phi^{(-n)} = 3^{-n}\gamma$, $n \geq 1$. This leads to

$$\phi^{(n)}(z) = \frac{(3^n+1)z + (3^n-1)}{(3^n+1) + (3^n-1)z} \tag{2}$$

and

$$\phi^{(-n)}(z) = \frac{(3^n+1)z - (3^n-1)}{(3^n+1) - (3^n-1)z} \tag{3}$$

for any positive integer n. We will need the following

Proposition 2.1 *If $z \in \mathbb{D}$, then $\phi^{(n)}(z) \to 1$ and $\phi^{(-n)}(z) \to -1$, nontangentially.*

Proof. Observe that $\phi^{(n)}(z) = \gamma^{-1}(3^n\gamma(z))$, $n \geq 1$. $3^n\gamma(z) \to \infty$ inside the right half-plane, along the line through the origin having direction vector $\gamma(z)$. But $\gamma^{-1}(\infty) = 1$, so $\phi^{(n)}(z) \to 1$. On the other hand, the line mentioned above is transformed by γ^{-1} into either a circle through -1 and 1 (different from the unit circle), or the real line (if $z \in \mathbb{R}$). Therefore $\phi^{(n)}(z) \to 1$ nontangentially. The proof of $\phi^{(-n)}(z) \to -1$ nontangentially is identical. $\square$

We can use this fact now and pretty much the same idea as in [8], in order to give an improved version of Theorem 1.1. For each $u \in H^2$, denote by $u(e^{i\theta})$, the nontangential limit function of u, which, as it is known, exists on the unit circle a.e. with respect to the Lebesgue arc-length measure $d\theta$. Observe that if $e^{i\theta} = \phi^{(n)}(e^{it})$, then

$$dt = |\left(\phi^{(-n)}(e^{i\theta})\right)'|d\theta \quad \text{i.e.} \quad dt = P(\phi^{(n)}(0), \theta)d\theta$$

where

$$P(z, \theta) = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}, \qquad z = re^{it} \tag{4}$$

is the usual Poisson kernel. In the sequel we shall often change the variables on the unit circle $\mathbb{T}$ in this manner.

Theorem 2.2 *If $u \in H^2$, α is one of the fixed points of ϕ, the nontangential limit of u exists at α and is nonzero, and $u(e^{i\theta})$ is essentially bounded on an open arc containing α, then $\mathcal{K}_u$ is minimal invariant if and only if u is constant.*

Proof. Suppose $\alpha = 1$. Chose $\delta > 0$ such that there is some $M > 0$ such that $|u(e^{i\theta})| \leq M$ a.e. for $|\theta| \leq \delta$. Denote $\phi^{(n)}(0) = \frac{3^n - 1}{3^n + 1} = a_n$. We can write

$$||C_\phi^n u||^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(e^{i\theta})|^2 P(a_n, \theta) d\theta =$$

$$= \frac{1}{2\pi} \int_{-\delta}^{\delta} |u(e^{i\theta})|^2 P(a_n, \theta) d\theta + \frac{1}{2\pi} \int_{|\theta| \geq \delta} |u(e^{i\theta})|^2 P(a_n, \theta) d\theta \leq$$

$$\leq M^2 + P(a_n, \delta) ||u||^2.$$

Since $P(a_n, \delta) \to 0$ as $n \to \infty$, this proves that the sequence $(u \circ \phi^{(n)})_n$ is a norm bounded sequence. By Proposition 2.1, this sequence tends pointwise to the nontangential limit of u at 1, $u(1)$. These two facts prove that this sequence converges weakly to $u(1)$. See [11] for a motivation of this statement. The weak closure and norm closure of $\mathcal{K}_u$ coincide, since this is a linear subspace, hence a convex set. Thus $\mathcal{K}_u$ contains the subspace $\mathbb{C}$ of the constant functions because $u(1)$ is nonzero. Obviously $\mathbb{C}$ is invariant under C_ϕ. The conclusion of the theorem follows. If the fixed point is -1, repeat the argument above with $\phi^{(-n)}$ instead of $\phi^{(n)}$, and recall that a minimal invariant subspace of an invertible operator is doubly invariant, that is invariant both under the operator and its inverse ([8]). $\square$

Proposition 2.3 *If u is an eigenfunction of C_ϕ, and the nontangential limit of u exists both at 1 and -1, then u is a constant function.*

Proof. Since u is an eigenfunction, there is some $z \in \mathbb{D}$ such that $u(z) \neq 0$. Let λ be the eigenvalue corresponding to u. We have that $u(\phi^{(n)}(z)) = \lambda^n u(z)$. Let now n tend to ∞ and use Proposition 2.1 to deduce that $\lim_{n \to \infty} \lambda^n = u(1)/u(z)$. Deduce that $\lim_{n \to \infty} \lambda^{-n} = u(-1)/u(z)$, by using the same proposition and the equality $u(\phi^{(-n)}(z)) = \lambda^{-n} u(z)$. Since both these limits exist λ must be 1. Hence, for each $w \in \mathbb{D}$ and each positive integer n, $u(\phi^{(n)}(w)) = u(w)$. Let now n tend to ∞, and deduce $u(w) = u(1)$ for each $w \in \mathbb{D}$. $\square$

3. The Inner Eigenfunctions of C_ϕ

We define the orbit of each point in $\mathbb{D}$ under ϕ. For each $\lambda \in \mathbb{D}$, the orbit of λ under ϕ is the set $\mathrm{Orb}(\lambda) = \{\phi^{(n)}(\lambda) : n \in \mathbb{Z}\}$; $\mathbb{Z}$ denotes the set of all integers. If ν is a positive, finite Borel measure on the unit circle $\mathbb{T}$ which is singular with respect to the Lebesgue arc-length measure we denote by S_ν the singular inner function induced by ν, that is

$$S_\nu(z) = \exp\left(-\int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\nu(\theta)\right) \qquad z \in \mathbb{D}.$$

If u and v are two inner functions which mutually divide each other we write $u \sim v$.

Lemma 3.1 *Suppose ν is a positive, finite Borel measure on $\mathbb{T}$, singular with respect to the Lebesgue arc-length measure; then there is a singular Borel measure μ such that $(S_\nu \circ \phi) \sim S_\mu$ and μ is determined by*

$$(5) \qquad \mu(\phi^{-1}(E)) = \int_E P(\phi(0), \theta) d\nu(\theta)$$

for each Borel subset E of $\mathbb{T}$.

Proof. The support of ν, suppν can be identified as "the set of all $e^{i\theta} \in \mathbb{T}$ where S_ν is not continuously extendable". $S_\nu \circ \phi$ is an inner function without zeros in $\mathbb{D}$. Thus there exists μ such that $S_\nu \circ \phi$ and S_μ divide each other. We readily see that supp$\mu = \phi^{-1}(\text{supp}\nu)$. We prove now (5) for Borel sets E which contain suppν. In this case, supp$\mu \subseteq \phi^{-1}(E)$ and we can write

$$e^{-\mu(\phi^{-1}(E))} = |S_\mu(0)| = |S_\nu(\phi(0))| = e^{-\int_{\mathbb{T}} P(\phi(0),\theta)d\nu(\theta)} = e^{-\int_E P(\phi(0),\theta)d\nu(\theta)}$$

which takes care of (5). In all these computations we used the identity $\text{Re}\frac{e^{i\theta}+z}{e^{i\theta}-z} = P(z,\theta)$, $\theta \in \mathbb{R}$, $z \in \mathbb{D}$. For an arbitrary Borel set E now, denote by ν_E the measure $\nu_E(A) = \nu(E \cap A)$, for each Borel set $A \subseteq \mathbb{T}$. We have that $(S_{\nu_E} \circ \phi) \sim S_{\mu_1}$ for some μ_1, and supp$\mu_1 \subseteq \phi^{-1}(E)$. On the other hand, $S_\mu \sim ((S_{\mu_1})(S_{\nu_{E^c}} \circ \phi))$ because $\nu = \nu_E + \nu_{E^c}$. E^c is the complement of E. So, if $(S_{\nu_{E^c}} \circ \phi) \sim S_{\mu_2}$ we have $\mu = \mu_1 + \mu_2$, supp$\mu_1 \subseteq \phi^{-1}(E)$, supp$\mu_2 \subseteq \phi^{-1}(E^c)$ i.e. $\mu_1 \perp \mu_2$. So $\mu_{\phi^{-1}(E)} = \mu_1$ and $\mu_{\phi^{-1}(E^c)} = \mu_2$. Thus

$$\mu(\phi^{-1}(E)) = \mu_1(\phi^{-1}(E)) = \int_E P(\phi(0),\theta)d\nu_E(\theta)$$

by what has already been proved. Since $\int_E P(\phi(0),\theta)d\nu_E(\theta) = \int_E P(\phi(0),\theta)d\nu(\theta)$, the desired conclusion follows. □

Theorem 3.2 *If ν is a positive, finite Borel measure on $\mathbb{T}$, singular with respect to the Lebesgue measure, then S_ν is an eigenfunction of C_ϕ if and only if the measure $\nu\phi^{-1}$ given by $\nu\phi^{-1}(E) = \nu(\phi^{-1}(E))$, is absolutely continuous with respect to ν, and its Radon-Nikodym derivative with respect to ν is $P(\phi(0),\theta)$.*

Proof. S_ν is an eigenfunction of C_ϕ if and only if $(S_\nu \circ \phi) \sim S\nu$. By Lemma 3.1 this is equivalent to

$$\nu(\phi^{-1}(E)) = \int_E P(\phi(0),\theta)d\nu(\theta)$$

for each Borel set $E \subseteq \mathbb{T}$, which means exactly that $d(\nu\phi^{-1})(\theta) = P(\phi(0),\theta)d\nu(\theta)$. □

Corollary 3.3 *If S_ν is a nonconstant, singular, inner eigenfunction of C_ϕ, then $\phi(\text{supp}\nu) = \text{supp}\nu$, $\{-1,1\} \subseteq \text{supp}\nu$, and $\nu(\{1\}) = \nu(\{-1\}) = 0$.*

Proof. If $K = \text{supp}\nu$, since $(S_\nu \circ \phi) \sim S_\nu$, like in the proof of Lemma 3.1 we deduce supp$\nu = \phi^{-1}(\text{supp}\nu)$ i.e. $\phi(K) = K$. If one of the fixed points of ϕ is not in K, we can extend $S_\nu(e^{i\theta})$ continuously at that point by a nonzero value. According to Theorem 1.1, $\mathcal{K}_{S_\nu}$ is nonminimal, which is a contradiction. Thus $\{-1,1\} \subseteq K$. If we suppose $\nu(\{1\}) = a > 0$, we have $a = \nu(\{\phi^{-1}(1)\}) = P(\phi(0),0)\nu(\{1\}) = P(\frac{1}{2},0)a$, according to (5) with $\mu = \nu$. We get $P(\frac{1}{2},0) = 1$, which is false. We admit therefore that $\nu(\{1\}) = 0$. Similarly $\nu(\{-1\}) = 0$. □

To see that nonzero measures inducing singular, inner eigenfunctions of C_ϕ do exist, we shall give the following

Example 3.4 Choose $\lambda \in \mathbb{T}$, $\lambda \neq \pm 1$. Set

$$w_n = \frac{a}{\prod_{k=1}^{n} P(\phi(0), \arg(\phi^{(k)}(\lambda)))}, \quad n \geq 1$$

$$w_{-n} = a \prod_{k=1}^{n-1} P(\phi(0), \arg(\phi^{(-k)}(\lambda))), \quad n > 1$$

and $w_0 = a$, where a is any fixed positive number, and arg designates the argument of a complex number. For each Borel set $A \subseteq \mathbb{T}$, if $A \cap \mathrm{Orb}(\lambda) \neq \emptyset$, denote $A \cap \mathrm{Orb}(\lambda) = \{\phi^{(j)}(\lambda) : j \in J\}$, $\quad J \subseteq \mathbb{Z}$, and define $\nu(A) = \sum_{j \in J} w_j$, and $\nu(A) = 0$ if $A \cap \mathrm{Orb}(\lambda) = \emptyset$. Obviously we obtain a singular, positive, Borel measure on $\mathbb{T}$. To see it is also finite, observe that

$$\frac{w_{n+1}}{w_n} = \frac{1}{P(\phi(0), \arg(\phi^{(n+1)}(\lambda)))} \to \frac{1}{P(\phi(0), 0)} = \frac{1}{3}.$$

Similarly $\frac{w_{-n-1}}{w_{-n}} \to \frac{1}{3}$, and by the quotient test $\sum_{-\infty}^{\infty} w_n < \infty$. Now, if μ is that singular Borel measure for which $S_\mu \sim (S_\nu \circ \phi)$, observe that $\mathrm{supp}\mu = \phi^{-1}(\mathrm{supp}\nu) = \phi^{-1}(\mathrm{Orb}(\lambda)) = \mathrm{Orb}(\lambda)$, and we have by (5)

$$\mu(\phi^{-1}(\{\phi^{(n)}(\lambda)\})) = P(\phi(0), \arg(\phi^{(n)}(\lambda)))\nu(\{\phi^{(n)}(\lambda)\}), \quad n \geq 1$$

that is

$$\mu(\{\phi^{(n-1)}(\lambda)\}) = P(\phi(0), \arg(\phi^{(n)}(\lambda))) \frac{a}{\prod_{k=1}^{n} P(\phi(0), \arg(\phi^{(k)}(\lambda)))}$$

hence

$$\mu(\{\phi^{(n-1)}(\lambda)\}) = \nu(\{\phi^{(n-1)}(\lambda)\}) \quad n \geq 1.$$

A similar computation shows that the equality above holds for each $n \leq 0$. Thus $\mu = \nu$, that is S_ν is a nonconstant, singular, inner eigenfunction of C_ϕ.

Each inner function can be factored in an essentially unique way in a product of a singular inner function and a Blaschke product. The natural question now is which Blaschke products are eigenfunctions of C_ϕ. If B is a Blaschke product and z a zero of B we denote by $\mathrm{mult}(z)$ the multiplicity of z.

Theorem 3.5 *A Blaschke product B is an eigenfunction of C_ϕ if and only if, either B is a unimodular constant, or B has the following properties. If $Z(B)$ denotes the set of all zeros of B, then for each $\lambda \in Z(B)$ we have that $\mathrm{Orb}(\lambda) \subseteq Z(B)$ and for each $z, w \in \mathrm{Orb}(\lambda)$, $\mathrm{mult}(z) = \mathrm{mult}(w)$.*

Proof. Suppose B is a Blaschke product, having infinitely many zeros. If $a \in \mathbb{D}$ denote $b = \phi^{-1}(a)$ and

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z} \quad , \quad z \in \mathbb{D}.$$

Observe that

$$\phi_a \circ \phi = e^{i\theta} \phi_b$$

for some unimodular $e^{i\theta}$. We see that, if $(z_n)_n$ is the sequence of the zeros of B, each one repeated according to its multiplicity, then $B \circ \phi$ is a Blaschke product having zeros $(\phi^{-1}(z_n))_n$. B is an eigenfunction of C_ϕ if and only if the Blaschke products B and $B \circ \phi$ mutually divide each other, as inner functions. This is equivalent with the fact that, B and $B \circ \phi$ have the same zeros with exactly the same multiplicities ([1]). In this situation is plain to see that for each integer n, $B \sim B \circ \phi^{(n)}$. The desired conclusion is now immediate. Indeed, if $\lambda \in Z(B)$ then $\mathrm{Orb}(\lambda)$ must be contained by $Z(B)$ because otherwise, for some n, B and $B \circ \phi^{(n)}$ would fail to have the same zeros. Also, on each orbit $\mathrm{Orb}(\lambda) \subseteq Z(B)$, the multiplicity function should be constant, since for each n, B and $B \circ \phi^{(n)}$ should have the same zeros with the same multiplicities. Clearly Blaschke products whose set of zeros is a union of full orbits and such that the zeros in each orbit have the same multiplicity are eigenfunctions of C_ϕ. Observe that nonconstant Blaschke products with finitely many factors cannot be eigenfunctions of C_ϕ by the same argument we use in the proof of Example 1.2. □

We wish to make the following coments here. Observe that the statement of this theorem says that the set of all zeros of B must be a union of full orbits and the zeros belonging to the same orbit should have the same multiplicity, i.e. the multiplicity function must be constant on each orbit, which of course does not exclude the possibility that the multiplicity function have distinct values on different orbits. Also observe that distinct orbits must be disjoint sets. Can the situation described in the theorem above occur frequently? Some examples might be in order here.

REMARK 1. For each $z \in \mathbb{D}$, we have that $\sum_{\lambda \in \mathrm{Orb}(z)}(1 - |\lambda|)$ is convergent, that is the Blaschke product having zeros $(\phi^{(n)}(z))_n$, each with multiplicity 1, is convergent.

Proof. It is easy to verify that for each positive integer n we have that

$$(6) \qquad 1 - |\phi^{(n)}(z)|^2 = \frac{4(1 - |z|^2)}{(1 - \frac{1}{3^n})^2|z|^2 + 2(1 - \frac{1}{3^{2n}})\mathrm{Re}z + (1 + \frac{1}{3^n})^2} \cdot \frac{1}{3^n}$$

and

$$(7) \qquad 1 - |\phi^{(-n)}(z)|^2 = \frac{4(1 - |z|^2)}{(1 - \frac{1}{3^n})^2|z|^2 - 2(1 - \frac{1}{3^{2n}})\mathrm{Re}z + (1 + \frac{1}{3^n})^2} \cdot \frac{1}{3^n}.$$

It is equally easy to deduce from (6) that the series $\sum_{n \geq 1}(1 - |\phi^{(n)}(z)|^2)$ is convergent since $\sum_{n \geq 1}\frac{1}{3^n} < \infty$. For the same reason $\sum_{n \geq 1}(1 - |\phi^{(-n)}(z)|^2) < \infty$. Observe that, if $(\lambda_n)_n$ is a sequence in $\mathbb{D}$, then $\sum_n(1 - |\lambda_n|) < \infty$ if and only if $\sum_n(1 - |\lambda_n|^2) < \infty$ because

$$1 - |\lambda_n|^2 \leq 2(1 - |\lambda_n|) \leq 2(1 - |\lambda_n|^2).$$

We deduce $\sum_{n \in \mathbb{Z}}(1 - |\phi^{(n)}(z)|) < \infty$ for each $z \in \mathbb{D}$. □

We have already observed that, if $z, w \in \mathbb{D}$ then $\mathrm{Orb}(z)$ and $\mathrm{Orb}(w)$ either coincide or are disjoint. In connection with that we wish to make the following

REMARK 2. There exist convergent Blaschke products B such that B is an eigenfunction of C_ϕ and $Z(B)$ is an infinite union of distinct orbits.

Proof. Choose a sequence $(z_n)_n$ in $\mathbb{D}$ such that for each $i, j \geq 0$, $i \neq j$, we have $\mathrm{Orb}(z_i) \neq \mathrm{Orb}(z_j)$ and such that both 1 and -1 are not accumulation points for the set $\{z_j : j \geq 1\}$. Taking into account this last condition and equalities (6) and (7) we deduce that there exist $c, c' > 0$ such that

$$(8) \qquad 1 - |\phi^{(n)}(z_j)|^2 \leq c \frac{1}{3^n}(1 - |z_j|^2) \quad n \geq 1, j \geq 1$$

and

$$(9) \qquad 1 - |\phi^{(-n)}(z_j)|^2 \leq c' \frac{1}{3^n}(1 - |z_j|^2) \quad n \geq 1, j \geq 1.$$

To see it is so, observe that

$$(3^n + 1) \pm (3^n - 1)z_j = (3^n - 1)(\frac{3^n + 1}{3^n - 1} \pm z_j).$$

As $n \to \infty$, $\frac{3^n + 1}{3^n - 1} \to 1$, but the set $\{z_j : j \geq 1\}$ is not arbitrarily close to 1 or -1. Now, if we require that $\sum_{j \geq 1}(1 - |z_j|) < \infty$, we deduce by (8) and (9) that the Blaschke product having zeros $(\phi^{(n)}(z_j))_{n \in \mathbb{Z}, j \geq 1}$ each with multiplicity 1, is convergent and is, according to Theorem 3.5 an eigenfunction of C_ϕ $\square$

These two theorems completely characterize those inner functions which are eigenfunctions of C_ϕ. Indeed, if u is inner, $u = BS_\nu$ where B is a (possibly constant) Blaschke product and ν is a singular, positive, finite, Borel measure on $\mathbb{T}$ (see [5], [6], [7], or [13]). This decomposition of u is unique up to multiplication with a unimodular constant. u is an eigenfunction of C_ϕ if and only if u and $u \circ \phi$ mutually divide each other as inner functions. This is true if and only if the Blaschke products, respectively the singular inner functions associated to u and $u \circ \phi$ mutually divide each other ([1]), i.e. if and only if both B and S_ν are eigenfunctions of C_ϕ. The description of the inner eigenfunctions of C_ϕ provided in this section works for any composition operator induced by a disk automorphism since in the proofs of the two theorems describing the singular inner eigenfunctions and the Blaschke products which are eigenfunctions we only used the fact that ϕ is a disk automorphism.

4. Some Applications

In this section we make use of the previously obtained results to decide if the cyclic, invariant subspaces of C_ϕ are minimal invariant or not.

Proposition 4.1 *If ν is a positive, finite, singular, Borel measure on $\mathbb{T}$, and $\nu(\{1\}) \neq 0$, then $\mathcal{K}_{S_\nu}$ is not minimal invariant.*

Proof. If $\nu(\{1\}) = a > 0$, $\exp(-a\frac{1+z}{1-z})$ is a common, inner divisor of the family of inner functions $\{S_\nu \circ \phi^{(n)} : n \geq 1\}$, and consequently divides the greatest common inner divisor S_μ, of the same family of inner functions. The fact that the greatest

inner divisor referred above is a singular inner function is clear, since it is an inner function without zeros in $\mathbb{D}$. To see that $\exp(-a\frac{1+z}{1-z})$ is a common divisor, observe that $\exp(-a\frac{1+z}{1-z}) \circ \phi^{(n)} = \exp(-3^n a\frac{1+z}{1-z})$. Therefore we have that $\mu(\{1\}) \geq a > 0$, (see [1]). By Corollary 3.3, S_μ cannot be an eigenfunction of C_ϕ, and by Theorem 1.3, $\mathcal{K}_{S_\nu}$ cannot be minimal invariant. $\qquad \square$

So if we need a minimal invariant cyclic subspace generated by a nonconstant, singular inner function S_ν, we should first make sure that 1 and -1 are in the support of ν, and $\nu(\{1\}) = 0$. Nevertheless, this is not enough.

Proposition 4.2 *Under the assumptions above on ν, if $\int_{\mathbb{T}} \frac{1}{\theta^2} d\nu(\theta) < \infty$ then $\mathcal{K}_{S_\nu}$ is not minimal invariant.*

Proof. Consider $z = r$, for $0 < r < 1$. We have

$$\frac{1 - |S_\nu(z)|}{1 - |z|} = \frac{1 - e^{-\int_{\mathbb{T}} P(z,\theta)d\nu(\theta)}}{1 - |z|} \leq \frac{\int_{\mathbb{T}} P(z,\theta)d\nu(\theta)}{1 - |z|}$$

because $e^x \geq x + 1$, for each $x \in \mathbb{R}$. So

$$\frac{1 - |S_\nu(z)|}{1 - |z|} \leq \int_{\mathbb{T}} \frac{P(r,\theta)}{1 - r} d\nu(\theta)$$

and for $0 < |\theta| \leq \pi$, we have that $P(r,\theta) \leq \frac{\pi^2}{\theta^2}(1 - r)$. Therefore

$$\frac{1 - |S_\nu(z)|}{1 - |z|} \leq \pi^2 \int_{\mathbb{T}} \frac{1}{\theta^2} d\nu(\theta) < \infty$$

for each r, $0 < r < 1$. Let now $r \to 1$, and deduce $\liminf_{z \to 1} \frac{1 - |S_\nu(z)|}{1 - |z|} < \infty$, so by the Julia-Carathéodory theorem ([4], Theorem 2.44), the nontangential limit of S_ν at 1 exists and is unimodular, and hence nonzero. The conclusion follows by Theorem 2.2. $\qquad \square$

5. Bilateral Orbits of Invertible Operators

The purpose of this section is to generalize the main idea in [2] to a statement valid for a larger class of invertible Banach space operators and then, use this result to give a shorter proof to a more general version of Theorem 1.4. Throughout this section $\mathcal{X}$ will denote a complex Banach space and T will be an invertible, bounded operator acting on $\mathcal{X}$. The bilateral orbit of $x \in \mathcal{X}$ under T is the set $\mathrm{Orb}(x) = \{T^n x : n \in \mathbb{Z}\}$. We say that the bilateral orbit of x approaches 0 exponentially if there exist constants $M > 0$ and a, $0 < a < 1$ such that $\|T^n x\| \le M a^n$ for each $n \ge 1$, or $\|T^{-n} x\| \le M a^n$ for each $n \ge 1$. The main idea in [2] was constructing eigenvectors of the form described in Theorem 1.4. This construction works in the following general framework.

Theorem 5.1 *If $x \ne 0$ and $\overline{\lim} \|T^{-n} x\|^{\frac{1}{n}} \, \overline{\lim} \|T^n x\|^{\frac{1}{n}} < 1$, then the point spectrum of the restriction of T to the invariant subspace $\vee_{n=-\infty}^{\infty} T^n x$ contains an open annulus centered at 0.*

Proof. Denote $l = \overline{\lim} \|T^n x\|^{\frac{1}{n}} < \dfrac{1}{\overline{\lim} \|T^{-n} x\|^{\frac{1}{n}}} = L$. Consider the vector $v_\lambda = \sum_{n=-\infty}^{\infty} \lambda^n T^{-n} x$. Observe that $\overline{\lim} \|\lambda^n T^{-n} x\|^{\frac{1}{n}} = |\lambda| \, \overline{\lim} \|T^{-n} x\|^{\frac{1}{n}} < 1$ if $|\lambda| < L$, and $\overline{\lim} \|\lambda^{-n} T^n x\|^{\frac{1}{n}} = |\lambda|^{-1} \overline{\lim} \|T^n x\|^{\frac{1}{n}} < 1$ if $l < |\lambda|$ so the series we considered converges for each λ in the annulus $A = \{z \in \mathbb{C} : l < |z| < L\}$. Clearly $T v_\lambda = \lambda v_\lambda$. The only thing there is to prove now is that $v_\lambda \ne 0$ for all λ in an open annulus. Since $x \ne 0$ we can choose φ a bounded linear functional on $\mathcal{X}$ such that $\varphi(x) \ne 0$. Therefore the function $f(\lambda) = \sum_{n=-\infty}^{\infty} \lambda^n \varphi(T^{-n} x) = \varphi(v_\lambda)$ is a nonzero function analytic on A and therefore its set of zeros will be a discrete subset of A. This means that v_λ can be zero only if λ belongs to this discrete set so the point spectrum of our operator will contain a full open annulus. $\square$

The statement in Theorem 1.4 can be generalized in this framework as follows.

Corollary 5.2 *If $x \ne 0$, $\mathrm{Orb}(x)$ is bounded and approaches 0 exponentially, then the point spectrum of the restriction of T to the invariant subspace $\vee_{n=-\infty}^{\infty} T^n x$ contains an open annulus centered at 0.*

Proof. If $\|T^n x\| \le M a^n$ then $\overline{\lim} \|T^n x\|^{\frac{1}{n}} \le a$. The fact that the bilateral orbit is bounded implies that $\overline{\lim} \|T^{-n} x\|^{\frac{1}{n}} \le 1$. Hence

$$\overline{\lim} \|T^{-n} x\|^{\frac{1}{n}} \, \overline{\lim} \|T^n x\|^{\frac{1}{n}} < 1.$$

If $\|T^{-n} x\| \le M a^n$ replace T by T^{-1} in the argument above. $\square$

We return now to the hyperbolic composition operator and give a short proof to a generalized version of Theorem 1.4.

Theorem 5.3 *If u is a nonzero function in H^2 such that there exist constants ε and δ, $-\frac{1}{2} < -\delta < \varepsilon < \frac{1}{2}$ so that*

$$(i) \qquad \frac{|u(e^{i\theta})|}{|1 - e^{i\theta}|^\varepsilon} \qquad \text{is essentially bounded on an open arc } C_1 \text{ containing } 1$$

(ii) $\qquad \dfrac{|u(e^{i\theta})|}{|1+e^{i\theta}|^{\delta}}$ $\quad$ *is essentially bounded on an open arc C_{-1} containing -1*

then the point spectrum of the restriction of C_{ϕ} to the invariant subspace $\vee_{n=-\infty}^{\infty} C_{\phi}^{n} u$ contains an open annulus centered at 0.

Proof. We shall prove $\overline{\lim}\,||C_{\phi}^{-n}u||^{\frac{1}{n}} \le 3^{-\delta} < 3^{\varepsilon} \le (\overline{\lim}\,||C_{\phi}^{n}u||^{\frac{1}{n}})^{-1}$. Like in the proof of Theorem 2.2 write

$$(10)\quad ||C_{\phi}^{n}u||^{2} = \frac{1}{2\pi}\int_{\phi^{(-n)}(C_1)}|u\circ\phi^{(n)}(e^{i\theta})|^2 d\theta + \frac{1}{2\pi}\int_{\phi^{(-n)}(\mathbb{T}\backslash C_1)}|u\circ\phi^{(n)}(e^{i\theta})|^2 d\theta =$$

$$= \frac{1}{2\pi}\int_{\phi^{(-n)}(C_1)}|u\circ\phi^{(n)}(e^{i\theta})|^2 d\theta + \frac{1}{2\pi}\int_{|\theta|>c}|u(e^{i\theta})|^2 P(\phi^{(n)}(0),\theta)d\theta \le$$

$$\le \frac{1}{2\pi}\int_{\phi^{(-n)}(C_1)}|u\circ\phi^{(n)}(e^{i\theta})|^2 d\theta + P(\phi^{(n)}(0),c)||u||^2$$

where $-c$ and c are the arguments of the endpoints of C_1. Solve now (2) for 3^n and get the identity

$$(11)\qquad\qquad \frac{(1-z)(1+\phi^{(n)}(z))}{(1+z)(1-\phi^{(n)}(z))} = 3^n.$$

Use now (i) and (11) to obtain that

$$|u(\phi^{(n)}(e^{i\theta}))|^2 = \frac{1}{3^{2n\varepsilon}}\left(\frac{|1-e^{i\theta}|}{|1+e^{i\theta}|}\right)^{2\varepsilon}|u(\phi^{(n)}(e^{i\theta}))|^2\left(\frac{|1+\phi^{(n)}(e^{i\theta})|}{|1-\phi^{(n)}(e^{i\theta})|}\right)^{2\varepsilon} \le$$

$$\le \frac{M}{3^{2n\varepsilon}}\left(\frac{|1-e^{i\theta}|}{|1+e^{i\theta}|}\right)^{2\varepsilon}\quad \text{a.e. on } \phi^{(-n)}(C_1)$$

for some $M > 0$. Taking (10) and this last inequality into consideration and writing the Poisson kernel explicitly we obtain

$$(12)\qquad\qquad ||C_{\phi}^{n}u||^{\frac{1}{n}} \le \frac{1}{3^{\varepsilon}}\left(\frac{1}{2\pi}M\int_{-\pi}^{\pi}\left(\frac{|1-e^{i\theta}|}{|1+e^{i\theta}|}\right)^{2\varepsilon}d\theta + \right.$$

$$\left. +4\frac{3^{2n\varepsilon+n}}{(3^n+1)^2(1-2|\phi^{(n)}(0)|\cos c + |\phi^{(n)}(0)|^2)}||u||^2\right)^{\frac{1}{2n}}$$

Observe now that $\int_{-\pi}^{\pi}(|1-e^{i\theta}|/|1+e^{i\theta}|)^{2\varepsilon}d\theta$ is finite if and only if $-\frac{1}{2} < \varepsilon < \frac{1}{2}$ and

$$\lim_{n\to\infty}\frac{3^{2n\varepsilon+n}}{(3^n+1)^2(1-2|\phi^{(n)}(0)|\cos c + |\phi^{(n)}(0)|^2)} < \infty$$

only if $\varepsilon \le \frac{1}{2}$. Now let n go to infinity in (12) and deduce $\overline{\lim}\,||C_{\phi}^{n}u||^{\frac{1}{n}} \le 3^{-\varepsilon}$ i.e. $3^{\varepsilon} \le 1/\overline{\lim}\,||C_{\phi}^{n}u||^{\frac{1}{n}}$. To obtain $\overline{\lim}\,||C_{\phi}^{-n}u||^{\frac{1}{n}} \le 3^{-\delta}$ repeat the same argument with -1 instead of 1, C_{ϕ}^{-n} instead of C_{ϕ}^{n}, C_{-1} instead of C_1, and δ instead of ε. $\square$

REMARK 3. To get Theorem 1.4 choose $\varepsilon > 0$ and $\delta = 0$ in Theorem 5.3. Also observe that, by the proof of Theorem 2.2 condition (a) in Theorem 1.4 is sufficient for $(C_\phi^{-n}u)_n$ to be a norm-bounded sequence whereas, by the proof of Theorem 5.3, condition (b) is sufficient for the sequence $(C_\phi^n u)_n$ to approach exponentially 0 with rate $a = 3^{-\varepsilon}$.

The essential boundedness conditions (i) and (ii) can be replaced by more general mean boundedness conditions.

Theorem 5.4 *If $u \in H^2$, $u \neq 0$ is such that there exist constants $\varepsilon, \delta, p_1, p_2$, and open arcs C_1 and C_{-1} containing 1 and -1, respectively, such that*

$(i)'$
$$\int_{C_1} \frac{|u(e^{i\theta})|^{2p_1}}{|1 - e^{i\theta}|^{2p_1\varepsilon}} P(\phi^{(n)}(0), \theta) d\theta \quad \text{is bounded for } n \geq 1$$

$(ii)'$
$$\int_{C_{-1}} \frac{|u(e^{i\theta})|^{2p_2}}{|1 + e^{i\theta}|^{2p_2\delta}} P(\phi^{(-n)}(0), \theta) d\theta \quad \text{is bounded for } n \geq 1$$

and if $-\delta < \varepsilon$, $-\frac{1}{2q_1} < \varepsilon < \frac{1}{2q_1}$, $-\frac{1}{2q_2} < \delta < \frac{1}{2q_2}$, where $q_1 = 1 - \frac{1}{p_1}$, $q_2 = 1 - \frac{1}{p_2}$, $1 < p_1, 1 < p_2$, then the point spectrum of the restriction of C_ϕ to the invariant subspace $\vee_{n=-\infty}^{\infty} C_\phi^n u$ contains an open annulus centered at 0.

Proof. The only difference between the proof of Theorem 5.3 and this proof is that here we use Hölder's inequality for $1 < p_i, q_i < \infty$, $i = 1, 2$ instead of using it for 1 and ∞ as we did in the other proof. Now we argue exactly as in the proof of Theorem 5.3, until we obtain

$$|u(\phi^{(n)}(e^{i\theta}))|^2 = \frac{1}{3^{2n\varepsilon}} \left(\frac{|1 - e^{i\theta}|}{|1 + e^{i\theta}|} \right)^{2\varepsilon} |u(\phi^{(n)}(e^{i\theta}))|^2 \left(\frac{|1 + \phi^{(n)}(e^{i\theta})|}{|1 - \phi^{(n)}(e^{i\theta})|} \right)^{2\varepsilon} \leq$$

$$\leq \frac{M}{3^{2n\varepsilon}} \left(\frac{|1 - e^{i\theta}|}{|1 + e^{i\theta}|} \right)^{2\varepsilon} \frac{|u \circ \phi^{(n)}(e^{i\theta})|^2}{|1 - \phi^{(n)}(e^{i\theta})|^{2\varepsilon}} \quad \text{a.e. on } \phi^{(-n)}(C_1)$$

for some $M > 0$. Integrate on $\phi^{(-n)}(C_1)$ and use Hölder's inequality to get

$$\int_{\phi^{(-n)}(C_1)} |u \circ \phi^{(n)}(e^{i\theta})|^2 \leq$$

$$\leq \frac{M}{3^{2n\varepsilon}} \left(\int_{\phi^{(-n)}(C_1)} \left(\frac{|1 - e^{i\theta}|}{|1 + e^{i\theta}|} \right)^{2q_1\varepsilon} d\theta \right)^{\frac{1}{q_1}} \left(\int_{\phi^{(-n)}(C_1)} \frac{|u \circ \phi^{(n)}(e^{i\theta})|^{2p_1}}{|1 - \phi^{(n)}(e^{i\theta})|^{2p_1\varepsilon}} d\theta \right)^{\frac{1}{p_1}}$$

The last quantity is equal to

$$\frac{M}{3^{2n\varepsilon}} \left(\int_{\phi^{(-n)}(C_1)} \left(\frac{|1 - e^{i\theta}|}{|1 + e^{i\theta}|} \right)^{2q_1\varepsilon} d\theta \right)^{\frac{1}{q_1}} \left(\int_{C_1} \frac{|u(e^{i\theta})|^{2p_1}}{|1 - e^{i\theta}|^{2p_1\varepsilon}} P(a_n, \theta) d\theta \right)^{\frac{1}{p_1}}$$

$$\leq \frac{M'}{3^{2n\varepsilon}} \left(\int_{\phi^{(-n)}(C_1)} \left(\frac{|1 - e^{i\theta}|}{|1 + e^{i\theta}|} \right)^{2q_1\varepsilon} d\theta \right)^{\frac{1}{q_1}}$$

for some $M' > 0$, where $a_n = \phi^{(n)}(0)$. We took of course condition $(i)'$ into consideration. We continue now exactly as in the proof of Theorem 5.3 replacing in (12) the integral $\int_{-\pi}^{\pi} \left(\frac{|1 - e^{i\theta}|}{|1 + e^{i\theta}|} \right)^{2\varepsilon} d\theta$ by the integral $\left(\int_{-\pi}^{\pi} \left(\frac{|1 - e^{i\theta}|}{|1 + e^{i\theta}|} \right)^{2q_1\varepsilon} d\theta \right)^{\frac{1}{q_1}}$. This last integral is finite only if $-\frac{1}{2q_1} < \varepsilon < \frac{1}{2q_1}$. Like in the proof of Theorem 5.3 we get $\overline{\lim} \, ||C_\phi^n u||^{\frac{1}{n}} \le 3^{-\varepsilon}$ and $\overline{\lim} \, ||C_\phi^{-n} u||^{\frac{1}{n}} \le 3^{-\delta}$. $\qquad\square$

Let's give an example where Theorem 5.4 can be applied. Suppose $p > 2$ and u is analytic on $\mathbb{D}$, such that $\int_{-\pi}^{\pi} |u(e^{i\theta})|^p / |1 - e^{i\theta}|^2 d\theta < \infty$. In this case $u \in H^p(\mathbb{D}) \subseteq H^2$. Furthermore, if $\mathbb{P}$ denotes the right open half-plane $\mathbb{P} = \{z \in \mathbb{C} : \mathrm{Re}\,z > 0\}$, then the function $v(w) = u((w-1)/(w+1))$, $w \in \mathbb{P}$, is in $H^p(\mathbb{P})$ and the converse is also true i.e. if $v \in H^p(\mathbb{P})$ and we define $u(z) = v((1+z)/(1-z))$, $z \in \mathbb{D}$, then $u \in H^p(\mathbb{D})$ and $\int_{-\pi}^{\pi} |u(e^{i\theta})|^p / |1 - e^{i\theta}|^2 d\theta < \infty$. We refer to [7] for these statements.

Theorem 5.5 *If for some $p > 2$ we have that $\int_{-\pi}^{\pi} |u(e^{i\theta})|^p / |1 - e^{i\theta}|^2 d\theta < \infty$, $u \ne 0$, and $u(e^{i\theta})$ is essentially bounded on an open arc containing -1, then the point spectrum of the restriction of C_ϕ to the invariant subspace $\vee_{n=-\infty}^{\infty} C_\phi^n u$ contains an open annulus centered at 0.*

Proof. The essential boundedness of $u(e^{i\theta})$ on an open arc containing -1 shows that condition $(ii)'$ holds for $\delta = 0$ for any $p_2 > 1$. Choose now $p_1 > 1$ and $\varepsilon > 0$ such that $2p_1 < p$ and $\varepsilon < \min(1/p, 1/2q_1)$ where as usual $\frac{1}{q_1} = 1 - \frac{1}{p_1}$. Clearly $-\delta < \varepsilon$. Consider now the sequence

$$(13) \qquad \int_{-\pi}^{\pi} \frac{|u(e^{i\theta})|^{2p_1}}{|1 - e^{i\theta}|^{2p_1\varepsilon}} P(a_n, \theta) d\theta, \quad n \ge 1$$

where $a_n = \frac{3^n - 1}{3^n + 1} = \phi^{(n)}(0)$. Set $e^{i\theta} = \frac{it-1}{it+1}$, observe that $d\theta = \frac{2}{1+t^2} dt$, $P(a_n, \theta)$ is transformed into $\frac{3^n(1+t^2)}{3^{2n}+t^2}$, and hence the sequence in (13) is bounded if and only if the following sequence is bounded.

$$(14) \qquad \int_{-\infty}^{\infty} \frac{|v(it)|^{2p_1} 3^n (1+t^2)^{p_1\varepsilon}}{3^{2n} + t^2} dt, \quad n \ge 1$$

where $v \in H^p(\mathbb{P})$ is obtained from u by $v(w) = u((w-1)/(w+1))$, $w \in \mathbb{P}$, and $v(it)$ is the limit function of v. Consider now $M > 0$ and set $\alpha = p/2p_1$, $1/\beta = 1 - 1/\alpha$. Since $\frac{3^n |t|}{(3^{2n}+t^2)} \le \frac{1}{2}$ for each $n \ge 1$ and $t \in \mathbb{R}$, we can write

$$\int_{|t| \ge M} \frac{|v(it)|^{2p_1} 3^n (1+t^2)^{p_1\varepsilon}}{3^{2n} + t^2} dt \le \frac{1}{2} \int_{|t| \ge M} \frac{|v(it)|^{2p_1} (1+t^2)^{p_1\varepsilon}}{|t|} dt \le$$

$$\frac{1}{2} \left(\int_{|t| \ge M} |v(it)|^{2p_1\alpha} dt \right)^{\frac{1}{\alpha}} \left(\int_{|t| \ge M} \frac{(1+t^2)^{\beta p_1\varepsilon}}{|t|^\beta} dt \right)^{\frac{1}{\beta}} \le$$

$$\frac{1}{2} \left(\int_{|t| \ge M} |v(it)|^p dt \right)^{\frac{1}{\alpha}} \left(\int_{|t| \ge M} \frac{(1 + \frac{1}{t^2})^{\beta p_1\varepsilon}}{|t|^{\beta - 2\beta p_1\varepsilon}} dt \right)^{\frac{1}{\beta}} < \infty$$

if $\beta - 2\beta p_1 \varepsilon > 1$ which is equivalent to $\varepsilon < \frac{1}{p}$. Of course, above we made use of Hölder's inequality and the fact that $v \in H^p(\mathbb{P})$. So all there is to prove now in order to show that p_1 and ε satisfy condition $(i)'$ is that the following sequence is bounded.

$$\int_{-M}^{M} \frac{|v(it)|^{2p_1} 3^n (1+t^2)^{p_1\varepsilon}}{3^{2n} + t^2} dt, \quad n \geq 1$$

There is no problem to see that there is some $c > 0$ such that

$$\frac{|v(it)|^{2p_1} 3^n (1+t^2)^{p_1\varepsilon}}{3^{2n} + t^2} \leq c|v(it)|^{2p_1} \quad \text{if} \quad |t| \leq M, n \geq 1$$

and

$$\int_{-M}^{M} |v(it)|^p dt < \infty \quad \text{implies} \quad \int_{-M}^{M} |v(it)|^{2p_1} dt < \infty$$

because $2p_1 < p$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

6. Final Comments

The following fact, probably known to many mathematicians is worth mentioning here.

Proposition 6.1 *If $f \in H^2$ and u, F are the inner, respectively the outer part of f then f is an eigenfunction of C_ϕ if and only if both u and F are eigenfunctions of C_ϕ.*

Proof. If φ is an inner function, we denote its radial limit also by φ. This function naturally induces a measure $m\varphi^{-1}$ on $\mathbb{T}$ by

$$(15) \qquad\qquad\qquad m\varphi^{-1}(E) = m(\varphi^{-1}(E))$$

where m is the normalized Lebesgue measure on $\mathbb{T}$. It is proved in [10] that this measure is absolutely continuous with respect to m. Consequently, if $\varphi = \phi$ and E is a Borel subset of $\mathbb{T}$ such that $m(E) = 0$, then $m(\phi(E)) = 0$. Suppose now g is any H^1-function such that $|g(e^{i\theta})| = |F \circ \phi(e^{i\theta})|$ a.e. on $\mathbb{T}$, then $|g \circ \phi^{-1}(e^{i\theta})| = |F(e^{i\theta})|$ a.e. on $\mathbb{T}$, and since $h \rightarrow h \circ \phi^{-1}$ is a bounded operator on H^1 ([10]), $g \circ \phi^{-1} \in H^1$. The fact that F is outer implies ([7], pag. 62), $|g \circ \phi^{-1}(z)| \leq |F(z)|$ for each $z \in \mathbb{D}$, so that $|g(z)| \leq |F \circ \phi(z)|$ for each $z \in \mathbb{D}$. This means $F \circ \phi$ is an outer function, ([7], pag. 62). $u \circ \phi$ is obviously inner. So if $f \circ \phi = \lambda f$, then $u(\lambda F) = (u \circ \phi)(F \circ \phi)$, and the unicity of the decomposition of an H^2-function in the product of its inner and outer factors implies that both u and F must be eigenfunctions of C_ϕ. The converse of this implication is trivial. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

To complete the characterization of the eigenfunctions of C_ϕ one should now answer the following

Question *Which outer functions are eigenfunctions of C_ϕ ?*

Schroeder's equation for our automorphism ϕ, that is the functional equation $u \circ \phi = \lambda u$ was solved in [3], Proposition 4.4. The solution provided there doesn't

tell us though which of those functions are H^2 functions i.e. eigenfunctions of C_ϕ, which are inner or outer. Nevertheless one can use that characterization of the solutions of Schroeder's equation in order to realize there are lots of outer eigenfunctions of C_ϕ and they can have a rather general form, as we can see in the following

Example 6.2 *For each function u analytic and bounded on the annulus*

$$A = \{z \in \mathbb{C} : e^{-\frac{\pi^2}{\log 3}} < |z| < e^{\frac{\pi^2}{\log 3}}\}$$

there is a constant $c \in \mathbb{C}$, such that

$$v(z) = \exp\left\{\frac{\alpha}{\log 3}\log\left(\frac{1+z}{1-z}\right)\right\}\left\{u\left(\exp\left\{\frac{2\pi i}{\log 3}\log\left(\frac{1+z}{1-z}\right)\right\}\right) + c\right\}, \quad z \in \mathbb{D}$$

is an outer eigenfunction of C_ϕ associated to the eigenvalue $\lambda = e^\alpha$, for each α, $-\frac{1}{2}\log 3 < \operatorname{Re}\alpha < \frac{1}{2}\log 3$, where $\log$ denotes the principal branch of the logarithm function acting on $\mathbb{P}$.

Proof. Since $\frac{1+\phi(z)}{1-\phi(z)} = 3\frac{1+z}{1-z}$ and $c \circ \phi = c$, it is easy to verify that $C_\phi v = \lambda v$. It is equally easy to verify that the function $\gamma(z) = \exp\left\{\frac{\alpha}{\log 3}\log\left(\frac{1+z}{1-z}\right)\right\}$ and its reciprocal $\frac{1}{\gamma(z)}$ are in H^2 (see [10], proof of Theorem 6 for the details). Hence γ is an outer H^2 function. Choose c such that the function $w(z) = u\left(\exp\left\{\frac{2\pi i}{\log 3}\log\left(\frac{1+z}{1-z}\right)\right\}\right) + c$ be bounded bellow, which is possible since u is a bounded function. So w is an analytic, bounded function on $\mathbb{D}$ and so is its reciprocal $\frac{1}{w}$. Hence w is a bounded outer function. The product of a bounded outer function and an H^2 outer function is an H^2 outer function. $\square$

REMARK 4. The spectrum of C_ϕ is the closed annulus $A = \{z \in \mathbb{C} : 3^{-\frac{1}{2}} \le |z| \le 3^{\frac{1}{2}}\}$, ([10], Theorem 6). This author observed in [9] that the point spectrum is the interior of this annulus. The outer eigenfunctions exhibited in Example 6.2 correspond to each eigenvalue in the point spectrum.

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