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WEIGHTED COMPOSITION OPERATORS ON THE HILBERT HARDY SPACE OF A HALF–PLANE

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Abstract. Operators of type $f \to \psi f \circ \varphi$ acting on function spaces are called weighted composition operators. If the weight function ψ is the constant function 1, then they are called composition operators. We consider weighted composition operators acting on the Hilbert Hardy space of a half–plane and study compactness, boundedness, invertibility, normality, and spectral properties of such operators.

1. Introduction

Reproducing kernel Hilbert spaces (RKHS), are Hilbert spaces H consisting of complex valued functions on a set S, so that the point evaluation functionals are continuous. For that reason, for each $w \in S$, there is an H–function k_w (called the reproducing kernel–function of index w), so that the, so called, reproducing identity holds:

$$f(w) = \langle f, k_w \rangle \qquad f \in H,\, w \in S. \tag{1.1}$$

The reader is referred to [1] or [16] for the basics on RKHS.

An operator $T_{\psi,\varphi}$ of type

$$T_{\psi,\varphi} f = \psi f \circ \varphi$$

acting on some function space is called a weighted composition operator. We will refer to the functions ψ and φ as the weight symbol respectively the composition symbol of $T_{\psi,\varphi}$.

Weighted composition operators have been noticed for quite a while. To give an example, Frank Forelli proved as early as 1964 that the isometries of non–Hilbert Hardy spaces over a disc need to be special weighted composition operators [10]. Despite all that, a systematic study of weighted composition operators on spaces of analytic functions started only about 10 years ago [12], [23]. A short collection of interesting results were published, e.g. [3], [5], [12], [14], [18], [20], [23]. The current paper is triggered by such papers as [20] or [8], were the underlying space is the Hilbert Hardy space of a half–plane and some results on weighted composition operators on that space are proved. This author noted that a primer on such operators does not exist in the literature, [26]. This paper is intended to be such a primer in the hope it will be a **useful** tool for researchers willing to study profoundly weighted composition operators.

It is well known that, if such operators are bounded on some RKHS H consisting of scalar valued functions on some set S, then the following equation (called the

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Caughran–Schwarz equality) holds, see [23, Theorem 5]:

$$T_{\psi,\varphi}^* k_x = \overline{\psi(x)} k_{\varphi(x)} \quad x \in S. \tag{1.2}$$

As an immediate consequence, note that, if one works on a RKHS where the null function is not a reproducing kernel function (as is the case of the two RKHS we work with in this paper), then $T_{\psi,\phi}$ is the null operator if and only if $\psi = 0$ and ϕ is any selfmap of the common domain of definition of the functions in the space. Thus, to avoid this trivial situation, we will say throughout this paper: let $T_{\psi,\phi}$ be a nonzero weighted composition operator, when we want to avoid it. It should be emphasized that, if two nonzero weighted composition operators T_{ψ_1,ϕ_1} and T_{ψ_1,ϕ_1}, acting on any of the RKHS we consider below, are equal, that fact can happen if and only if $\psi_1 = \psi_2$ and $\phi_1 = \phi_2$ (see the proof of Theorem 5), and so those operators are uniquely identified by their inducing symbols.

If $\psi = 1$, we call $T_{1,\varphi}$ the composition operator induced by φ (or with symbol φ) and denote it C_φ since

$$C_\varphi f = f \circ \varphi \qquad f \in H.$$

Clearly, in this particular case, the Caughran–Schwarz equation looks as follows:

$$C_\varphi^* k_x = k_{\varphi(x)} \quad x \in S. \tag{1.3}$$

Also, note that, if $\varphi(z) = z$ is the identity function, then $T_{\psi,\varphi}$ coincides with M_ψ, the multiplication operator with symbol ψ

$$M_\psi f = \psi f \qquad f \in H. \tag{1.4}$$

We will be concerned in this paper with two RKHS, namely the Hilbert Hardy space over the open unit disc $\mathbb{U}$, a space we denote H^2, and the Hilbert Hardy space over the right open half–plane Π^+, which we denote $H^2(\Pi^+)$, with emphasis on the latter space. It is known that H^2 consists of all analytic functions on $\mathbb{U}$ having square summable Maclaurin coefficients. The Hilbert norm of that space can be described as follows:

$$\|f\|_2 = \sqrt{\sum_{n=0}^{\infty} |c_n|^2} = \sup_{0 \le r < 1} \sqrt{\int_{\mathbb{T}} |f(ru)|^2 \, dm(u)} \qquad f(z) = \sum_{n=0}^{\infty} c_n z^n \in H^2, \tag{1.5}$$

where $\mathbb{T} = \partial \mathbb{U}$ is the unit circle and $dm = d\theta/2\pi$, the normalized arclength measure on $\mathbb{T}$.

The space $H^2(\Pi^+)$ consists of all analytic functions on Π^+ with the property

$$\|f\| = \sup \left\{ \sqrt{\frac{1}{\pi} \int_{-\infty}^{+\infty} |f(x+iy)|^2 \, dy} : x > 0 \right\} < \infty.$$

The above quantity describes the Hilbert norm of $H^2(\Pi^+)$. For the theory of Hardy spaces, the reader is referred to [7], [11], or [17].

All composition operators on H^2 are bounded and a lot of them can be compact. The space $H^2(\Pi^+)$ is very different. This author proved that $H^2(\Pi^+)$ cannot support compact composition operators [21]. Compact operators are bounded, since they transform the unit ball of the space on which they act into a relatively compact (hence bounded) set. Thus, it was not unusual to investigate compactness prior to boundedness. The characterization of boundedness followed later [23], more exactly it was proved that ϕ, an analytic selfmap of Π^+, induces a bounded composition operator on $H^2(\Pi^+)$ if and only if ϕ fixes the point at infinity and has a finite angular derivative there.

More formally, let ϕ be an analytic selfmap of Π^+, $\gamma(z) = (1 + z)/(1 - z)$, the **Cayley** transform of $\mathbb{U}$ onto Π^+, and $\varphi = \gamma^{-1} \circ \phi \circ \gamma$ the conformal conjugate of ϕ by **Cayley's** transform and its inverse.

If the angular limit $\varphi(1)$ of φ at 1 equals 1, then the angular derivative $\varphi'(1)$ is known to exist if and only if and only if

$$C := \sup \left\{ \frac{\Re w}{\Re \phi(w)} : w \in \Pi^+ \right\} < \infty,$$

in which case, the equality $\varphi'(1) = C$ holds and the angular limit $\phi(\infty)$ of ϕ at infinity exists and satisfies the condition $\phi(\infty) = \infty$. For that reason, we denote $C = \phi'(\infty)$ and call it the angular derivative of ϕ at infinity **whether** finite or not. The reader is referred to [27] for angular limits and angular derivatives.

Thus, according to [23], ϕ induces a bounded composition operator on $H^2(\Pi^+)$, if and only if $\phi(\infty) = \infty$ and $\phi'(\infty) < \infty$. If that fact happens, then:

Theorem 1 ([9])**.**

$$\|C_\phi\| = r(C_\phi) = \|C_\phi\|_e = \sqrt{\phi'(\infty)}, \tag{1.6}$$

where $\|C_\phi\|_e$ denotes the essential norm of C_ϕ.

What makes weighted composition operators worth studying in a space where bounded unweighted composition operators are few and none are compact is, among other things, the fact that they can be compact and the differences from the weighted composition operators on H^2 are interesting. Here are some simple examples. If $T_{\psi,\varphi}$ is bounded on H^2 then ψ must belong to H^2 since $T_{\psi,\varphi}(1) = \psi$. On the other hand if $T_{\psi,\phi}$ is bounded on $H^2(\Pi^+)$, then it does not follow that $\psi \in H^2(\Pi^+)$, which is easy to see, since $T_{1,\phi} = C_\phi$, but $1 \notin H^2(\Pi^+)$. Nevertheless, Corollary 2 provides a large class of compact operators $T_{\psi,\phi}$ on $H^2(\Pi^+)$ with the properties $\psi \in H^2(\Pi^+)$ and C_ϕ is unbounded. The class of bounded weighted composition operators on $H^2(\Pi^+)$ contains a "unitarily equivalent copy" of the class of composition operators on H^2 (Example 2), which can be used to address unitarily invariant properties of weighted composition operators on $H^2(\Pi^+)$, by reducing them to the similar properties of the (intensely investigated), composition operators on H^2. A weighted composition operator $T_{\psi,\phi}$ on $H^2(\Pi^+)$ can be isometric although the composition operator C_ϕ is unbounded, or bounded, but not isometric (Remark 2).

In this introductory section, we chose to introduce notation and the main concepts. Also, we will briefly describe the results contained by the next sections. In section 2 we address the questions of when weighted composition operators on $H^2(\Pi^+)$ are compact, and more generally, bounded, providing necessary or sufficient conditions for boundedness or compactness (Propositions 2, 3, Theorems 2, 3, 4). We also study when such operators are invertible (Theorem 4), or isometric (Theorems 6, 7). Invertible isometries are called unitary operators. Such operators are characterized in Theorem 5. Unitary operators are normal. The study of normal weighted composition operators on $H^2(\Pi^+)$ is mainly done in section 3. Section 4 is dedicated to the study of spectral properties of weighted composition operators acting on $H^2(\Pi^+)$ and the description of numerical ranges. Recall that the numerical range $W(T)$ of a Hilbert space operator T is the set $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$, (see [16] for more on numerical ranges). Throughout this paper, the notation $\| \quad \|_\infty$ designates the supremum norm.

2. Boundedness and compactness

There has been some work on understanding weighted composition operators on H^2. One can obtain information on those acting on $H^2(\Pi^+)$ using that work and the following:

Proposition 1. *Let ϕ be an analytic selfmap of Π^+ and ψ an analytic map on the same set. Let $\varphi = \gamma^{-1} \circ \phi \circ \gamma$ be the conformal conjugate of ϕ by the* ***Cayley*** *transform and its inverse. Finally, denote by $\tilde{\psi}$ the map*

$$\tilde{\psi}(z) = \psi \circ \gamma(z) \frac{1 - \varphi(z)}{1 - z} \qquad z \in \mathbb{U}. \tag{2.1}$$

Then the operators $T_{\psi,\phi}$ and $T_{\tilde{\psi},\varphi}$ are unitarily equivalent.

Proof. The following are two unitary operators between the spaces H^2 and $H^2(\Pi^+)$, which are inverse to each other:

$$Uf := \frac{2g(z)}{1-z} \qquad f \in H^2(\Pi^+), g(z) = f\left(\frac{1+z}{1-z}\right), z \in \mathbb{U}$$

and

$$Vg := \frac{f(w)}{1+w} \qquad g \in H^2, f(w) = g\left(\frac{w-1}{w+1}\right), w \in \Pi^+.$$

To see these operators are isometric, recall that the way to transform $dm = d\theta/2\pi$, the normalized arclength measure, into dt, the Lebesgue measure on the vertical axis is

$$\frac{d\theta}{2\pi} = \frac{dt}{\pi(1+t^2)} \tag{2.2}$$

(see [17, page 243]). Therefore

$$\|Vg\|_2^2 = \int_{-\infty}^{+\infty} \frac{1}{|1+it|^2} \left| g\left(\frac{it-1}{it+1}\right)\right|^2 \frac{dt}{\pi} = \int_{-\pi}^{\pi} |g(e^{i\theta})|^2 \frac{d\theta}{2\pi}$$

and so, V is isometric from $H^2(\Pi^+)$ into H^2 and it is easy to check that V and U are each other's inverse. A routine computation shows that $UT_{\psi,\phi}V = T_{\tilde{\psi},\varphi}$. □

As a fast application, we record the following consequence:

Corollary 1. *If ψ is an analytic map on Π^+ then M_ψ, the multiplication operator induced by ψ, is bounded on $H^2(\Pi^+)$ if and only if ψ is a bounded analytic function, and, in that case, the spectrum of M_ψ is the closure of the range of ψ and the numerical range of the same operator equals the range of ψ.*

Indeed, using the same notations as in Proposition 1, let ϕ be the identity of Π^+, hence φ is the identity of $\mathbb{U}$ and so, by Proposition 1, $M_{\tilde{\psi}} = T_{\tilde{\psi},\varphi}$ is unitarily equivalent to $T_{\psi,\phi} = M_\psi$ so, the boundedness and descriptions of the spectrum and numerical range of M_ψ follow from the corresponding properties of multiplication operators acting on H^2, which are well known (see [19] for the numerical range).

Easy necessary conditions for the boundedness, respectively compactness, of weighted composition operators are obtained by evaluating their adjoints on kernel functions and using the Caughran–Schwartz equality (1.2). In the case of the RKHS, $H^2(\Pi^+)$, this results in the following:

Proposition 2. *Let ϕ be an analytic selfmap of Π^+ and ψ a map analytic on Π^+. In order that $T_{\psi,\phi}$ be bounded the following condition must hold*

$$M_{\psi,\phi} := \sup\left\{|\psi(w)|\sqrt{\frac{\Re w}{\Re\phi(w)}} : w \in \Pi^+\right\} < \infty. \tag{2.3}$$

If ψ is constant, then $T_{\psi,\phi}$ is bounded if and only if condition (2.3) holds and $\|T_{\psi,\phi}\| = M_{\psi,\phi}$. In order that $T_{\psi,\phi}$ be compact, the following condition must hold:

$$m_{\psi,\phi} := \limsup_{\Re w \to 0} |\psi(w)|\sqrt{\frac{\Re w}{\Re\phi(w)}} = 0. \tag{2.4}$$

If $T_{\psi,\phi}$ is compact and $\psi \neq 0$, then condition

$$\Re\phi(it)| > 0 \qquad \text{a.e.} \tag{2.5}$$

must be satisfied.

Proof. A straightforward computation with the operators U, V and the Szegö kernels

$$k_\alpha(z) = \frac{1}{1-\overline{\alpha}z} \qquad \alpha, z \in \mathbb{U}$$

which are well known to be the reproducing kernel–functions of the RKHS H^2, shows that the reproducing kernel functions of $H^2(\Pi^+)$ are the functions

$$K_\beta(w) = \frac{1}{2(w+\overline{\beta})} \qquad w, \beta \in \Pi^+.$$

By the Caughran–Schwarz equation (1.2) and the reproducing property (1.1) one can write:

$$M_{\psi,\phi} = \sup\left\{\frac{\|T^*_{\psi,\phi}K_\beta\|}{\|K_\beta\|} : \beta \in \Pi^+\right\} \leq \|T_{\psi,\phi}\|. \tag{2.6}$$

Thus, $M_{\psi,\phi}$ must be finite if $T_{\psi,\phi}$ is bounded. If $M_{\psi,\phi} = \|T_{\psi,\phi}\|$, we say that $T_{\psi,\phi}$ attains its norm on kernels. As one can see from Theorem 1, this happens if $\psi = 1$ and hence it will be true when ψ is any constant function.

One can see that the family of normalized kernels

$$\mathcal{F} = \left\{\frac{K_\beta}{\|K_\beta\|} : \beta \in \Pi^+\right\}$$

is a family of norm bounded functions and $\frac{K_\beta}{\|K_\beta\|}$ tends to 0 pointwise when $\Re\beta \to 0$. Since, $H^2(\Pi^+)$ is a RKHS, this means that $\frac{K_\beta}{\|K_\beta\|}$ tends weakly to 0 when $\Re\beta \to 0$. Therefore, if $T_{\psi,\phi}$ is compact, then

$$\frac{\|T^*_{\psi,\phi}K_\beta\|}{\|K_\beta\|} \to 0 \quad \text{if} \quad \Re\beta \to 0,$$

that is $m_{\psi,\phi} = 0$.

Finally, $T_{\psi,\phi}$ is compact if and only if $T_{\tilde{\psi},\varphi}$ is compact, and $\psi \neq 0$ if and only if $\tilde{\psi} \neq 0$, where the notation in the proof of Proposition 1 is used. In that case, condition

$$|\varphi| < 1 \quad \text{a.e.} \quad \text{on} \quad \mathbb{T}$$

must hold [23, Theorem 10]. Hence, given equality (2.2), condition (2.5) must be satisfied.

□

As we observed, the space $H^2(\Pi^+)$ does not support compact composition operators, so a weighted composition operator $T_{\psi,\phi}$ with constant symbol ψ is compact if and only if $\psi = 0$ and hence $T_{\psi,\phi}$ is the null operator. Nevertheless, nonzero compact weighted composition operators on $H^2(\Pi^+)$ do exist, as we show in the following.

2.1. **Compact operators.** One approach to understanding when operators in some class are compact is to check when they are Hilbert–Schmidt, mostly if their action on a complete orthonormal basis of that space is easy to understand. This is the approach followed in [23].

Theorem 2. *The operator $T_{\psi,\phi}$ has Hilbert–Schmidt norm $\| \quad \|_{HS}$ computable with the formula:*

$$\|T_{\psi,\phi}\|_{HS} = \sqrt{\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{|\psi(it)|^2}{\Re\phi(it)}\, dt}. \tag{2.7}$$

In particular, if there is some $c > 0$ so that

$$\Re\phi(w) \geq c \qquad w \in \Pi^+ \tag{2.8}$$

and $\psi \in H^2(\Pi^+)$, then $T_{\psi,\phi}$ is Hilbert–Schmidt (that is, the Hilbert–Schmidt norm of $T_{\psi,\phi}$ is finite).

Proof. If $T_{u,v}$ is a weighted composition operator on H^2, then according to [23, Theorem 9], its Hilbert–Schmidt norm is

$$\|T_{u,v}\|_{HS} = \sqrt{\int_{\mathbb{T}} \frac{|u|^2}{1-|v|^2}\, dm}. \tag{2.9}$$

Let ϕ, φ, ψ, and $\tilde{\psi}$ be as in Proposition 1. Then, by that proposition, and (2.9), one has that the operators $T_{\psi,\phi}$ and $T_{\tilde{\psi},\varphi}$ have the same Hilbert–Schmidt norm, namely

$$\|T_{\psi,\phi}\|_{HS} = \sqrt{\int_{\mathbb{T}} \left|\frac{1-\varphi(u)}{1-u}\right|^2 \left|\psi\left(\frac{1+u}{1-u}\right)\right|^2 \frac{dm(u)}{1-|\varphi(u)|^2}}. \tag{2.10}$$

Therefore one can write

$$\begin{aligned}
&\|T_{\psi,\phi}\|_{HS}^2 = \\
&\int_{-\infty}^{\infty} \left|1-\frac{\phi(it)-1}{\phi(it)+1}\right|^2 \left|1-\frac{it-1}{it+1}\right|^{-2} |\psi(it)|^2 \left(1-\left|\frac{\phi(it)-1}{\phi(it)+1}\right|^2\right)^{-1} \frac{dt}{\pi(1+t^2)} \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{4}{|1+\phi(it)|^2}|\psi(it)|^2 \frac{1+t^2}{4} \frac{|1+\phi(it)|^2}{|1+\phi(it)|^2-|\phi(it)-1|^2} \frac{dt}{1+t^2} \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{|\psi(it)|^2}{\Re\phi(it)}\, dt.
\end{aligned}$$

Hence (2.7) holds. □

Corollary 2. *Let $\psi \in H^2(\Pi^+)$ and let ϕ be a bounded analytic selfmap of Π^+, so that the closure of $\phi(\Pi^+)$ is a subset of Π^+. Then $T_{\psi,\phi}$ is Hilbert-Schmidt (and hence bounded), but C_ϕ is unbounded.*

Indeed, condition (2.8) is satisfied, but $\phi(\infty) \neq \infty$. Relative to the situation in Corollary 2, let us note that:

Proposition 3. *If $T_{\psi,\phi}$ is bounded, then $\psi/(1+\phi) \in H^2(\Pi^+)$. In particular, if ϕ is a bounded analytic selfmap of Π^+, ψ is analytic on Π^+, and $T_{\psi,\phi}$ is bounded, then ψ must belong to $H^2(\Pi^+)$.*

Proof. Indeed, it is known that the function $1/(1+w)$ is in $H^2(\Pi^+)$ (since $1/(1+w) = 2K_1$), so, if $T_{\psi,\phi}$ is bounded, then $T_{\psi,\varphi}(1/(1+w)) = \psi/(1+\phi) \in H^2(\Pi^+)$. If ϕ is bounded, then there is some $c > 0$ so that

$$|\psi| \le c\frac{|\psi|}{|1+\phi|}$$

which ends the proof. □

Theorem 3. *If $\psi/(1+\phi) \in H^2(\Pi^+)$ and*

$$\lim_{\delta\to 0^+}\left(\operatorname{ess\,sup}\left\{\frac{|\psi(it)|^2(1+t^2)}{|1+\phi(it)|^2} : t\in\mathbb{R}\,,\, \frac{|\phi(it)-1|}{|\phi(it)+1|} \ge 1-\delta\right\}\right) = 0, \tag{2.11}$$

then $T_{\psi,\phi}$ is compact.

Proof. Operator $T_{\psi,\phi}$ is compact if and only if the unitarily equivalent operator $T_{\tilde\psi,\varphi}$ is compact, where the same notations as in Proposition 1 are used. By [23, Theorem 12], the later operator is compact if $\tilde\psi \in H^2$ and the following condition holds

$$\lim_{\delta\to 0^+}\left(\operatorname{ess\,sup}\left\{|\tilde\psi(\zeta)|^2 : \zeta\in\mathbb{T}\,,\, |\varphi(\zeta)| \ge 1-\delta\right\}\right) = 0. \tag{2.12}$$

As one can easily check, $V\tilde\psi = \psi/(1+\phi)$ and so, $\psi/(1+\phi) \in H^2(\Pi^+)$ if and only if $\tilde\psi \in H^2$.

In that case, it is easy to see condition (2.11) holds if and only if (2.12) holds. □

To give a simple example:

Example 1. *Let $\psi(w) = \frac{1}{w+1}$ and $\phi(w) = w+1$, $w \in \Pi^+$. One can easily check $T_{\psi,\phi}$ satisfies the condition in* Theorem 3 *and is therefore compact, but C_ϕ is not compact because it is bounded, since $\phi(\infty) = \infty$ and $\phi'(\infty) = 1 < \infty$.*

It is equally easy to check that the operator in the above example is actually Hilbert–Schmidt by Theorem 2, so the immediate question would be if compact, non–Hilbert–Schmidt weighted composition operators acting on $H^2(\Pi^+)$ exist. The answer is affirmative, and one way to get it is the following:

Example 2. *Given any fixed analytic selfmap ϕ of Π^+, let*

$$\tilde\phi(w) = \frac{1+\phi(w)}{1+w} \qquad w \in \Pi^+. \tag{2.13}$$

*The operators $T_{\tilde\phi,\phi}$ and C_φ are unitarily equivalent (where, as usual, φ denotes the conformal conjugate of ϕ via **Cayley's** transform).*

Indeed, the fact above follows by a straightforward application of Proposition 1. Consequently, the known properties of composition operators on H^2 can be used via Example 2, to provide examples of weighted composition operators on $H^2(\Pi^+)$ with similar properties. For instance, one can give examples of weighted composition operators on $H^2(\Pi^+)$ which are compact but not Hilbert–Schmidt, essentially quasinilpotent without being power compact, and so on (see [6] and [27] for more examples of unitarily invariant properties of composition operators on H^2).

2.2. **Invertible and isometric operators.** The general boundedness criteria for weighted composition operators on H^2 are not easy to apply, since they are often in terms of Carleson measures [4], [20], but checking that a given measure is Carleson may be as challenging as finding the norm of a weighted composition operator with the definition. In some particular cases, easier criteria can be derived. We use them in the following to prove when certain weighted composition operators on $H^2(\Pi^+)$ are bounded. In the study of weighted composition isometries on H^2 the notion of inner function is important. Let us recall that an analytic selfmap of the unit disc is called inner if its radial limit function which necessarily exists a.e. on $\mathbb{T}$ is unimodular a.e. Also, unimodular constant functions are considered inner. We will say that an analytic selfmap ϕ of Π^+ is inner, if its conformal conjugate φ is an inner function (which means that ϕ is an analytic selfmap of Π^+ with the property $\Re\phi(it) = 0$ a.e.). A weighted composition operator $T_{\psi,\phi}$ cannot be isometric if ϕ is not an inner function. Some of the simplest half–plane inner functions are the conformal conjugates of the disc automorphisms (called half–plane automorphisms). If different from the identity map, they come in three flavors: *elliptic* (those fixing just one point in Π^+), *parabolic* (those fixing exactly one point situated on the boundary $\partial\Pi^+$ of Π^+, possibly ∞), and *hyperbolic* (those fixing two distinct points situated on the boundary $\partial\Pi^+$ of Π^+). Recall that a finite product of disc **automorphisms** times a unimodular constant is called a finite Blaschke product. With this terminology we prove the following:

Theorem 4. *If ϕ is an analytic selfmap of Π^+, ψ is analytic on Π^+, and φ, the conformal conjugate of ϕ, is a finite Blaschke product, then $T_{\psi,\phi}$ is bounded if and only if the function*

$$F(w) := \frac{(w+1)\psi(w)}{1+\phi(w)} \qquad w \in \Pi^+ \tag{2.14}$$

is bounded. A weighted composition operator $T_{\psi,\phi}$ on $H^2(\Pi^+)$ is invertible if and only if ϕ is a half–plane automorphism and the function F is both bounded and bounded away from 0. The operator $T_{\psi,\phi}$ on $H^2(\Pi^+)$ is a Fredholm operator if and only if it is invertible.

Proof. Let $\tilde{\psi}$ be the function in (2.1). Then, by Proposition 1, $T_{\psi,\phi}$ is bounded if and only if the operator $T_{\tilde{\psi},\varphi}$ is bounded. Now according to [23, Theorem 6], the operator $T_{\tilde{\psi},\varphi}$ is bounded if and only if $\tilde{\psi}$ is a bounded analytic function on the unit disc. By an easy computation, one can see that the range of $\tilde{\psi}$ is the same as the range of F.

The operator $T_{\psi,\phi}$ is invertible if and only if the operator $T_{\tilde{\psi},\varphi}$ is invertible. According to [13, Theorem 2.0.1], the later fact happens if and only if φ is a disc automorphism and $\tilde{\psi}$ is both bounded and bounded away from 0, that is ϕ must be a half–plane automorphism and the function F must be bounded and bounded away from 0.

Operator $T_{\psi,\phi}$ is Fredholm if and only if the operator $T_{\tilde{\psi},\varphi}$ is Fredholm. According to [18, Theorem 3.5], $T_{\tilde{\psi},\varphi}$ is Fredholm if and only if φ is a disc automorphism and the multiplication operator $M_{\tilde{\psi}}$ is a Fredholm operator. Now a multiplication operator M_u on H^2 is Fredholm if and only if M_u is invertible, that is if and only if u is both bounded and bounded away from zero. Indeed u must be bounded since M_u needs to be a bounded operator. Also u needs to be bounded away from zero in order that M_u be Fredholm. Indeed, if arguing by contradiction one assumes M_u

is Fredholm, although u is not bounded away from zwero, one gets a contradiction, as follows. In that case, M_u is not invertible and so, the spectrum $\sigma(M_u)$ of M_u, equals the closure of the range of u, which is the closure of a nonempty open subset of the complex plane (in the interesting case when $u \neq 0$), and $0 \in \sigma(M_u)$. Such a set cannot be the spectrum of a Fredholm operator, since those operators are invertible modulo the compact operators and so, by Weyl's theorem ([16, Problem 143]), their spectrum can differ from the spectrum of an invertible operator, by containing 0 and some eigenvalues. But if u is not constant, M_u has no eigenvalues, which is very easy to check. One obtains a contradiction, which ends the proof. □

Corollary 3. *If $\phi(\infty) = \infty$, $\phi'(\infty) < \infty$, and $\|\psi\|_\infty < \infty$, then $T_{\psi,\phi}$ is bounded. If ϕ is a half–plane automorphism with property $\phi(\infty) = \infty$, then $T_{\psi,\phi}$ is bounded if and only if ψ is a bounded analytic function.*

Indeed, if $\phi(\infty) = \infty$, $\phi'(\infty) < \infty$, and $\|\psi\|_\infty < \infty$, then by Theorem 1 and Corollary 1, operators C_ϕ and M_ψ are bounded and so $T_{\psi,\phi} = M_\psi C_\phi$ is bounded. In case ϕ is a half–plane automorphism with property $\phi(\infty) = \infty$ and $T_{\psi,\phi}$ is bounded, then the map F in (2.14) is bounded, hence the map $\psi \circ \gamma(z)(1 - \varphi(z))/(1 - z)$ is a bounded analytic map on $\mathbb{U}$. Given that φ, the conformal conjugate of ϕ, is a disc automorphism fixing 1, the map $(1 - \varphi(z))/(1 - z)$ is both bounded and bounded away from 0. Thus $\psi \circ \gamma$ is a bounded analytic function and so, $\|\psi\|_\infty < \infty$.

An easy example of isometric weighted composition operator on $H^2(\Pi^+)$ is the following:

Example 3. *If $|\psi(it)| = 1$ a.e. and ϕ is inner, $\phi(\infty) = \infty$, and $\phi'(\infty) = 1$, then $T_{\psi,\phi}$ is isometric.*

Indeed, this author proved that C_ϕ is isometric on $H^2(\Pi^+)$ if and only if ϕ has the properties above [23]. On the other hand, if $|\psi(it)| = 1$ a.e., then M_ψ is also isometric, and clearly, $T_{\psi,\phi} = M_\psi C_\phi$.

If ψ is constant, then $T_{\psi,\phi}$ is unitary if and only if that constant is unimodular and ϕ is either the identity or a parabolic or hyperbolic automorphism fixing ∞ [25]. If we allow ψ to be nonconstant, the situation changes. More formally, we can prove:

Theorem 5. *The unitary weighted composition operators $T_{\psi,\phi}$ on $H^2(\Pi^+)$ are those induced by half–plane automorphisms ϕ paired with weight symbols ψ having the following form:*

$$\psi(w) = c\sqrt{\Re q}\frac{1 + \phi(w)}{\overline{q} + w} \qquad w \in \Pi^+, \tag{2.15}$$

where c is any unimodular constant and $q = \phi^{-1}(1)$.

If ϕ is a half–plane automorphism, then $T_{\psi,\phi}$ is an isometry acting on $H^2(\Pi^+)$ if and only if, ψ has the form

$$\psi(w) = \sqrt{\Re q}\, u(w)\frac{1 + \phi(w)}{\overline{q} + w} \qquad w \in \Pi^+, \tag{2.16}$$

where u is an analtic function on Π^+, unimodular a.e. on the imaginary axis, and $q = \phi^{-1}(1)$.

Proof. Using the same notation as in Proposition 1, one has that $T_{\psi,\phi}$ is unitary if and only if $T_{\tilde{\psi},\varphi}$ is unitary. According to [3], the latter fact is equivalent to φ being a disc automorphism paired with a function of type

$$\tilde{\psi}(z) = \frac{c\sqrt{1-|p|^2}}{1-\overline{p}z} \qquad z \in \mathbb{U},$$

where $p = \varphi^{-1}(0) = (\phi^{-1}(1) - 1)/(\phi^{-1}(1) + 1)$ and $|c| = 1$. By a straightforward computation, one gets that the above means that ψ must have form

$$\psi(w) = \frac{c\sqrt{1-|p|^2}}{1+\overline{p}} \frac{1+\phi(w)}{1-\frac{\overline{p}-1}{\overline{p}+1}w} \qquad w \in \Pi^+, \tag{2.17}$$

where c is any unimodular constant and $p = (q-1)/(q+1)$, which means ψ has form (2.15). Indeed, note that, one must have that

$$\psi(w)\frac{w+1}{\phi(w)+1} = \frac{c\sqrt{1-|p|^2}(w+1)}{1+\overline{p}+w(1-\overline{p})} \qquad w \in \Pi^+,$$

hence (2.17) must hold. One the other hand, the equality

$$\frac{\sqrt{1-|p|^2}}{1+\overline{p}} = \frac{(1+\overline{q})\sqrt{\Re q}}{|1+q|}$$

is valid as well. Thus $T_{\psi,\phi}$ is unitary if and only if

$$\psi(w) = c\frac{(1+\overline{q})\sqrt{\Re q}}{|1+q|} \frac{1+\phi(w)}{\overline{q}+w} \qquad w \in \Pi^+,$$

holds, that is if and only if (2.15) holds (since $|(1+\overline{q})/|1+q|| = 1$).

Operator $T_{\psi,\phi}$ is isometric if and only if $T_{\tilde{\psi},\varphi}$ is isometric and ϕ is a half–plane automorphism if and only if φ is a disc automorphism. On the other hand, $T_{\tilde{\psi},\varphi}$ is isometric if and only if representable as $T_{\tilde{\psi},\varphi} = M_v U$ where v is an inner function on the unit disc, M_v the (necessarily isometric) multiplication operator induced by it, and U a unitary, weighted composition operator on H^2, [24, Theorem 10]. On the other hand, two nonzero weighted composition operators on H^2, say T_{ψ_1,φ_1} and T_{ψ_2,φ_2}, are equal if and only if $\psi_1 = \psi_2$ and $\varphi_1 = \varphi_2$, a fact that follows easily from the necessary equalities

$$\psi_1 = T_{\psi_1,\varphi_1}(1) = T_{\psi_2,\varphi_2}(1) = \psi_2$$

and

$$\psi_1\varphi_1 = T_{\psi_1,\varphi_1}(z) = T_{\psi_2,\varphi_2}(z) = \psi_2\varphi_2.$$

Therefore, $\tilde{\psi}$ must have the form

$$\tilde{\psi} = v(z)\frac{c\sqrt{1-|p|^2}}{1-\overline{p}z} \qquad z \in \mathbb{U}$$

where v is inner and $\varphi(p) = 0$. By our computations above, that fact happens if and only if ψ has form (2.16) with $u = v \circ \gamma^{-1}$, an analytic function on Π^+, which is unimodular a.e. on the imaginary axis.

□

To give an example, note that:

Remark 1. *If ϕ is an elliptic half–plane automorphism, fixing* 1*, C_ϕ is unbounded, but*

$$T_{c\frac{1+\phi(w)}{1+w},\phi}$$

is unitary (where c is any unimodular constant).

Unitary operators are onto isometries. This author proved that a composition operator on $H^2(\Pi^+)$ is isometric if and only if it is induced by an inner function ϕ fixing ∞ and having property $\phi'(\infty) = 1$, [23, Proposition 4]. Recent results of this author make easy characterizing the isometric weighted composition operators on $H^2(\Pi^+)$ as follows:

Theorem 6. *In order that $T_{\psi,\phi}$ be an isometric weighted composition operator on $H^2(\Pi^+)$ it is necessary that ϕ be inner and $\psi/(1+\phi)$ have norm 1. Assuming that the previous necessary conditions hold, then $T_{\psi,\phi}$ is isometric if and only if the following condition holds*

$$\int_X \frac{|\psi(it)|^2}{|1+\phi(it)|^2}\,dt = \frac{1}{\Re\phi(1)}\int_X \frac{|\phi(1)-\phi(it)|^2}{1+t^2}\,dt \qquad X = \phi^{-1}(Y) \quad Y \in \mathcal{L}, \tag{2.18}$$

where $\mathcal{L}$ denotes the set of Lebesgue measurable subsets of $\mathbb{R}$. If ϕ is inner and ψ is unimodular a.e. on the imaginary axis, then $T_{\psi,\phi}$ is isometric if and only if $\phi(1) = 1$.

Proof. Let φ and $\tilde{\psi}$ be the same functions as in Proposition 1. By Proposition 1, $T_{\psi,\phi}$ is isometric if and only if $T_{\tilde{\psi},\varphi}$ is isometric, so by [24, Theorem 4], φ needs to be inner and hence, ϕ must be inner too. On the other hand, $f(w) = \frac{1}{1+w}$ is a norm one function in $H^2(\Pi^+)$, and so, if $T_{\psi,\phi}$ is isometric, then $T_{\psi,\phi}f = \frac{\psi}{1+\phi}$ must have norm 1 as well.

Now, if ϕ is inner and $\psi/(1+\phi)$ has norm 1, then, according to [24, Theorem 5], $T_{\tilde{\psi},\varphi}$ is isometric if and only if

$$\int_X |\tilde{\psi}|^2\,dm = \int_X \frac{dm(u)}{P(\varphi(0),\varphi(u))} \qquad X = \varphi^{-1}(Y) \qquad Y \in \tilde{\mathcal{L}}. \tag{2.19}$$

where $P(z,u)$, $z \in \mathbb{U}$, $u \in \mathbb{T}$ is the usual Poisson kernel, $dm = d\theta/2\pi$, and $\tilde{\mathcal{L}}$ denotes the set of Lebesgue measurable subsets of $\mathbb{T}$.

If $X = \varphi^{-1}(Y)$, for some $Y \in \tilde{\mathcal{L}}$, then, by a straightforward computation, one obtains that

$$\int_X |\tilde{\psi}|^2\,dm = \frac{1}{\pi}\int_{\phi^{-1}(\gamma(Y))} \frac{|\psi(it)|^2}{|1+\phi(it)|^2}\,dt. \tag{2.20}$$

Indeed, recall the change of measure formula $dm = dt/(\pi(1+t^2))$ and the identities $\tilde{\psi}(z) = \psi\circ\gamma(z)(1-\varphi(z))/(1-z)$, and $\phi = \gamma\circ\varphi\circ\gamma^{-1}$. Based on those equalities, one can write, (using notation Υ_S for the characteristic function of set S):

$$\int_X |\tilde{\psi}|^2\,dm = \int_{\mathbb{T}} |\psi\circ\gamma|^2 \left|\frac{1-\gamma^{-1}\circ\phi}{1-\gamma^{-1}}\right|^2 \circ\gamma\Upsilon_{\phi^{-1}\circ\gamma(Y)}\circ\gamma\,dm =$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} |\psi(it)|^2 \frac{1+t^2}{|1+\phi(it)|^2}\Upsilon_{\phi^{-1}\circ\gamma(Y)}\frac{dt}{1+t^2},$$

which proves that (2.20) holds. Let X and Y be as above. We claim that

$$\int_X \frac{dm(u)}{P(\varphi(0),\varphi(u))} = \frac{1}{\pi\Re\phi(1)}\int_{\phi^{-1}(\gamma(Y))} \frac{|\phi(1)-\phi(it)|^2}{1+t^2}\,dt. \tag{2.21}$$

Once the above claim is proved, (2.19), (2.20), and (2.21) combine into showing that $T_{\psi,\phi}$ is isometric if and only if (2.18) holds, since γ induces a bijection between $\tilde{\mathcal{L}}$ and $\mathcal{L}$.

To prove (2.21), note that

$$\int_X \frac{dm(u)}{P(\varphi(0),\varphi(u))} = \int_{\mathbb{T}} \left(\frac{1}{P(\varphi(0),\varphi(u))} \circ \gamma^{-1}\right) \circ \gamma \left(\Upsilon_X \circ \gamma^{-1}\right) \circ \gamma \, dm(u) =$$

$$\int_{\phi^{-1}(\gamma(Y))} \frac{|\frac{\phi(1)-1}{\phi(1)+1} - \frac{\phi(it)-1}{\phi(it)+1}|^2}{1-\left|\frac{\phi(1)-1}{\phi(1)+1}\right|^2} \frac{dt}{\pi(1+t^2)} = \frac{1}{\pi \Re\phi(1)} \int_{\phi^{-1}(\gamma(Y))} \frac{|\phi(1)-\phi(it)|^2}{1+t^2} \, dt.$$

Finally, if ϕ is inner and ψ is unimodular a.e. on the imaginary axis, then both φ and $\tilde{\psi}$ are inner and so, $T_{\tilde{\psi},\varphi}$ is isometric if and only if $\varphi(0) = 0$, [24, Theorem 5]. By Proposition 1, $T_{\psi,\phi}$ is isometric if and only if $\varphi(0) = 0$, which is equivalent to $\phi(1) = 1$. $\square$

As an interesting fact, observe that:

Remark 2. *If $|\psi(it)| = 1$ a.e., ϕ is an inner selfmap of Π^+, not the identity or an elliptic automorphism, and $\phi(1) = 1$, then $T_{\psi,\phi}$ is isometric but C_ϕ is either unbounded or bounded, but not an isometry.*

Indeed, if ϕ does not fix the point at infinity having a finite angular derivative there, C_ϕ is unbounded. Else, C_ϕ is bounded, but $\phi'(\infty) > 1$ and so, C_ϕ cannot be isometric [23, Proposition 4]. The inequality $\phi'(\infty) > 1$ is a consequence of the Denjoy–Wolff theorem and the fact that ϕ fixes a point in Π^+. For the aforementioned theorem, the reader is **referred** to [6], [27], or Theorem 15 in the current paper.

In the following, we will mention the so called, Wold decomposition of an isometry. We recall that a Hilbert space isometry V is called a forward unilateral **shift** if the sequence $\{V^{*n}\}$ tends strongly to 0. Given any Hilbert space isometry V acting on some space H, there is a unique associated direct sum decomposition $H = H_0 \oplus H_1$ so that $V|H_0$ is unitary and $V|H_1$ is a unilateral forward shift (with the understanding that the first or second subspace in the aforementioned direct sum, may be null). This direct sum decomposition is called the Wold decomposition of V.

We will consider in the following the situation of analytic selfmaps of Π^+ having a fixed point. Relative to that situation, it is **useful** to record the following technical result.

Lemma 1. *Assume the maps ϕ, ψ, φ, and $\tilde{\psi}$ are related as in Proposition 1, and $\phi(p) = p$ for some $p \in \Pi^+$. Then*

$$\tilde{\psi}(q) = \psi(p) \quad \text{and} \quad \phi'(p) = \varphi'(q)$$

where

$$q = \frac{p-1}{p+1}.$$

Proof. The proof is left to the reader, being a straightforward computation with the maps ϕ, ψ, φ, and $\tilde{\psi}$. $\square$

We saw that elliptic automorphisms ϕ of Π^+ can induce unitary weighted composition operators $T_{\psi,\phi}$ if paired with the right kind of function ψ. More generally:

Theorem 7. *If ϕ fixes a point $p \in \Pi^+$ and $T_{\psi,\phi}$ is a contraction, then necessarily $|\psi(p)| \leq 1$. If the isometric weighted composition operator $T_{\psi,\phi}$ is induced by some (necessarily inner) ϕ fixing $p \in \Pi^+$ then ψ must have the form*

$$\psi(w) = c\frac{\overline{p} + \phi(w)}{\overline{p} + w} \qquad w \in \Pi^+, \tag{2.22}$$

where $|c| = 1$, and all $T_{\psi,\phi}$ induced by some inner function ϕ fixing $p \in \Pi^+$, paired with ψ of form (2.22) are isometric operators. Those operators are unilateral forward shifts if and only $|\psi(p)| < 1$. If $|\psi(p)| = 1$ then $T_{\psi,\phi}$ is either unitary or is representable as the direct sum of a rank–one unitary operator acting on the subspace of $H^2(\Pi^+)$ spanned by the function $1/(\overline{p} + w)$ and a unilateral forward shift.

Proof. Like before, we consider the weighted composition operator $T_{\tilde{\psi},\varphi}$ which is unitarily equivalent to $T_{\psi,\phi}$, and is therefore a contraction. On the other hand, φ fixes $q = (p-1)/(p+1)$ and so, one has that $|\tilde{\psi}(q)| = |\psi(p)| \leq 1$, by [24, Proposition 1]. The operator $T_{\psi,\phi}$ is isometric, if and only if $T_{\tilde{\psi},\varphi}$ is an isometry. Then, by [24, Theorem 9], $\tilde{\psi}$ must have form

$$\tilde{\psi}(z) = c\frac{1 - \overline{q}\varphi(z)}{1 - \overline{q}z} \qquad z \in \mathbb{U}, \tag{2.23}$$

where $c \in \mathbb{T}$, which is equivalent to ψ having form (2.22). Indeed, (2.23) holds if and only if

$$\frac{w+1}{\phi(w)+1}\psi(w) = c\frac{w+1}{\phi(w)+1}\,\frac{\phi(w)(1-\overline{q}) + 1 + \overline{q}}{w(1-\overline{q}) + 1 + \overline{q}} \qquad w \in \Pi^+,$$

which is equivalent to (2.22).

If $T_{\psi,\phi}$ is isometric, then that operator is a unilateral forward shift if and only if $T_{\tilde{\psi},\varphi}$ is such a shift, a fact that happens if and only if $|\tilde{\psi}(q)| = |\psi(p)| < 1$, by [24, Theorem 7]. In case $|\tilde{\psi}(q)| = |\psi(p)| = 1$ and $T_{\psi,\phi}$ is isometric but not unitary, then the Wold decomposition of the isometric operator $T_{\tilde{\psi},\varphi}$ is $H^2 = (\mathrm{Span}(K_q)) \oplus (\mathrm{Span}(K_q))^{\perp}$, where $K_q(z) = 1/(1 - \overline{q}z)$, [24, Theorem 9]. In that case, the Wold decomposition of $T_{\psi,\phi}$ is $H^2(\Pi^+) = (V\mathrm{Span}(K_q)) \oplus (V\mathrm{Span}(K_q))^{\perp}$, where V is the unitary operator in the proof of Proposition 1. But

$$VK_q(w) = \frac{1}{1 - \overline{q}\frac{w-1}{w+1}} = \frac{1}{1-\overline{q}}\,\frac{1}{w+\overline{p}} \qquad w \in \Pi^+$$

and hence, $V\mathrm{Span}(K_q) = \mathrm{Span}(1/(\overline{p} + w))$, which ends the proof. □

3. Normal weighted composition operators

We already saw that weighted composition operators on $H^2(\Pi^+)$ can be normal (e.g. when they are unitary). Also, even unweighted composition operators on $H^2(\Pi^+)$ can be normal [25]. Here is another simple example:

Example 4. *The weighted composition operator $T_{c\tilde{\phi},\phi}$, $c \in \mathbb{C}$, $c \neq 0$ is normal if and only if ϕ is the identity or ϕ has the form*

$$\phi(w) = \frac{1 + \lambda w}{\lambda + w} \qquad w \in \Pi^+ \tag{3.1}$$

for some finite $\lambda \in \overline{\Pi^+}$ (*where* ϕ *and* $\tilde{\phi}$ *are related as in* Example 2 *and the upper bar denotes closure*).

Given that $T_{c\tilde{\phi},\phi} = cT_{\tilde{\phi},\phi}$, the fact stated above follows immediately from Example 2 and the well known fact that a composition operator on H^2 is normal if and only if its symbol φ has the form $\varphi(z) = kz$, $z \in \mathbb{U}$, for some constant $k \in \overline{\mathbb{U}}$.

Relative to the terminology "finite λ" used in Example 4, note that like before, the closure of Π^+ is considered in the extended complex plane (or equivalently on the Riemann sphere) and so, that closure contains ∞.

The following is a complete characterization of normal weighted composition operators on $H^2(\Pi^+)$ induced by composition symbols with a fixed point in Π^+.

Theorem 8. *A nonzero weighted composition operator* $T_{\psi,\phi}$ *induced by an analytic selfmap* ϕ *of* Π^+ *fixing some point* $p \in \Pi^+$ *is normal if and only if* ψ *has form*

$$\psi(w) = c\frac{\overline{p} + \phi(w)}{\overline{p} + w} \qquad w \in \Pi^+, \tag{3.2}$$

where c *is any nonzero constant in* $\mathbb{C}$, *and* ϕ *is the identity or a linear fractional map of form*

$$\phi(w) = \frac{(\rho\Re p + i\Im p)w + |p|^2}{w + \rho\Re p - i\Im p} \qquad w \in \Pi^+. \tag{3.3}$$

where ρ *is any finite constant in* $\overline{\Pi^+}$.

Proof. By Proposition 1, $T_{\psi,\phi}$ is normal if and only if $T_{\tilde{\psi},\varphi}$ is normal and ϕ fixes $p \in \Pi^+$ if and only if φ fixes $q = (p-1)/(p+1)$. Keeping all that in mind, recall that, by [3, Theorem 10], operator $T_{\tilde{\psi},\varphi}$ is normal if and only if $\tilde{\psi}$ has form

$$\tilde{\psi} = c\frac{k_q}{k_q \circ \varphi} \tag{3.4}$$

with $c \in \mathbb{C}$, $k_q(z) = 1/(1 - \overline{q}z)$, and φ of form

$$\varphi = \alpha_q \circ m_\delta \circ \alpha_q \tag{3.5}$$

where

$$\alpha_q(z) = \frac{q - z}{1 - \overline{q}z} \qquad z \in \mathbb{U}$$

$\delta \in \overline{\mathbb{U}}$, and

$$m_\delta(z) = \delta z \qquad z \in \mathbb{U}.$$

Then, relative to the map ψ, one can write

$$c\frac{k_q(z)}{k_q \circ \varphi(z)} = \psi \circ \gamma(z)\frac{1 - \varphi(z)}{1 - z} \qquad z \in \mathbb{U}$$

that is

$$c\frac{k_q(\gamma^{-1}(w))}{k_q \circ \varphi(\gamma^{-1}(w))} = \psi(w)\frac{1 + w}{1 + \phi(w)} \qquad w \in \Pi^+$$

or equivalently

$$c\frac{1 + w}{1 + \phi(w)}\,\frac{\phi(w)(1 - \overline{q}) + 1 + \overline{q}}{w(1 - \overline{q}) + 1 + \overline{q}} = \psi(w)\frac{1 + w}{1 + \phi(w)} \qquad w \in \Pi^+$$

which is equivalent to (3.2).

Turning to map ϕ now, one can write

$$\phi = \gamma \circ \varphi \circ \gamma^{-1} = (\gamma \circ \alpha_q \circ \gamma^{-1}) \circ (\gamma \circ m_\delta \circ \gamma^{-1}) \circ (\gamma \circ \alpha_q \circ \gamma^{-1}). \tag{3.6}$$

Denote $m = \gamma \circ m_\delta \circ \gamma^{-1}$ and $\alpha = \gamma \circ \alpha_q \circ \gamma^{-1}$. Assume first that $\delta \neq 1$. One has that

$$\gamma \circ m_\delta \circ \gamma^{-1}(w) = \frac{\frac{1+\delta}{1-\delta}w + 1}{w + \frac{1+\delta}{1-\delta}} \qquad w \in \Pi^+.$$

Hence map m must have form

$$m(w) = \frac{1+\rho w}{\rho + w} \qquad w \in \Pi^+ \tag{3.7}$$

where

$$\rho = \frac{1+\delta}{1-\delta}.$$

And so, since δ can be any constant in $\overline{\mathbb{U}} \setminus \{1\}$, then ρ is any finite constant in $\overline{\Pi^+}$. In the case $\delta = 1$, φ is the identity and hence so is ϕ. Therefore $T_{\psi,\phi}$ is in that case a multiplication operator with constant symbol, hence a scalar multiple of the identity operator, which is obviously a normal operator.

Let us consider map α now, with the intention of finding the expression of ϕ. One can write

$$\alpha \circ \gamma(z) = \frac{1+\alpha_q(z)}{1-\alpha_q(z)} = \frac{(1+q)-(1+\overline{q})z}{(1-q)+(1-\overline{q})z}$$

so

$$\alpha(w) = \frac{(1+q)-(1+\overline{q})\frac{w-1}{w+1}}{(1-q)+(1-\overline{q})\frac{w-1}{w+1}} = \frac{(iy)w+(1+x)}{(1-x)w+(-iy)}$$

where $q = x + iy$. To find the expression of ϕ we will work matricially and note that

$$\begin{bmatrix} iy & 1+x \\ 1-x & -iy \end{bmatrix} \begin{bmatrix} \rho & 1 \\ 1 & \rho \end{bmatrix} \begin{bmatrix} iy & 1+x \\ 1-x & -iy \end{bmatrix} =$$
$$\begin{bmatrix} \rho(1-x^2-y^2)+2iy & 1+2x+x^2+y^2 \\ 1-2x+x^2+y^2 & \rho(1-x^2-y^2)-2iy \end{bmatrix}.$$

The consequence is that ϕ has expression

$$\phi(w) = \frac{(\rho(1-|q|^2)+2i\Im q)w + |1+q|^2}{|1-q|^2 w + \rho(1-|q|^2) - 2i\Im q} \qquad w \in \Pi^+. \tag{3.8}$$

Since, as observed before, $q = (p-1)/(p+1)$, one clearly has that

$$|1+q|^2 = \frac{4|p|^2}{|1+p|^2},\ |1-q|^2 = \frac{4}{|1+p|^2}, \tag{3.9}$$

$$1+|q|^2 = \frac{2(|p|^2+1)}{|1+p|^2}, \quad 1-|q|^2 = \frac{4\Re p}{|1+p|^2}, \quad \text{and} \quad \Im q = \frac{2\Im p}{|1+p|^2}. \tag{3.10}$$

Equations (3.8), (3.9), and (3.10) combine into showing that

$$\phi(w) = \frac{(\rho\Re p + i\Im p)w + |p|^2}{w + \rho\Re p - i\Im p} \qquad w \in \Pi^+. \tag{3.11}$$

□

Corollary 4. *The only nonzero, normal, weighted composition operators induced by a composition symbol ϕ with the property $\phi(1) = 1$ are those described in Example 4.*

Results in [5] provide an if and only if characterization of Hermitian weighted composition operators on H^2 in the case when ψ, the weight symbol, is a bounded analytic function. This seems to leave open the possibility that $T_{\psi,\varphi}$ be Hermitian on H^2 although ψ (which must belong to H^2 since $T_{\psi,\varphi}1 = \psi$) is an unbounded H^2–function. We wish to observe in the following that the aforementioned situation cannot occur. More formally, we prove:

Proposition 4. *In order that $T_{\psi,\varphi}$ be Hermitian on H^2, it is necessary that the following estimate hold:*

$$\|\psi\|_\infty \leq \frac{|\psi(0)|}{1-|\varphi(0)|}. \tag{3.12}$$

Proof. Indeed, if $T_{\psi,\varphi}$ is Hermitian, one has that

$$\langle T_{\psi,\varphi}1, g\rangle = \langle \psi, g\rangle = \langle 1, T_{\psi,\varphi}g\rangle = \overline{\psi(0)g\circ\varphi(0)} \qquad g \in H^2.$$

On the other hand, if $g = k_c$ is the reproducing kernel function of index c, that is $k_c = 1/(1-\bar{c}z)$, $z \in \mathbb{U}$, then

$$|\psi(c)| = |\langle \psi, k_c\rangle| = \frac{|\psi(0)|}{|1-c\overline{\varphi(0)}|} \qquad c \in \mathbb{U},$$

which implies (3.12), since

$$1-|\varphi(0)| \leq 1-|\bar{c}\varphi(0)| \leq |1-\bar{c}\varphi(0)| \qquad c \in \mathbb{U}.$$

□

We want to recall that:

Lemma 2 ([5, Corollary 2.3])**.** *A linear fractional map of the form*

$$\varphi(z) = a + \frac{bz}{1-\bar{a}z}$$

where a is a constant in $\mathbb{U}$ and $b \in \mathbb{R}$ maps $\mathbb{U}$ into itself if and only if

$$-(1-|a|^2) \leq b \leq (1-|a|)^2. \tag{3.13}$$

Proposition 4, Lemma 2, and [5, Theorem 2.1] combine into proving the following:

Corollary 5. *A nonzero weighted composition operator $T_{\tilde{\psi},\varphi}$ is Hermitian on H^2 if and only if the functions $\tilde{\psi}$ and φ have forms*

$$\varphi = a + \frac{bz}{1-\bar{a}z} \qquad z \in \mathbb{U} \tag{3.14}$$

where $a \in \mathbb{U}$ and $b \in \mathbb{R}$ satisfy condition (3.13) *and*

$$\tilde{\psi}(z) = \frac{c}{1-\bar{a}z}, \tag{3.15}$$

where $c \in \mathbb{R}$, $c \neq 0$.

Indeed, by [5, Theorem 2.1], if $\tilde{\psi}$ is a bounded analytic function, then $T_{\tilde{\psi},\varphi}$ is normal and nonzero if and only if φ is an analytic selfmap of $\mathbb{U}$ having form (3.14), for some $a \in \mathbb{U}$ and $b \in \mathbb{R}$, (a fact which is equivalent to a and b satisfying (3.13) by Lemma 2), and $\tilde{\psi}$ has form (3.15). As we saw, the restriction that $\tilde{\psi}$ be bounded is superfluous (by Proposition 4).

Based on Corollary 5, we can provide a complete characterization of the Hermitean weighted composition operators on $H^2(\Pi^+)$:

Theorem 9. *A nonzero weighted composition operator $T_{\psi,\phi}$ on $H^2(\Pi^+)$ is Hermitian if and only if ψ and ϕ are maps with the following properties. The map ϕ should have form*

$$\phi(w) = \frac{A + Bw}{\overline{B} + Cw} \qquad w \in \Pi^+ \tag{3.16}$$

where

$$A = |1 + a|^2 - b,\ B = 1 + b + 2i\Im a - |a|^2,\ C = |1 - a|^2 - b, \tag{3.17}$$

and

$$a \in \mathbb{U},\ b \in \mathbb{R}, \tag{3.18}$$

satisfy condition (3.13). *The map ψ must have form*

$$\psi(w) = c\frac{1 + \phi(w)}{1 + \overline{a} + w(1 - \overline{a})} \qquad w \in \Pi^+, \tag{3.19}$$

where a and ϕ satisfy the conditions above, and $c \in \mathbb{R}$, $c \neq 0$.

Proof. By Proposition 1, $T_{\psi,\varphi}$ is Hermitian if and only if $T_{\tilde{\psi},\varphi}$ is Hermitian. Therefore φ and $\tilde{\psi}$ must satisfy the conditions in Corollary 5. The above conditions are equivalent to the following:

$$\frac{\phi(w) - 1}{\phi(w) + 1} = a + \frac{b\frac{w-1}{w+1}}{1 - \overline{a}\frac{w-1}{w+1}} \qquad w \in \Pi^+ \tag{3.20}$$

and

$$\psi(w)\frac{w + 1}{\phi(w) + 1} = \frac{c}{1 - \overline{a}\frac{w-1}{w+1}} \qquad w \in \Pi^+ \tag{3.21}$$

for some $a \in \mathbb{U}$, $b \in \mathbb{R}$ which satisfy (3.13), and some $c \in \mathbb{R}$, $c \neq 0$.

As the reader can easily check, (3.20) holds if and only if ϕ has form (3.16) with coefficients satisfying (3.17), whereas (3.21) holds if and only if ψ has form (3.19). □

The previous theorem does not show explicitly which nonzero weighted composition operators induced by a composition symbol with a fixed point in Π^+ are Hermitian. We prefer to obtain that piece of information by using our considerations on numerical ranges, rather than using Theorem 9 (see Corollary 7 in the next section).

4. Spectral properties

Let $\sigma(T)$ denote the spectrum of the operator T. The spectrum of some compact weighted composition operators acting on H^2 is characterized in the following:

Theorem 10 ([12])**.** *If $T_{\psi,\varphi}$ is compact on H^2 and φ fixes some $q \in \mathbb{U}$, then the spectrum $\sigma(T_{\psi,\varphi})$ of $T_{\psi,\varphi}$ is given by formula*

$$\sigma(T_{\psi,\varphi}) = \{0, \psi(q), \psi(q)\varphi'(q), \psi(q)(\varphi'(q))^2, \dots, \psi(q)(\varphi'(q))^n, \dots\}. \tag{4.1}$$

The above theorem combines with Proposition 1 and Lemma 1 into proving a very similar spectral formula for compact weighted composition operators on $H^2(\Pi^+)$ induced by composition symbols with a fixed point.

Theorem 11. *Assume ϕ is an analytic selfmap of Π^+ so that $\phi(p) = p$ for some $p \in \Pi^+$, ψ is analytic on Π^+, and $T_{\psi,\phi}$ is compact. Then*

$$\sigma(T_{\psi,\phi}) = \{0, \psi(p), \psi(p)\phi'(p), \psi(p)(\phi'(p))^2, \dots, \psi(p)(\phi'(p))^n, \dots\}. \tag{4.2}$$

If $T_{\psi,\phi}$ is a (not necessarily compact), normal operator, and $\phi(p) = p$ then

$$\sigma(T_{\psi,\phi}) = \overline{\{\psi(p), \psi(p)\phi'(p), \psi(p)(\phi'(p))^2, \dots, \psi(p)(\phi'(p))^n, \dots\}}, \tag{4.3}$$

where the upper bar denotes closure. Finally, if $T_{\psi,\phi}$ is a unitary operator induced by some ϕ which fixes no point in Π^+, then

$$\sigma(T_{\psi,\phi}) = \mathbb{T}. \tag{4.4}$$

Proof. Let the maps ϕ, ψ, φ, and $\tilde{\psi}$ be related as in Proposition 1. Then, formula (4.2) is the immediate consequence of (4.1), Lemma 1, and Proposition 1, whereas relations (4.3) and (4.4) are consequences of Lemma 1, Proposition 1, and [3, Proposition 11 and Theorem 7]. $\square$

In case the composition symbol ϕ of $T_{\psi,\phi}$ is a half–plane automorphism, then $\sigma(T_{\psi,\phi})$ can be determined in multiple particular cases. Here are some examples. Denote $i\mathbb{R}$ the set of purely imaginary numbers (i.e. the vertical axis). Recall that, a parabolic half–plane automorhism is a conformal half–plane automorphism fixing one point situated on $\partial\Pi^+ = i\mathbb{R} \cup \{\infty\}$ and no point in Π^+.

Theorem 12. *Assume ϕ is a parabolic half–plane automorphism. If $\phi(\infty) = \infty$ and ψ is an analytic map on Π^+ with the property*

$$\psi(s) := \lim_{w \to s} \psi(w) \text{ exists, is finite} \qquad s \in i\mathbb{R} \cup \{\infty\}, \tag{4.5}$$

and there is $c > 0$ so that

$$c \leq |\psi(w)| \qquad w \in \Pi^+, \tag{4.6}$$

then $T_{\psi,\phi}$ is bounded and

$$\sigma(T_{\psi,\phi}) = |\psi(\infty)|\mathbb{T}. \tag{4.7}$$

If $\phi(p) = p$, but $p \neq \infty$, then there is a unique $it \in i\mathbb{R}$ so that $\phi(it) = \infty$. If $\psi/(1+\phi)$ is bounded away from zero, condition (4.5) holds for all $s \in i\mathbb{R} \setminus \{it\}$, and

$$\lim_{w \to s} \frac{(1+w)\psi(w)}{1+\phi(w)} \text{ exists and is finite, for } s \in \{it, \infty\}, \tag{4.8}$$

then $T_{\psi,\phi}$ is bounded and

$$\sigma(T_{\psi,\phi}) = |\psi(p)|\mathbb{T}. \tag{4.9}$$

Proof. As usual, we consider the maps φ (the conformal conjugate of ϕ) and $\tilde{\psi}(z) = \psi \circ \gamma(z)(1 - \varphi(z))/(1 - z)$. If $\phi(\infty) = \infty$, relation (4.7) is a direct consequence of [18, Theorem 4.3]. Indeed, $\tilde{\psi}$ is a map belonging to the disc algebra, (by (4.5)), and **hence** a bounded analytic map. One should note that $\tilde{\psi}(1) = \psi(\infty)\varphi'(1) = \psi(\infty)$, since φ is a parabolic disc automorphism with fixed point 1. This makes $T_{\tilde{\psi},\varphi}$, bounded and so, $T_{\psi,\phi}$ is bounded too and the two operators have identical spectrum equal to

$$|\tilde{\psi}(1)|\mathbb{T} = |\psi(\infty)|\mathbb{T},$$

provided that $\tilde{\psi}$ is bounded away from zero ([18, Theorem 4.3]), a fact that holds, given (4.6).

In case ϕ fixes $p \neq \infty$, relation (4.9) follows as a consequence of [18, Theorem 4.3] as well, by a rather similar proof. Indeed, by a straightforward computation, one establishes that the range of $\tilde{\psi}$ is the same as that of $(1+w)\psi(w)/(1+\phi(w))$, $w \in \Pi^+$ and so, $\tilde{\psi}$ is bounded away from zero if and only if $\psi/(1+\phi)$ is bounded away from zero. Also, the necessary condition that $\tilde{\psi}$ extends by continuity to the boundary of the unit disc is equivalent to the fact that condition (4.8) holds and condition (4.5) is satisfied for all $s \in i\mathbb{R} \setminus \{it\}$. □

Next, recall that hyperbolic half-plane automorphisms are conformal automorphisms of Π^+ with two distinct fixed points, both situated on $i\mathbb{R} \cup \{\infty\}$ and no fixed point in Π^+. If ϕ is such an automorphism and ψ belongs to a particular class of analytic maps on Π^+, then $\sigma(T_{\psi,\phi})$ is easy to find. Indeed:

Theorem 13. *Let ϕ be a hyperbolic half–plane automorphism with fixed points α and β. If the analytic map ψ is such that $\psi/(1+\phi)$ is bounded away from zero and continuously extensible to $\Pi^+ \cup i\mathbb{R} \cup \{\infty\}$, then $T_{\psi,\phi}$ is bounded and its spectral radius $r(T_{\psi,\phi})$ is given by the formula*

$$r(T_{\psi,\phi}) = \max\left\{\frac{|\psi(\alpha)|}{\sqrt{\phi'(\alpha)}}, \frac{|\psi(\beta)|}{\sqrt{\phi'(\beta)}}\right\}, \tag{4.10}$$

if $\alpha \neq \infty$ and $\beta \neq \infty$. In case one of the fixed points, say α, is ∞, then the spectral radius is

$$r(T_{\psi,\phi}) = \max\left\{|\psi(\infty)|\sqrt{\phi'(\infty)}, \frac{|\psi(\beta)|}{\sqrt{\phi'(\beta)}}\right\}. \tag{4.11}$$

If $\alpha \neq \infty$, $\beta \neq \infty$, and

$$\frac{|\psi(\alpha)|}{\sqrt{\phi'(\alpha)}} = \frac{|\psi(\beta)|}{\sqrt{\phi'(\beta)}}, \tag{4.12}$$

then

$$\sigma(T_{\psi,\phi}) = r(T_{\psi,\phi})\mathbb{T}. \tag{4.13}$$

If $\alpha = \infty$ and $|\psi(\infty)| = |\psi(\beta)|$, then (4.13) *holds.*

Proof. The proof consists in applying [18, Theorems 4.6 and 4.8] to the operator $T_{\tilde{\psi},\varphi}$ which is unitarily equivalent to $T_{\psi,\phi}$ and note that, the fixed points of the hyperbolic disc automorphism φ are $a = \gamma^{-1}(\alpha)$ and $b = \gamma^{-1}(\beta)$. Like in Lemma 1, one has that $\tilde{\psi}(a) = \psi(\alpha)$ and $\tilde{\psi}(b) = \psi(\beta)$ and so, the statements in this theorem follow by [18, Theorems 4.6 and 4.8] if $\alpha \neq \infty$, $\beta \neq \infty$. However, if $\alpha = \infty$, then $a = 1$ and, in this case, one has that $\tilde{\psi}(1) = \psi(\infty)\varphi'(1) = \psi(\infty)\phi'(\infty)$, which combines with [18, Theorems 4.6] into proving formula (4.11). If the fixed points of ϕ are finite, then formula (4.13) follows by [18, Theorem 4.8]. In order to prove (4.13) in the particular case $\alpha = \infty$, it should be noted that the necessary equality (4.12) is equivalent, in that case to

$$|\psi(\infty)|\sqrt{\varphi'(1)} = \frac{|\psi(\beta)|}{\sqrt{\varphi'(b)}}$$

hence to $|\psi(\infty)| = |\psi(\beta)|$, since $\varphi'(a)\varphi'(b) = 1$, an equality valid for all hyperbolic disc automorphisms having fixed points a and b [25, Remark 3]. By [18, Theorem 4.8], this ends the proof. □

We turn now to weighted composition operators whose composition symbol is an elliptic half–plane automorphism. If ϕ is such an automorphism, that is if its conformal conjugate φ is a disc automorphism, not the identity, with one fixed point

$q \in \mathbb{U}$, then let $\alpha_q(z) = (q - z)/(1 - \overline{q}z)$. This is a selfinverse disc automorphism and, visibly, the conformal conjugate $\alpha_q \circ \varphi \circ \alpha_q$ of φ is a disc automorphism fixing the origin, hence a rotation. Two situations arise: either that rotation is by a root of unity, which happens if and only if there is some natural number n so that $\varphi \circ \cdots \circ \varphi$, ($n$ times) is the identity, but $\varphi \circ \cdots \circ \varphi$, $(n-1)$ times, is not the identity, or that rotation is not by a root of unity. Keeping all that in mind, one can use results in [18] to determine the spectra of certain invertible weighted composition operators whose composition symbols are elliptic automorphisms:

Theorem 14. *Let ϕ be an elliptic, half–plane automorphism fixing $p \in \Pi^+$ and ψ an analytic map on Π^+ such that $\psi/(1+\phi)$ is bounded away from zero and continuously extendable to $\Pi^+ \cup i\mathbb{R} \cup \{\infty\}$. If φ, the conformal conjugate of ϕ via* ***Cayley's*** *transform, is also conformally conjugated to a rotation by a unimodular number, other than a root of unity, then $T_{\psi,\phi}$ has circular spectrum, namely:*

$$\sigma(T_{\psi,\phi}) = |\psi(p)|\mathbb{T}. \tag{4.14}$$

Else, φ must be conformally conjugated to a rotation by a root of unity of order n. In that case, denote, as usual $\sqrt[n]{}$ the multi–valued, complex n–th root. One has that

$$\sigma(T_{\psi,\phi}) = \overline{\sqrt[n]{(\psi\psi \circ \phi\psi \circ \phi^{[2]} \dots \psi \circ \phi^{[n-1]})(\Pi^+)}}, \tag{4.15}$$

where the upper bar denotes closure and $\phi^{[k]}$ is the k–fold iterate of ϕ.

Proof. Equality (4.14) results by applying to the unitarily equivalent operator $T_{\tilde{\psi},\varphi}$ [18, Theorems 4.14] and Lemma 1.

For the case when φ is conformally conjugated to a rotation by a root of unity of order n now, [18, Theorem 4.11] says that

$$\sigma(T_{\tilde{\psi},\varphi}) = \overline{\sqrt[n]{(\tilde{\psi}\tilde{\psi} \circ \varphi\tilde{\psi} \circ \varphi^{[2]} \dots \tilde{\psi} \circ \varphi^{[n-1]})(\mathbb{U})}}. \tag{4.16}$$

On the other hand, one notes that

$$\begin{gathered}
(\tilde{\psi}\tilde{\psi} \circ \varphi\tilde{\psi} \circ \varphi^{[2]} \dots \tilde{\psi} \circ \varphi^{[n-1]})(z) = \\
\psi \circ \gamma(z)\frac{1-\varphi(z)}{1-z}\psi \circ \gamma \circ \varphi(z)\frac{1-\varphi^{[2]}(z)}{1-\varphi(z)} \dots \\
\dots \psi \circ \gamma \circ \varphi^{[n-1]}(z)\frac{1-z}{1-\varphi^{[n-1]}(z)} = \\
\psi \circ \gamma(z)\psi \circ \gamma \circ \varphi(z) \dots \psi \circ \gamma \circ \varphi^{[n-1]}(z) = \\
\psi(w)\psi \circ \phi(w) \dots \psi \circ \phi^{[n-1]}(w) \qquad w = \gamma(z),\ z \in \mathbb{U},
\end{gathered}$$

which proves equality (4.15). ☐

Corollary 6. *The spectrum of a unitary weighted composition operator $T_{\psi,\phi}$ acting on $H^2(\Pi^+)$ with elliptic composition symbol can be circular, namely*

$$\sigma(T_{\psi,\phi}) = \sqrt{\Re q}\left|\frac{1+p}{\overline{q}+p}\right|\mathbb{T} \tag{4.17}$$

if the composition symbol ϕ is an elliptic half–plane automorphism fixing p, $q = \phi^{-1}(1)$, and the conformal conjugate φ of ϕ is not a map conformally conjugated to a rotation by a root of 1. In case φ is a map conformally conjugated to a rotation *by a root of* 1 *of order* $n \geq 2$*, then let* G_n *be the subgroup of* $\mathbb{T}$ *generated by that root of unity. One has that*

$$\sigma(T_{\psi,\phi}) = cG_n \tag{4.18}$$

for some unimodular constant c*.*

Indeed, equality (4.17) is an immediate consequence of Theorem 14 and relation (2.15). In the case when φ is a map conformally conjugated to a rotation by a root of 1 of order $n \geq 2$, the map under the radical sign in formula (4.15), must be a unimodular constant, since the spectrum of a unitary operator is a nonempty compact subset of the unit circle and hence, it cannot be a set with interior points. Thus $\sigma(T_{\psi,\phi})$ must have form (4.18).

Given that isometric weighted composition operators were studied in previous sections (see e.g. Theorems 6, 7), we wish to note the following, as a last remark about their spectra:

Remark 3. *By the Wold decomposition theorem and the well known fact that the spectrum of a forward shift is the closed unit disc, the spectrum of* $T_{\psi,\phi}$ *equals that disc in case* $T_{\psi,\phi}$ *is a non–unitary isometry.*

As we know, the closure of the numerical rage of an operator is a superset of the convex hull of its spectrum [16]. Prior to recording the numerical range computations for weighted composition operators, which can be obtained from the literature via Proposition 1, we need to recall a classical function theory theorem, very useful for those who study composition operators. It is called, the Denjoy–Wolff theorem and the reader is referred to [6] or [27] for it.

Theorem 15 (Denjoy–Wolff)**.** *The iterates of an analytic selfmap* φ *of* $\mathbb{U}$ *other than the identity or an elliptic disc automorphism converge uniformly on the compact subsets of* $\mathbb{U}$ *to a a point* p *necessarily situated in the closure of* $\mathbb{U}$*.*

We refer to p as the Denjoy–Wolff point of φ. If $p \in \mathbb{T}$, then the angular derivative $\varphi'(p)$ of φ at p must exist, be a real, positive, number, and satisfy $\varphi'(p) \leq 1$. Inspired from the properties of conformal disc automorphisms, the maps φ with property $\varphi'(p) < 1$ are called maps of hyperbolic type, whereas those with property $\varphi'(p) = 1$ are called maps of parabolic type.

With this terminology, we can summarize in the following some computations of numerical ranges of weighted composition operators on $H^2(\Pi^+)$.

Theorem 16. *Let* $T_{\psi,\phi}$ *be a bounded weighted composition operator on* $H^2(\Pi^+)$*. Then:*

(a) *If* $T_{\psi,\phi}$ *is a non–unitary isometric composition operator, induced by some inner* ϕ *fixing some* $p \in \Pi^+$*, then*

$$W(T_{\psi,\phi}) = \mathbb{U} \qquad \text{if} \qquad |\psi(p)| < 1, \tag{4.19}$$

and

$$W(T_{\psi,\phi}) = \{\psi(p)\} \cup \mathbb{U} \qquad \text{if} \qquad |\psi(p)| = 1. \tag{4.20}$$

(b) *If* $T_{\psi,\phi}$ *is bounded and* ϕ *is an inner function of parabolic automorphic or of hyperbolic type, then* $W(T_{\psi,\phi})$ *is a circular disc centered at the origin. In particular, all isometric weighted composition operators whose composition symbol* *is an inner function of parabolic automorphic or of hyperbolic type, have numerical range equal to* $\mathbb{U}$.

(c) *If* $T_{\psi,\phi}$ *is unitary,* ϕ *has a fixed point* $p \in \Pi^+$ *and* $\phi'(p)$ *is not a root of unity, then* $W(T_{\psi,\phi})$ *is the union of* $\mathbb{U}$ *and a countable dense subset of* $\mathbb{T}$*, more exactly*

$$W(T_{\psi,\phi}) = \mathbb{U} \cup \{\psi(p)(\phi'(p))^n : n = 0, 1, 2, 3, \dots\}. \tag{4.21}$$

If $\phi'(p)$ *is a root of unity of order* $n > 2$*, then*

$$W(T_{\psi,\phi}) = \psi(p)P_n \tag{4.22}$$

where P_n *is the regular, closed polygonal region (boundary, plus interior), with* n *sides, inscribed in the unit circle and having a vertex at* 1*. If* $n = 2$*, then* $W(T_{\psi,\phi}) = \psi(p)[-1, 1]$.

(d) *If* $T_{\psi,\phi}$ *is normal, but not unitary and* ϕ*, not the identity, has a fixed point* $p \in \Pi^+$*, then* $W(T_{\psi,\phi})$ *is a closed convex polygonal region or a line segment. More formally, if* $\phi'(p)$ *is not a real number, then* $W(T_{\psi,\phi})$ *is a closed, convex, polygonal region whose vertices form a finite subset of the set* $\{\psi(p)(\phi'(p))^n : n = 0, 1, 2, 3, \dots\}$*. If* $\phi'(p)$ *is a real number, then*

$$W(T_{\psi,\phi}) = \psi(p)[\phi'(p), 1] \quad \text{if} \quad \phi'(p) \leq 0 \tag{4.23}$$

respectively

$$W(T_{\psi,\phi}) = \psi(p)(0, 1] \quad \text{if} \quad \phi'(p) > 0. \tag{4.24}$$

(e) *If* $T_{\psi,\phi}$ *is normal and* ϕ*, not the identity, has a fixed point* $p \in \Pi^+$*, then* $T_{\psi,\phi}$ *is Hermitian if and only if* $\psi(p)$ *and* $\phi'(p)$ *are real numbers and, in that case* $W(T_{\psi,\phi})$ *is given by* (4.23) *and* (4.24).

Proof. In case (a), equation (4.19) is a direct consequence of Proposition 1, [24, Theorem 8], Lemma 1, and the well known fact that the numerical range of a forward shift equals $\mathbb{U}$, whereas equality (4.20) is a consequence of Proposition 1, [24, Theorem 13], and Lemma 1.

The statements in (b) are direct consequences of Proposition 1 and [24, Theorem 15]. The statements ϕ is an inner function of parabolic or of hyperbolic type mean, of course, that the conformal conjugate φ of ϕ is an inner function of parabolic respectively of hyperbolic type. By parabolic automorphic type, we mean that φ has separated orbits with respect to the pseudohyperbolic metric. The reader is also referred to [2] for more details on the Denjoy–Wolff theorem and the notions related to it.

In order to prove (c) now, note that, by Proposition 1, Lemma 1, and the proof of [3, Proposition 11], $T_{\psi,\phi}$ is unitarily equivalent to $\psi(p)C_{\phi'(p)z}$, acting on H^2. Then one must have that $|\psi(p)| = |\phi'(p)| = 1$, since $T_{\psi,\phi}$ is unitary and so, its spectrum must be a subset of the unit circle, whereas $\psi(p)C_{\phi'(p)z}$ is a diagonal operator with diagonal entries $\{\psi(p)(\phi'(p))^n : n = 0, 1, 2, 3, \dots\}$. Also, by a well known property of numerical ranges, one has that

$$W(\psi(p)C_{\phi'(p)z}) = \psi(p)W(C_{\phi'(p)z}). \tag{4.25}$$

On the other hand, according to [22, Proposition 2.1], $W(C_{\phi'(p)z}) = \mathbb{U} \cup \{(\phi'(p))^n : n = 0, 1, 2, 3, \dots\}$ if the unimodular number $\phi'(p)$ is not a root of unity, whereas $W(C_{\phi'(p)z})$ is the regular, closed, polygonal region with n sides, inscribed in $\mathbb{T}$ and having a vertex at 1 if $\phi'(p)$ is a root of unity of order $n \geq 2$. By equality (4.25), one gets that (4.21) and (4.22) hold.

To prove (d), use an argument identical to the one used to prove (c) and [22, Proposition 2.2].

A bounded operator is Hermitian if and only if its numerical range is a subset of the real line. Then (e) is an immediate consequence of (c). □

As a consequence, we obtain the following characterization of Hermitian weighted composition operators whose composition symbol has a fixed point.

Corollary 7. *Let $p \in \Pi^+$. Then the nonzero Hermitian weighted composition operators on $H^2(\Pi^+)$ induced by selfmaps ϕ of Π^+ fixing p are those whose composition symbol ϕ has form* (3.3) *for some real $\rho > 0$ paired with weight symbols ψ satisfying* (3.2) *for some real, nonzero constant c.*

Finally, given an analytic selfmap ϕ of Π^+, let φ denote the selfmap of the disc, conformally conjugated to ϕ and let ϕ and $\tilde{\phi}$ be related as in Example 2, then by that example, one has that

$$\sigma(C_\varphi) = \sigma(T_{\tilde{\phi},\phi}), \quad \sigma_e(C_\varphi) = \sigma_e(T_{\tilde{\phi},\phi}), \quad \text{and} \quad W(C_\varphi) = W(T_{\tilde{\phi},\phi})$$

where $\sigma_e(T)$ denotes the essential spectrum of the operator T. Thus, the determination of the aforementioned sets for weighted composition operators of type $T_{\tilde{\phi},\phi}$ is immediate, provided it is known in the case of the composition operators C_φ, unitarily equivalent to them. For a good source of information about when spectra of composition operators are known, the reader is referred to [6].

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