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COMPOSITION OPERATORS AND A PULL-BACK MEASURE FORMULA

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A pull-back measure formula obtained in some particular cases by E. A. Nordgren and this author is generalized in the framework of boundary measures for zero-free Nevanlinna class functions on the unit polydisk. The formula is used to characterize the zero-free Nevanlinna class functions which are solutions of Schröder's equation induced by a polydisk automorphism ϕ (i.e. to determine the zero-free functions f belonging to the Nevanlinna class which are solutions of the functional equation $f \circ \phi = \lambda f$, for some constant λ), thus generalizing earlier results obtained by R. Mortini and this author.

1. INTRODUCTION

Our aim is to derive a formula that describes how certain measures on the distinguished boundary of the unit polydisk are transformed under composition by a proper holomorphic self-map of the polydisk. Let $\mathbf{U}$ denote the unit disk in $\mathbf{C}$ and $\mathbf{T}$ its boundary, the unit circle. For n a positive integer, we consider the measures on $\mathbf{T}^n$ associated with the Nevanlinna class $N(\mathbf{U}^n)$. By definition $N(\mathbf{U}^n)$ consists of those holomorphic functions f on $\mathbf{U}^n$ such that $\log^+ |f|$ has an n-harmonic majorant in $\mathbf{U}^n$. For each such f, the least n-harmonic majorant of $\log |f|$ in $\mathbf{U}^n$ is equal to the Poisson integral of a uniquely determined real Borel measure β_f on $\mathbf{T}^n$, called the boundary measure of f, [1] and [8]. The Poisson kernel for $\mathbf{U}^n$ will be denoted by P. We recall that P is a function on $\mathbf{U}^n \times \mathbf{T}^n$. Its value at (z, w) is the product of the Poisson kernels for $\mathbf{U}$ evaluated at the respective coordinates of z and w.

We shall be concerned with the relation between β_f and $\beta_{f\circ\phi}$ when f is a zero-free Nevanlinna class function, and ϕ is a proper holomorphic self-map of $\mathbf{U}^n$. The latter means that each of the coordinate functions ϕ_j is a finite Blaschke product depending on exactly one of the variables $z_1, \dots, z_n$, and no two such coordinate functions depend on the same variable; see also [8, Theorem 7.3.3]. We note that if $f \in N(\mathbf{U}^n)$ and ϕ is proper, then automatically $f \circ \phi \in N(\mathbf{U}^n)$.

THEOREM *If $f \in N(\mathbf{U}^n)$ is zero-free and ϕ is a proper holomorphic self-map of $\mathbf{U}^n$, then*

$$d\beta_{f\circ\phi}\phi^{-1}(w) = P(\phi(0), w)d\beta_f(w). \tag{1}$$

In the preceding formula, the measure on the left side is the pull-back of $\beta_{f\circ\phi}$ under ϕ, the measure whose value on any Borel set E is $\beta_{f\circ\phi}(\phi^{-1}(E))$. The proof of the theorem will consist of checking that the measures involved in equality (1) have identical Fourier coefficients. The second section of this paper is dedicated to the proof.

Special cases of the theorem have appeared before. When $n = 1$ and f is the constatnt function $f \equiv e$, the theorem reduces to a result of E.A.Nordgren [7]. When $n = 1$ and f is a singular inner function, the theorem reduces to a previous result of the author [5]. Now we wish to prove an immediate consequence of the theorem. First let us recall that the functional equation

$$f \circ \phi = \lambda f \tag{2}$$

is called Schröder's equation. In equation (2) λ denotes a fixed complex number.

COROLLARY *If ϕ is a polydisk automorphism, then a zero-free function $f \in N(\mathbf{U}^n)$ is a solution of Schröder's equation for some constant λ if and only if $\beta_f\phi^{-1} << \beta_f$ and the Radon-Nykodim derivative $d\beta_f\phi^{-1}/d\beta_f$ equals $P(\phi(0), w)$.*

PROOF. By [1, Theorem 2.2], a zero-free function $f \in N(\mathbf{U}^n)$ is a solution of Schröder's equation if and only if $\beta_{f\circ\phi} = \beta_f$. Since ϕ is an automorphism, this is equivalent to $(\beta_{f\circ\phi})\phi^{-1} = \beta_f\phi^{-1}$ which by the theorem above happens if and only if $d\beta_f\phi^{-1}(w) = P(\phi(0), w)d\beta_f(w)$. □

This corollary is a generalization of [5, Theorem 3.2], which says that a singular inner function S_ν is an eigenfunction of a composition operator C_ϕ whose symbol ϕ is a disk automorphism if and only if $d(\nu\phi^{-1})(w) = P(\phi(0), w)d\nu(w)$. In that paper this author gave examples of such singular measures ν on $\mathbf{T}$ for the disk automorphism $\varphi(z) = (2z+1)/(z+2)$; see [5, Example 3.4]. To build another simple example, on the bidisk this time, let's consider S_ν as above, denote by λ_1 the eigenvalue of C_φ corresponding to the eigenfunction S_ν, consider another singular inner eigenfunction S_μ of C_φ corresponding to the eigenvalue λ_2 and construct now the bidisk automorphism $\phi(z_1, z_2) = (\varphi(z_1), \varphi(z_2))$. Clearly $f(z_1, z_2) = S_\mu(z_1)S_\nu(z_2)$ is an eigenfunction of C_ϕ corresponding to the eigenvalue $\lambda = \lambda_1\lambda_2$. Therefore its boundary measure β_f will satisfy the condition described in the corollary. This fact can be also obtained as a direct consequence of [5, Theorem 3.2]. Indeed, observe that

$$d\beta_f(w) = (-d\mu) \times (dm_1)(w) + (dm_1) \times (-d\nu)(w) \quad w = (w_1, w_2) \in \mathbf{T}^2.$$

Now, by [5, Theorem 3.2], one obtains that $d\beta_f\phi^{-1}(w) = P(\phi(0), w)d\beta_f(w)$. Earlier than [5], R. Mortini obtained particular results in the same direction. In [6] he characterized the singular inner functions induced by discrete, singular Borel measures on $\mathbf{T}$ that are eigenfunctions of a hyperbolic composition operator.

2. THE PULL-BACK MEASURE FORMULA

This section is dedicated to proving the theorem. In order to make the proof easier to read we need three preliminary lemmas. We begin by introducing some necessary notations.

For any $z = (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n$, and any $k = (k_1, k_2, \ldots, k_n) \in \mathbf{Z}^n$, we denote $z^k = z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n}$, whenever the product makes sense. By $|k|$ we mean $|k| = |k_1|+|k_2|+\ldots+|k_n|$. The notation $\bar{z}^k$ means $\bar{z}^k = \bar{z}_1^{k_1} \bar{z}_2^{k_2} \ldots \bar{z}_n^{k_n}$. For any $\lambda \in \mathbf{C}$ and any $k \in \mathbf{Z}$, $\tilde{\lambda}^k$ will denote λ^k if $k \geq 0$, respectively $\bar{\lambda}^{|k|}$ if $k < 0$. For any $z \in \mathbf{C}^n$ and any $k \in \mathbf{Z}^n$ we denote $\tilde{z}^k = \tilde{z}_1^{k_1} \tilde{z}_2^{k_2} \ldots \tilde{z}_n^{k_n}$. It is both easy and useful to observe that if $w = (w_1, w_2, \ldots, w_n) \in \mathbf{T}^n$ and $k \in \mathbf{Z}^n$, then $\tilde{w}^k = w^k$. For any Borel measure μ on $\mathbf{T}^n$, and any $k \in \mathbf{Z}^n$, $c_k(\mu)$ designates the Fourier coefficient of μ of index k, i.e.

$$c_k(\mu) = \int_{\mathbf{T}^n} \bar{w}^k d\mu(w).$$

For absolutely continuous measures $f(w)dm_n(w)$, we write $c_k(f)$ instead of $c_k(f(w)dm_n(w))$. For each function f on $\mathbf{U}^n$ and each r, $0 < r < 1$, f_r designates the function on $\mathbf{T}^n$ given by $f_r(w) = f(rw)$, $w \in \mathbf{T}^n$. Finally, for each complex Borel measure μ on $\mathbf{T}^n$, P_μ denotes its Poisson integral.

LEMMA 1 *Let μ be any complex Borel measure on $\mathbf{T}^n$, $u = P_\mu$, and $k \in \mathbf{Z}^n$, $k = (k_1, k_2, \ldots, k_n)$. For each fixed r, $0 < r < 1$,*

$$c_k(u_r) = r^{|k|} c_k(\mu). \tag{3}$$

PROOF. Recall first that for each $v, w \in \mathbf{T}^n$, and r, $0 < r < 1$, we have

$$P(rv, w) = P(rw, v) = \sum_{j \in \mathbf{Z}^n} r^{|j|} \bar{w}^j v^j$$

and the series converges absolutely and uniformly as $w, v \in \mathbf{T}^n$. Therefore we can write

$$c_k(u_r) = \int_{\mathbf{T}^n} u(rv) \bar{v}^k dm_n(v) = \int_{\mathbf{T}^n} \int_{\mathbf{T}^n} P(rv, w) \bar{v}^k d\mu(w) dm_n(v) =$$

$$\int_{\mathbf{T}^n} \int_{\mathbf{T}^n} \sum_{j \in \mathbf{Z}^n} r^{|j|} \bar{w}^j v^{(j-k)} dm_n(v) d\mu(w) =$$

$$\sum_{j \in \mathbf{Z}^n} r^{|j|} \int_{\mathbf{T}^n} \bar{w}^j \left(\int_{\mathbf{T}^n} v^{(j-k)} dm_n(v) \right) d\mu(w) = r^{|k|} \int_{\mathbf{T}^n} \bar{w}^k d\mu(w) = r^{|k|} c_k(\mu)$$

since $\int_{\mathbf{T}^n} v^{(j-k)} dm_n(v) = \delta_{jk}$. □

For any $k \in \mathbf{Z}^n$, $k = (k_1, \ldots, k_n)$, we denote by $\tilde{\phi}^k$, the function $\tilde{\phi}^k(z) = \tilde{\phi}_1^{k_1}(z) \tilde{\phi}_2^{k_2}(z) \ldots \tilde{\phi}_n^{k_n}(z)$, $z \in \overline{\mathbf{U}}^n$. By a previous observation, if $\phi(z) \in \mathbf{T}^n$, then $\tilde{\phi}^k(z) = \phi^k(z)$. Suppose now that ϕ is a self-map of $\mathbf{T}^n$ with the property that there is $N > n/2$ such that for each $j = 1, 2, \ldots, n$, $\phi_j \in \mathcal{C}^N(\mathbf{T}^n)$.

LEMMA 2 *For any real Borel measure μ on $\mathbf{T}^n$, any ϕ as above, and any $k \in \mathbf{Z}^n$,*

$$c_k(\mu\phi^{-1}) = \sum_{s \in \mathbf{Z}^n} c_s(\mu) \bar{c}_s(\phi^k) \tag{4}$$

and the convergence in (4) is absolute.

PROOF. By [3], pp. 163 and the fact that μ is real we have

$$c_k(\mu\phi^{-1}) = \int_{\mathbf{T}^n} \bar{w}^k d\mu\phi^{-1}(w) = \overline{\int_{\mathbf{T}^n} \phi^k(w)d\mu(w)}.$$

Now $\phi \in \mathcal{C}^N(\mathbf{T}^n)$ so, by [10, Ch. VII, Corollary 1.9], ϕ^k has absolutely summable Fourier coefficients, and one is allowed to integrate its Fourier series termwise with respect to μ. One obtains

$$c_k(\mu\phi^{-1}) = \overline{\sum_{s\in\mathbf{Z}^n} c_s(\phi^k)\bar{c}_s(\mu)} = \sum_{s\in\mathbf{Z}^n} c_s(\mu)\bar{c}_s(\phi^k).$$

The sequence $\{c_s(\phi^k)\}_{s\in\mathbf{Z}^n}$ is absolutely summable, and the sequence $\{c_s(\mu)\}_{s\in\mathbf{Z}^n}$ is bounded, so the convergence in (4) is absolute. □

In the next lemma we calculate the Fourier coefficients of $\beta_{f\circ\phi}$.

LEMMA 3 *If $f \in N(\mathbf{U}^n)$ is zero-free, and ϕ is a proper holomorphic self-map of $\mathbf{U}^n$, then for each $k \in \mathbf{Z}^n$*

$$c_k(\beta_{f\circ\phi}) = \sum_{s\in\mathbf{Z}^n} c_s(\beta_f)c_k(\phi^s), \tag{5}$$

and the series in (5) *converges absolutely.*

PROOF. Since $f\circ\phi$ is a zero-free Nevanlinna class function, $\log|f\circ\phi|$ is n-harmonic, because there is some holomorphic h such that $f\circ\phi = \exp(h)$ and hence $\log|f\circ\phi| = \mathrm{Re}h$. We deduce that $\log|f\circ\phi| = P_{\beta_{f\circ\phi}}$ and, by the same kind of argument, $\log|f| = P_{\beta_f}$. By Lemma 1 we can write

$$c_k(\beta_{f\circ\phi}) = r^{-|k|}c_k(\log|f\circ\phi|_r) \tag{6}$$

for any r, $0 < r < 1$. Let's recall that the Poisson kernel is

$$P(z,w) = \sum_{s\in\mathbf{Z}^n} \tilde{z}^s\bar{w}^s \tag{7}$$

and the convergence in (7) is absolute and uniform with respect to $w \in \mathbf{T}^n$ if z stays in any fixed compact subset of $\mathbf{U}^n$. Integrating (7) with respect to $d\beta_f(w)$ one gets

$$\log|f(z)| = \sum_{s\in\mathbf{Z}^n} \tilde{z}^s c_s(\beta_f)$$

so

$$\log|f\circ\phi(rw)| = \sum_{s\in\mathbf{Z}^n} c_s(\beta_f)\tilde{\phi}^s(rw)$$

with absolute and uniform convergence as $w \in \mathbf{T}^n$ if r is fixed. So one can multiply the equality above by $\bar{w}^k$, integrate termwise the resulting series dm_n, and get

$$c_k(\log|f\circ\phi|_r) = \sum_{s\in\mathbf{Z}^n} c_s(\beta_f)\int_{\mathbf{T}^n} \tilde{\phi}_r^s(w)\bar{w}^k dm_n(w) =$$

$$\sum_{s\in\mathbf{Z}^n} c_s(\beta_f)c_k(\tilde{\phi}_r^s),$$

and the convergence above is absolute. On the other hand, one can easily see that

$$c_k(\tilde{\phi}_r^s) = r^{|k|}c_k(\phi^s)$$

so

$$r^{-|k|}c_k(\log|f\circ\phi|_r) = \sum_{s\in\mathbf{Z}^n} c_s(\beta_f)c_k(\phi^s)$$

which by (6) concludes the proof. □

Denote by $< .,. >$ the inner product in $L^2_{\mathbf{T}^n}(dm_n)$. We are ready to prove formula (1).

PROOF. For any fixed $k\in\mathbf{Z}^n$ we can write

$$c_k(\beta_{f\circ\phi}\phi^{-1}) = \sum_{s\in\mathbf{Z}^n} c_s(\beta_{f\circ\phi})\bar{c}_s(\phi^k) =$$

$$\sum_{s\in\mathbf{Z}^n}\sum_{t\in\mathbf{Z}^n} c_t(\beta_f)c_s(\phi^t)\bar{c}_s(\phi^k) = \sum_{t\in\mathbf{Z}^n} c_t(\beta_f)\left(\sum_{s\in\mathbf{Z}^n} c_s(\phi^t)\bar{c}_s(\phi^k)\right) =$$

$$\sum_{t\in\mathbf{Z}^n} c_t(\beta_f) < \phi^t,\phi^k >= \sum_{t\in\mathbf{Z}^n} c_t(\beta_f)\tilde{\phi}^{t-k}(0).$$

Above we used Lemmas 2 and 3, and the fact that

$$< \phi^t,\phi^k >= \int_{\mathbf{T}^n}\phi^t(w)\bar{\phi}^k(w)dm_n(w) =$$

$$\int_{\mathbf{T}^n}\phi^{t-k}(w)dm_n(w) = \tilde{\phi}^{t-k}(0),$$

since each component of ϕ is an inner function, holomorphic on an open neighbourhood of $\overline{\mathbf{U}}^n$, and depending on exactly one variable without repetition. Substitute now z by $\phi(0)$ in (7), multiply by $\bar{w}^k$ and integrate $d\beta_f(w)$. One obtains

$$c_k(P(\phi(0),w)d\beta_f(w)) = \sum_{t\in\mathbf{Z}^n} c_t(\beta_f)\tilde{\phi}^{t-k}(0),$$

so the measures $\beta_{f\circ\phi}\phi^{-1}$ and $P(\phi(0),w)d\beta_f(w)$ have identical Fourier coefficients. □

The assumption that ϕ is proper seems pretty restrictive at first glance. Let's observe that it is essential, because if one drops it, formula (1) is not true any more, as one can see from the following example.

EXAMPLE *Let $f\equiv e$ and $\phi(z_1,z_2) = (z_1,z_1)$, $(z_1,z_2)\in\mathbf{U}^2$. Formula (1) is not true for this choice of f and ϕ.*

PROOF. Let's observe that $f\circ\phi\equiv e$, so both f and $f\circ\phi$ are zero-free Nevanlinna class functions, and ϕ leaves $\mathbf{U}^2$ invariant, so the only missing assumption in the hypothesis of the theorem is the fact that ϕ is proper. It is obvious that $\beta_f = \beta_{f\circ\phi} = m_2$. Therefore $\beta_{f\circ\phi}\phi^{-1} = m_2\phi^{-1}$ and $P(\phi(0), w)d\beta_f(w) = m_2$. These measures are different because they have different Fourier coefficients. Indeed, $c_{(1,-1)}(m_2) = 0$ but

$$c_{(1,-1)}(m_2\phi^{-1}) = \int_{\mathbf{T}^2} \bar{w}_1 w_2 dm_2\phi^{-1}(w) = \int_{\mathbf{T}^2} \bar{w}_1 w_1 dm_2(w) = 1 \neq 0$$

$\square$

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