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SPECTRAL PROPERTIES OF OPERATORS HAVING DENSE ORBITS

Valentin Matache

1. Introduction

In the following H will denote a separable, infinite-dimensional, complex Hilbert space. The term operator will always mean linear, bounded operator on H . By invariant subspace we mean closed, invariant linear manifold. For a given operator T , the set of all invariant subspaces of T will be denoted $\text{Lat}T$, since obviously it is a lattice. The set of all operators commuting with T is denoted $\{T\}'$. A subspace will be called *hyperinvariant* for T if it is invariant under any operator in $\{T\}'$.

If x is a vector in H and T an operator on H the set $\{x, Tx, T^2x, \dots, T^n x, \dots\}$ will be called the orbit of x under T . If some orbit is dense in H then T will be called a *hypercyclic* operator and x a *hypercyclic vector* of T .

The statement *any Hilbert space operator acting on an infinitely-dimensional, separable, complex Hilbert space has proper invariant subspaces* will be referred to as *the invariant subspace problem* since it is an open problem in operator theory. In connection with that one could ask if any Hilbert space operator has proper, closed invariant sets (not necessarily subspaces). The corresponding problem for Banach space operators has been given a negative answer in [16]. As for Hilbert space operators, this is also an open problem. Obviously this problem has a negative answer if and only if there is some hypercyclic operator T on H such that any nonzero x in H is a hypercyclic vector of T . This author calls in [11] such an operator *everywhere-hypercyclic*.

Examples of hypercyclic operators can be found in [3], [6], [7] and [11]. We content ourselves to mention here that some composition operators and some adjoints of Toeplitz operators are hypercyclic. If S is a coshift and λ a complex number such that $|\lambda| > 1$ then λS is also a hypercyclic operator.

We shall list here some results in [11] that will be frequently used in this paper. The following is Lemma 2.3 in [11].

Theorem 1.1 *If T is a hypercyclic operator then the compression of T to the orthocomplement of any invariant subspace of T is hypercyclic.*

We denote by D the complex, open unit disc and call its boundary ∂D , the unit circle. If K is a nonvoid, compact subset of the complex plane and K is disconnected then the pair (K_1, K_2) of nonvoid, compact, mutually disjoint subsets of K is called a Riesz decomposition of K if $K = K_1 \cup K_2$. The following is Theorem 2.5 in [11].

Theorem 1.2 *If T is hypercyclic then the spectrum of T has nonvoid intersection with the unit circle. If T has disconnected spectrum and (K_1, K_2) is a Riesz decomposition of the spectrum of T then both K_1 and K_2 have nonvoid intersection with the unit circle.*

The terms C_0 -contraction and C_0 -contraction are used below in the sense of [18], that is $A \in C_0$ means that A is a contraction such that the sequence (A^n) tends strongly to 0, and $A \in C_0$ means that A is a completely nonunitary contraction such that for some inner function u , $u(T) = 0$ in the sense of the Nagy-Foiaş functional calculus for contractions. The following theorem contains the statements of Proposition 2.13 and Proposition 2.14 in [11].

Theorem 1.3 *If A is a contraction acting on H and λ a complex number such that λA is hypercyclic then A is in C_0 - C_0 .*

Finally we recall Corollary 2.7 in [11].

Theorem 1.4 *Any hypercyclic operator has no nontrivial finite-codimensional invariant subspace.*

A Hilbert space operator U is called universal if for any Hilbert space operator T there is some complex $\lambda \neq 0$ and some M in $\text{Lat}U$ such that the restriction $U|_M$ is similar to λT . Obviously the invariant subspace problem has a negative answer if and only if for any fixed universal operator U , $\text{Lat}U$ contains an infinitely-dimensional atom. The following nice result in [4] is a source of examples of universal operators.

Theorem 1.5 *Any onto Hilbert space operator having infinitely-dimensional kernel is universal.*

An operator T is called quasitriangular if there exists a sequence $\{P_n\}_n$ of projections of finite rank which converges to the identity in the strong operator topology such that $\|P_n T P_n - T P_n\| \rightarrow 0$. If both T and its adjoint T^* are quasitriangular we say T is biquasitriangular. The following celebrated result in [1] will be frequently needed.

Theorem 1.6 *If T is not biquasitriangular then either T or T^* has an eigenvalue.*

Consequently, in case T has an eigenvalue, the corresponding eigenspace is a nontrivial, hyperinvariant subspace of T . If T^* has an eigenvalue the orthocomplement of the eigenspace is hyperinvariant for T .

The merit of proving **Theorem 1.2** for the first time belongs to C. Kitai who did it in her unpublished thesis [10]. The author of this present paper discovered the same thing later, independently. C. Kitai was also the first to prove that hypercyclic operators are not hyponormal which implies that hypercyclic operators cannot be quasinormal and, in particular, they cannot be normal.

2. The spectrum of a hypercyclic operator

In the following $\sigma(T)$ and $\sigma_p(T)$ will denote the spectrum and the point spectrum of T respectively.

Proposition 2.1 *If T is any hypercyclic operator and $|\sigma(T)|$ is countable then $\sigma(T) \subseteq \partial D$.*

Proof. Since $|\sigma(T)|$ is compact and countable then it contains at least an isolated point [5, Problem 2.25]. Suppose now λ is an isolated point in $|\sigma(T)|$ and $\lambda \neq 1$. Denote K_1 the intersection of $\sigma(T)$ and the circle centered at 0, having radius λ and $K_2 = \sigma(T) - K_1$. Obviously K_2 is nonvoid since $\lambda \neq 1$ and, by **Theorem 1.2** $\sigma(T)$ has nonvoid intersection with the unit circle. (K_1, K_2) is a Riesz decomposition of $\sigma(T)$ and $K_1 \cap \partial D = \emptyset$ which is absurd by **Theorem 1.2**. Consequently we deduce that the set of isolated points in $|\sigma(T)|$ is $\{1\}$. If we denote now $K_1 = \sigma(T) \cap \partial D$ and $K_2 = \sigma(T) - K_1$ we must admit $K_2 = \emptyset$ for if not, we obtain a Riesz decomposition of $\sigma(T)$ with $K_2 \cap \partial D = \emptyset$ which is absurd. We deduce $\sigma(T) = K_1$ that is $\sigma(T) \subseteq \partial D$. □

Corollary 2.2 *Noninvertible hypercyclic operators have uncountable spectra.*

Proof. By **Proposition 2.1**, if we suppose $\sigma(T)$ is countable then so is $|\sigma(T)|$ and hence $\sigma(T) \subseteq \partial D$. We deduce $0 \notin \sigma(T)$ that is T is invertible. □

For any operator T we denote by $\sigma_e(T)$, $\sigma_{le}(T)$, $\sigma_{re}(T)$, and $\omega(T)$ the essential spectrum, the left essential spectrum, the right essential spectrum and the Weyl spectrum of T respectively. $r_e(T)$ denotes the essential spectral radius of T , $r(T)$ is the spectral radius. If \mathbf{C} is the complex plane we call *hole* any bounded, connected component of $\mathbf{C} - \sigma_e(T)$ and *pseudohole* any connected component of $\sigma_e(T) - \sigma_{le}(T)$ or $\sigma(T) - \sigma_{re}(T)$. It is well-known that the Fredholm index is constant on holes and pseudoholes. We refer the reader to [14] for all the notions and assertions above.

Theorem 2.3 *If T is a hypercyclic operator then any operator in the uniformly closed, unital algebra generated by T is quasitriangular.*

Proof. Suppose p is a nonconstant polynomial and $A = p(T)$. Denote $q(z) = p(\bar{z})$, $z \in \mathbf{C}$. Obviously $A^* = q(T^*)$. If A^* has eigenvalues suppose v is an eigenvector of A^* . Then the cyclic subspace $\bigvee_{n \geq 0} (T^*)^n v$ is a nontrivial, finite-dimensional subspace in $\text{Lat} T^*$. Its orthocomplement would be a nontrivial, finite-codimensional subspace in $\text{Lat} T$. This is absurd by **Theorem 1.4**. We deduce A^* has no eigenvalues. Consequently for any λ in \mathbf{C} , $\text{Ker}(A^* - \lambda) = 0$. It is clear now that the Fredholm index is nonnegative on any hole or pseudohole of A , that is $\dim(\text{Ker}(A - \lambda)) - \dim(\text{Ker}(A^* - \bar{\lambda})) \geq 0$ for λ in any hole or pseudohole. By Theorem 1.31 in [14] we can conclude now that A is quasitriangular. Since the set of all quasitriangular operators is norm-closed, [14, Proposition 4.15], the desired result follows. □

C. Kitai observed in [10] that hypercyclic operators are quasitriangular.

Proposition 2.4 *If T is hypercyclic and $A \in \{T\}'$ then A^* has no eigenvalues of finite multiplicity.*

Proof. Suppose λ is an eigenvalue of A^* having finite multiplicity. Denote M the corresponding eigenspace. $H \ominus M$ is a finite-codimensional, nontrivial subspace in $\text{Lat}T$. This is absurd by **Theorem 1.4**. □

Here is an interesting application of this result.

Corollary 2.5 *Any compact operator commuting with a hypercyclic operator is quasinilpotent.*

Proof. Suppose T is a hypercyclic operator and K is a compact operator in $\{T\}'$. Suppose λ is a nonzero, complex number in $\sigma(K)$. Then K^* is compact and consequently $\bar{\lambda}$ is an eigenvalue of K^* having finite multiplicity. By **Proposition 2.4** this is impossible. We conclude $\sigma(K) = \{0\}$. □

In connection with that we can prove

Proposition 2.6 *If T is hypercyclic, A is in $\{T\}'$ and A is polynomially compact then $\sigma(A)$ is finite.*

Proof. Suppose p is a nonzero polynomial such that $p(A)$ is compact. $A \in \{T\}'$ implies $p(A) \in \{T\}'$. By **Corollary 2.5** $\sigma(p(A)) = \{0\}$. By the spectral mapping theorem $\sigma(p(A)) = p(\sigma(A))$, so $\sigma(A)$ is contained in the set of the zeros of p . We deduce $\sigma(A)$ is finite. □

Suppose T is hypercyclic, A is in $\{T\}'$, f is an analytic function on a Cauchy domain containing $\sigma(A)$ and $f(A)$ is to be considered in the sense of the Riesz - Dunford functional calculus for operators (see [15, Ch. 2]). We can prove

Corollary 2.7 *Under the assumptions above, if $\sigma(A)$ is infinite then $f(A)$ is compact if and only if $f(A) = 0$.*

Proof. Suppose f is nonconstant and $f(A)$ is compact. By [15, Proposition 2.13] A is polynomially compact, hence $\sigma(A)$ is finite which is absurd. □

According to [9] for any fixed, unimodular, complex λ , there is some compact operator K such that $\lambda + K$ is a hypercyclic operator. Consequently, polynomially compact hypercyclic operators do exist. In connection with that class of operators we prove

Theorem 2.8 *If T is a hypercyclic operator which is also polynomially compact and $\sigma(T) = \{\lambda_1, \lambda_2, \dots, \lambda_j\}$, then there exist a decomposition of H in algebraic (not necessarily orthogonal) direct sum of j subspaces*

$$H = H_1 \oplus H_2 \oplus \dots \oplus H_j$$

and there is some positive integer n such that

$$[T - (\lambda_1 1 \oplus \lambda_2 1 \oplus \dots \oplus \lambda_j 1)]^n = K$$

where K is a dense range, compact, quasinilpotent operator.

Proof. Suppose first the spectrum of T is a singleton, $\sigma(T) = \{\lambda\}$. Obviously $|\lambda| = 1$. $A = T - \lambda$ is quasinilpotent. There exists a nonzero polynomial p such that $p(T)$ is compact. Denote $q(z) = p(z + \lambda)$, $z \in \mathbf{C}$. q is a nonzero polynomial and $p(T) = q(A)$ is a compact operator, that is A is polynomially compact. Since $q(A)$ is a compact operator commuting with T , we deduce by **Corollary 2.5** $q(A)$ is quasinilpotent. By the spectral mapping theorem we deduce

$$\sigma(q(A)) = \{0\} = q(\sigma(A)) = \{q(0)\}.$$

Hence $q(0) = 0$. Therefore there exists another polynomial s such that $q(z) = z^n s(z)$, $z \in \mathbf{C}$ and $s(0) \neq 0$.

$$\sigma(s(A)) = \{s(0)\} \neq \{0\}$$

implies $s(A)$ is invertible. $q(A)$ is compact, hence $A^n = q(A)[s(A)]^{-1}$ is compact, that is $K = (T - \lambda)^n$ is compact and obviously nilpotent because it commutes with T . If K is not a dense-range operator, then $\text{Ker}(T^* - \bar{\lambda})^n \neq 0$, and we immediately obtain that $\text{Lat}T$ contains nontrivial finite-codimensional subspaces, which contradicts **Theorem 1.4**. This ends the proof if $\sigma(T)$ is a singleton. Let

$$\sigma(T) = \{\lambda_1, \dots, \lambda_j\}.$$

Clearly $|\lambda_i| = 1$, $i = 1, 2, \dots, j$, according to **Proposition 2.1**. Set $K_i = \{\lambda_i\}$, $i = 1, \dots, j$. The theory of spectral decomposition built by Riesz for operators with disconnected spectra ([15, Theorem 2.10]), can be used to get the nontrivial Banach space projections P_1, \dots, P_j commuting with each other and with T and such that, $P_i P_k = 0$ if $i \neq k$ and $P_1 + P_2 + \dots + P_j = 1$. Denote H_1, \dots, H_j the ranges of these projections and T_1, \dots, T_j the restrictions of T to those ranges respectively. According to [11] T_k are hypercyclic operators acting on H_k , and having spectra $\sigma(T_k) = \{\lambda_k\}$, $k = 1, 2, \dots, j$. Consider the algebraic direct sum

$$H = H_1 \oplus \dots \oplus H_j$$

and observe that

$$T = T_1 \oplus \dots \oplus T_j$$

and, if p is a nonzero polynomial such that $p(T)$ is compact, then $p(T) = p(T_1) \oplus \dots \oplus p(T_j)$, hence T_1, \dots, T_j are polynomially compact. By what has already been proved for polynomially compact, hypercyclic operators having the spectrum equal

to a singleton we can chose the positive integers n_1, \dots, n_j such that $(T_k - \lambda_k)^{n_k}$ be compact for each $k = 1, 2, \dots, j$. Consequently, if n is greater than each n_k , then $(T_k - \lambda_k)^n$ is compact. Chose such a n and observe that

$$[(T_1 - \lambda_1)^n \oplus \dots \oplus (T_j - \lambda_j)^n] = [T - (\lambda_1 1 \oplus \dots \oplus \lambda_j 1)]^n$$

The desired conclusion is now clear. □

3. The essential spectrum

The essential spectra of operators commuting with a hypercyclic operator have interesting properties.

Theorem 3.1 *If T is hypercyclic and A is in $\{T\}'$ then*

$$(1) \quad \partial\sigma(A) \subseteq \partial\sigma_e(A)$$

$$(2) \quad \sigma(A) = \omega(A)$$

Proof. Denote $\sigma_{ap}(A^*)$ the approximate point spectrum of A^* . Chose λ in $\sigma_{ap}(A^*)$. If $\text{Ker}(A^* - \lambda) \neq 0$ then by **Proposition 2.4**, $\text{Ker}(A^* - \lambda)$ is infinitely-dimensional and, by [14, Proposition 2.15] we deduce $\lambda \in \sigma_{le}(A^*)$, hence $\lambda \in \sigma_e(A^*)$. If $\text{Ker}(A^* - \lambda) = 0$, then $A^* - \lambda$ is one-to-one, and by [19, Lemma 4.4.5], $A^* - \lambda$ is not bounded from below if and only if the range of $A^* - \lambda$ is not closed. Again by [14, Proposition 2.15] we deduce $\lambda \in \sigma_{le}(A^*)$. We have proved $\sigma_{ap}(A^*) \subseteq \sigma_{le}(A^*)$. Since $\partial\sigma(A^*) \subseteq \sigma_{ap}(A^*)$ and $\sigma_e(A^*) \subseteq \sigma(A^*)$ we deduce $\partial\sigma(A^*) \subseteq \partial\sigma_e(A^*)$. Taking complex conjugates one obtains (1).

To prove (2), suppose λ is in $\sigma(A) - \omega(A)$. Then λ is in E , a connected component of $\mathbf{C} - \sigma_e(A)$ and the Fredholm index is 0 on E . $A - \lambda$ is a Fredholm operator hence $\text{Ker}(A^* - \bar{\lambda})$ is finite-dimensional. By **Proposition 2.4** we deduce $\text{Ker}(A^* - \bar{\lambda}) = 0$. So $A - \lambda$ has dense range and since $A - \lambda$ is Fredholm $A - \lambda$ also has closed range. Consequently $A - \lambda$ is onto that is $A - \lambda$ is invertible. This is absurd. We deduce (2) holds. □

Corollary 3.2 *If A commutes with a hypercyclic operator then $r(A) = r_e(A)$.*

Proof. This is an immediate consequence of (1). □

Corollary 3.3 *If A commutes with a hypercyclic operator T and $\sigma(A)$ is not equal to any of the sets $\sigma_{le}(A)$, $\sigma_{re}(A)$ and $\sigma_e(A)$, then T has proper invariant subspaces.*

Proof. A is biquasitriangular if and only if $\omega(A) = \sigma_e(A) = \sigma_{le}(A) = \sigma_{re}(A)$, [14, Theorem 6.1]. By **Theorem 3.1** if $\sigma(A)$ does not equal each of the sets $\sigma_{le}(A)$,

$\sigma_{re}(A)$ and $\sigma_e(A)$ then A is not biquasitriangular so by **Theorem 1.6**, A has a nontrivial hyperinvariant subspace and consequently $\text{Lat}T$ is nontrivial. \square

The authors of [13] observe that the invariant subspace problem is equivalent to the following statement. *All the minimal invariant subspaces of some hyperbolic composition operator are one-dimensional.* This remark can be generalized to

Proposition 3.4 *If T is any hypercyclic operator such that $\sigma_e(T) - \sigma_{re}(T)$ is nonvoid then the invariant subspace problem is equivalent to the fact that all the minimal invariant subspaces of T are one-dimensional.*

Proof. $\lambda \in \sigma_e(T) - \sigma_{re}(T)$ if and only if $T - \lambda$ is semi-Fredholm and the index of $T - \lambda$ is $+\infty$, [14, Proposition 1.20]. Since by **Theorem 1.4**, $\text{Ker}(T^* - \bar{\lambda}) = 0$ and $T - \lambda$ has closed range, then $T - \lambda$ is universal according to **Theorem 1.5**. Since $\text{Lat}T = \text{Lat}(T - \lambda)$ the desired result follows. \square

Proposition 3.5 *If T is hypercyclic and one of the sets $\partial\sigma_e(T)$, $\sigma_{re}(T)$, $\sigma_{le}(T)$ and $\sigma_e(T)$ has void intersection with the unit circle then $\sigma(T)$ is connected.*

Proof. Since T is hypercyclic $\sigma_e(T) = \sigma_{le}(T)$. Indeed, by [14, Proposition 1.20], $\lambda \in \sigma_e(T) - \sigma_{le}(T)$ if and only if $T - \lambda$ is semi-Fredholm, having index $-\infty$. But by **Theorem 1.4** $\text{Ker}(T^* - \bar{\lambda}) = 0$ for any complex λ . We deduce $\sigma_e(T) = \sigma_{le}(T)$. The following inclusion is valid for any operator [14, Proposition 1.8]. $\partial\sigma_e(T) \subseteq \sigma_{le}(T) \cap \sigma_{re}(T)$. If T is hypercyclic we deduce by (1) that

$$\partial\sigma(T) \subseteq \partial\sigma_e(T) \subseteq \sigma_{le}(T) \cap \sigma_{re}(T) \subseteq \sigma_{re}(T) \subseteq \sigma_e(T) = \sigma_{le}(T).$$

So if the assumptions of this proposition hold then $\partial\sigma(T)$ has void intersection with the unit circle ∂D . By **Theorem 1.2** $\partial D \cap \sigma(T) \neq \emptyset$. If we suppose $\partial D \cap (\mathbf{C} - \sigma(T)) \neq \emptyset$ it follows $\partial\sigma(T) \cap \partial D \neq \emptyset$ since ∂D is connected. We deduce $\partial D \subseteq \sigma(T)$. Suppose $\sigma(T)$ is disconnected and (K_1, K_2) is a Riesz decomposition of $\sigma(T)$. Since ∂D is compact and connected, either $\partial D \subseteq K_1$ or $\partial D \subseteq K_2$. We deduce $K_2 \cap \partial D = \emptyset$ or $K_1 \cap \partial D = \emptyset$ which is impossible since T is hypercyclic, according once more to **Theorem 1.2**. We conclude $\sigma(T)$ is connected. \square

One of the most celebrated applications of the Riesz-Dunford functional calculus for operators is the fact it can be used to show that operators with disconnected spectra have proper hyperinvariant subspaces. Obviously if a hypercyclic operator satisfies the conditions in the proposition above this result is useless. Still the following remark can be made.

Remark *If T is hypercyclic and one of the sets $\sigma_{re}(T)$, $\sigma_{le}(T)$, and $\sigma_e(T)$ has void intersection with the unit circle, then T has proper hyperinvariant subspaces.*

Proof. Since $\sigma(T) \cap \partial D \neq \emptyset$ we deduce that one of the sets above is not equal to $\sigma(T)$. Exactly as in the proof of **Corollary 3.3** we deduce that T is not biquasitriangular and conclude by **Theorem 1.6** that T has proper hyperinvariant subspaces. □

We observe that hypercyclic operators satisfying the hypothesis in **Proposition 3.5** exist. For example one can take $T = \lambda S$ where $\lambda \in \mathbf{C}$, S is a coshift, and $|\lambda| > 1$.

Theorem 3.6 *If T is hypercyclic and $\sigma(T)$ has no interior point then T is biquasitriangular.*

Proof. If $\sigma(T)$ has no interior point, then $\sigma(T) = \partial\sigma(T)$. By **Theorem 3.1**, (1), we can write

$$\sigma(T) \subseteq \partial\sigma_e(T) \subseteq \sigma_{le}(T) \cap \sigma_{re}(T) \subseteq \sigma_{re}(T) \subseteq \sigma_e(T) \subseteq \sigma(T).$$

Taking **Theorem 3.1**, (2), and the above relation into consideration, we deduce

$$\sigma(T) = \sigma_e(T) = \omega(T) = \sigma_{re}(T).$$

We have already observed in the proof of **Proposition 3.5** that $\sigma_e(T) = \sigma_{le}(T)$. We conclude T is biquasitriangular. □

Suppose now H^2 is the classical Hardy space of all functions analytic on D , having square-summable Taylor coefficients. If ϕ is an analytic selfmap of D then the transform $f \rightarrow f \circ \phi$, $f \in H^2$ is called the *composition operator* induced by ϕ . If ϕ is a parabolic or hyperbolic Möbius transform the operator is frequently called *parabolic*, respectively *hyperbolic* composition operator. Parabolic and hyperbolic composition operators are hypercyclic. See [3] for a proof. The spectrum of a parabolic composition operator is ∂D according to [12]. We deduce

Corollary 3.7 *Parabolic composition operators are biquasitriangular, hypercyclic operators.*

4. Hypercyclic scalar multiples

Suppose A is a contraction acting on H . If for some complex λ , $T = \lambda A$ is hypercyclic we shall say that A has hypercyclic *scalar multiples*. We are perfectly aware of the fact that this terminology is not exactly the best, since scalar multiple might be misinterpreted. However, lacking a better short term we decided to use it with care. As we mention in the introduction, any coshift is a simple example of a contraction having hypercyclic scalar multiples. Observe that any coshift is a C_0 -contraction. According to **Theorem 1.3**, we proved that each contraction

having hypercyclic scalar multiples has this property. Among other things, in this paragraph, we continue this research line, and investigate to which extent the basic properties of a coshift hold for these contractions. For instance, if A is a coshift, then, since A^* is an isometry, we can easily see that one of the defect indices of A is finite, and that for each nonzero vector h , the sequence $((A^*)^n h)$ does not tend to 0. We will show that this last property holds for each contraction having hypercyclic scalar multiples if one of the defect indices is finite (**Theorem 4.5**). We will also prove in this paragraph that the only contractions having quasinormal adjoint and a finite defect index, which possess hypercyclic scalar multiples are the coshifts (**Theorem 4.6**).

We shall denote D_A and D_{A^*} the defect subspaces associated with A . δ_A and δ_{A^*} are the defect indices of A and Θ_A is the characteristic function of A . We refer the reader to [18] for all these notions which are well-known to anybody familiar with dilation theory. According to [18], we should replace the short notation Θ_A with $\{D_A, D_{A^*}, \Theta_A(\lambda)\}$, $|\lambda| < 1$, to point out that the characteristic function is an analytic, purely contractive function and for each λ with $|\lambda| < 1$, $\Theta_A(\lambda)$ is a contraction acting between the defect subspaces of A . We recall that Θ_A admits an *analytic scalar multiple* when there is a nonzero complex function δ , analytic on the open unit disc and a contractive analytic function $\{D_{A^*}, D_A, \Omega(\lambda)\}$, $|\lambda| < 1$, such that

$$(3) \quad \Omega(\lambda)\Theta_A(\lambda) = \delta(\lambda)1_{D_A} \quad , \quad |\lambda| < 1$$

and

$$(4) \quad \Theta_A(\lambda)\Omega(\lambda) = \delta(\lambda)1_{D_{A^*}} \quad , \quad |\lambda| < 1.$$

If the spectrum of A is not the whole, closed unit disc and if $1 - A^*A$ is a nuclear operator then A is usually called a *weak contraction*. The characteristic functions of weak contractions admit analytic scalar multiples. We refer to [18, Chap. VIII] for details. Having recalled these facts, we prove

Theorem 4.1 *If A is a contraction on H and Θ_A admits analytic scalar multiples then for each complex λ , λA is not hypercyclic.*

Proof. If A is not completely nonunitary then for each λ , λA is not completely nonnormal and fails being hypercyclic. We shall suppose A is completely nonunitary and the analytic scalar function δ is an analytic scalar multiple of Θ_A . The characteristic function of A^* is given by $\Theta_{A^*}(\lambda) = (\Theta_A(\bar{\lambda}))^*$, $|\lambda| < 1$. One can easily deduce that Θ_{A^*} admits analytic scalar multiples. Suppose that for some complex λ , λA is hypercyclic. By **Theorem 1.3** we deduce $A \in C_0$, that is $A^* \in C_{0,0}$ so by [18, Ch. VI Proposition 3.5] Θ_{A^*} is inner and consequently Θ_{A^*} is bilaterally inner, [18, Ch. V, Theorem 6.2]. We deduce $A \in C_{00}$. According to [18, Ch. VI, Theorem 5.1] we obtain $A \in C_0$ which is absurd.

□

Above we have used the notation $A \in C_{0.}$. We recall this means that $A^* \in C_0$. We also recall that $C_0 \cap C_{0.} = C_{00}$.

Corollary 4.2 *Weak contractions do not have hypercyclic scalar multiples.*

Corollary 4.3 *If the contraction A has hypercyclic scalar multiples and one of the defect indices is finite then the spectrum of A is the whole, closed unit disc.*

Proof. If we suppose that the spectrum of A is not the whole closed, unit disc then either A or A^* is weak, and consequently Θ_A has analytic scalar multiples. □

Corollary 4.4 *If an invertible contraction A has hypercyclic scalar multiples then both defect indices are infinite.*

We recall that a contraction A is in $C_{.1}$ if for each nonzero $h \in H$ the sequence $((A^*)^n h)$ does not tend to 0. We usually denote $C_{01} = C_0 \cap C_{.1}$.

Theorem 4.5 *If the contraction A has hypercyclic scalar multiples and one of the defect indices of A is finite then $A \in C_{01}$.*

Proof. Suppose first δ_{A^*} is finite. Since $A \in C_0$, by [18, Ch. II, Theorem 4.1] A has a triangularization of type

$$(5) \quad \begin{pmatrix} C_{01} & * \\ 0 & C_{00} \end{pmatrix}$$

Denote by B the C_{00} term in the triangularization above. Clearly B^* is the restriction of A^* to an invariant subspace. According to [18, Ch. VII, Proposition 3.6], $\delta_{B^*} \leq \delta_{A^*}$. So $B^* \in C_{00}$ and $\delta_B = \delta_{B^*}$ is finite. By [18, Ch. VI, Theorem 5.2] we deduce $B \in C_0$. By **Theorem 1.1** B also has hypercyclic scalar multiples which is absurd unless $A \in C_{01}$. Since $A \in C_0$, we deduce Θ_{A^*} is inner, so $\Theta_{A^*}(\lambda)$ maps isometrically D_{A^*} into D_A a.e. for $|\lambda| = 1$. Consequently $\delta_{A^*} \leq \delta_A$. So if δ_A is finite, so is δ_{A^*} . □

Theorem 4.6 *If A is a quasinormal contraction such that one of the defect indices is finite then A^* has hypercyclic scalar multiples if and only if A is a unilateral shift.*

Proof. Clearly any unilateral shift has the required property. For the converse observe that if A^* has hypercyclic scalar multiples then as in the proof of **Theorem 4.5** we deduce $\delta_A \leq \delta_{A^*}$ so we may suppose that δ_A is finite. By **Theorem 4.5** $A^* \in C_{01}$ so $A \in C_{10}$. We deduce $\text{Ker} A = 0$. It is well-known (see [18, Ch. I, (3.7)]) that $D_A = \overline{A^* D_{A^*}} \oplus \text{Ker} A = \overline{A^* D_{A^*}}$. Since δ_A is finite we may write $D_A = A^* D_{A^*}$. On the other hand $D_{A^*} = \overline{A D_A} \oplus \text{Ker} A^* = A D_A \oplus \text{Ker} A^*$ so

$A^*D_{A^*} = A^*AD_A$ that is $A^*AD_A = D_A$. Since A is one - to - one we deduce that the restriction of A^*A to D_A which we consider as a transform of D_A onto itself is invertible. Denote it by V . The restriction of A to $(D_A)^\perp$ transforms isometrically $(D_A)^\perp$ onto $(D_{A^*})^\perp$. The details of this rather well-known fact can be found in [18, proof of Theorem 4.1 in Ch. VI]. Likewise A^* transforms isometrically $(D_{A^*})^\perp$ onto $(D_A)^\perp$. So the restriction of A^*A to $(D_A)^\perp$ can be viewed as a unitary operator acting on $(D_A)^\perp$. Denote this operator by U . With respect to the decomposition $H = D_A \oplus (D_A)^\perp$ A^*A may be written $A^*A = V \oplus U$. Clearly both V and U are positive. Since U is unitary $U^2 = 1_{(D_A)^\perp}$. We deduce $U = 1_{(D_A)^\perp}$. So $A^*A = V \oplus 1$. If we prove $D_A = 0$ then $A^*A = 1$ so A is an isometry and since $A^* \in C_0$, A is a shift [2, Corollary 2.4]. Suppose $D_A \neq 0$. If we show this is absurd then the proof is over. Clearly the spectrum of $V^{\frac{1}{2}}$ does not coincide with $\{1\}$ because this means $V = 1$ and hence $A^*A = 1$ that is $D_A = 0$. Since D_A is finitely dimensional we may chose α in $(0, 1)$ an eigenvalue of $V^{\frac{1}{2}}$ associated to the eigenvector $h \neq 0$. Clearly $(A^*A)^{\frac{1}{2}} = V^{\frac{1}{2}} \oplus 1$. Suppose $A = W(A^*A)^{\frac{1}{2}}$ is the polar decomposition of A . Since A is quasinormal W and $(A^*A)^{\frac{1}{2}}$ commute, so $A^n = W^n(V^{\frac{1}{2}} \oplus 1)^n$ for any positive integer n . We deduce $\|A^n h\| = \alpha^n \|W^n h\| \leq \alpha^n \|h\| \rightarrow 0$ so $A \notin C_{10}$ which is absurd. \square

Theorem 4.7 *If a contraction A has at least one finite defect index then for any complex λ the operator $T = \lambda A$ has a proper, closed invariant subset.*

Proof. If A has no hypercyclic scalar multiple the conclusion is immediate. Suppose the contrary now. Then, according to **Theorem 4.5**, $A \in C_{01}$ and hence $\delta_{A^*} \leq \delta_A$. If one of the defect indices is finite then $\delta_{A^*} < \delta_A$. Indeed if δ_A is infinite then clearly δ_{A^*} must be finite and the inequality holds. If δ_A is finite then so is δ_{A^*} . Suppose $\delta_A = \delta_{A^*}$. We show this is absurd. Indeed, $A \in C_0$ implies that $(\Theta_A(e^{-it}))^*$ is a.e. an isometry of D_{A^*} into D_A . Since $\delta_A = \delta_{A^*}$ is finite we deduce that actually $(\Theta_A(e^{-it}))^*$ is a.e. a onto isometry. Consequently, by [18, Ch. VI, Proposition 3.5] $A \in C_{00}$ which is absurd. We admit $\delta_{A^*} < \delta_A$. Since $D_A = \overline{A^*D_{A^*}} \oplus \text{Ker}A$, we deduce $\text{Ker}A \neq 0$. $\text{Ker}A$ is a proper invariant subspace of T . \square

We have shown in this paper that if A is a contraction such that A^* is quasinormal and one of the defect indices is finite then the only possibility that A have hypercyclic scalar multiples is that A be a coshift. We can show that if $\delta_A \leq 1$ then the quasinormality assumption is superfluous.

Proposition 4.8 *The only contractions A admitting hypercyclic scalar multiples and such that $\delta_A \leq 1$ are the coshifts of multiplicity 1.*

Proof. If $\delta_A = 0$ then A is isometric and clearly has no hypercyclic scalar multiples. If $\delta_A = 1$ suppose A has hypercyclic scalar multiples. As in the proof of **Theorem**

4.7 deduce $\delta_{A^*} < \delta_A$ hence $\delta_{A^*} = 0$ which means A^* is isometric and since $A \in C_0$, we deduce A^* is a shift having multiplicity δ_A . \square

5. Compalence and hypercyclicity

Recall that two operators A and B are compalent if there is some unitary operator U and some compact operator K such that $U^*AU = B + K$. We refer the reader to [14] for more about compalence.

Theorem 5.1 *If A commutes with a hypercyclic operator T , then A is compalent with an invertible operator if and only if A is invertible.*

Proof. Suppose B is invertible and compalent with A . Then $\sigma(A) \subseteq \sigma(B)$. Indeed if λ is in $\sigma(A^*) - \sigma(B^*)$ then by [14, Theorem 2.4], λ is an eigenvalue of A^* of finite multiplicity. By **Proposition 2.4** in this paper such eigenvalues do not exist. We deduce $\sigma(A^*) \subseteq \sigma(B^*)$ and taking complex conjugates, $\sigma(A) \subseteq \sigma(B)$. If we suppose A is not invertible we deduce $0 \in \sigma(B)$ that is B is not invertible, which is absurd. The reverse implication is trivial. \square

Theorem 5.2 *If T is hypercyclic, A is in $\{T\}'$, and A is compalent with a normal operator then there is a normal operator N such that A is compalent with N and $\sigma(A) = \sigma(N)$.*

Proof. Suppose A is compalent with the normal operator N' . Set $K = \sigma_e(N')$ and chose $E \subseteq K$, a countable set, dense in K . Repeat each element in E infinitely-many times and denote $(d_n)_n$ the sequence one obtains this way. Chose a Hilbert basis in H and denote N the diagonal operator with respect to this basis, having diagonal $(d_n)_n$. Clearly $\sigma_p(N) = E$ and no eigenvalue has finite multiplicity. By [14, Proposition 2.16], we deduce $E \subseteq \sigma_e(N) \subseteq \sigma(N)$. Since $\sigma(N)$ is the closure of E , [8, Problem 63], we deduce $\sigma_e(N) = \sigma(N) = K$. Since $\sigma_e(N) = \sigma_e(N')$, N and N' are compalent according to [14, Corollary 2.13]. Compalence is an equivalence so A is compalent with N . We deduce by **Theorem 3.1** that $\sigma(A) = \omega(A) = \omega(N)$. Since $\sigma_e(N) \subseteq \omega(N) \subseteq \sigma(N)$ we also deduce $\sigma(N) = \omega(N)$ hence $\sigma(N) = \sigma(A)$. \square

Theorem 5.3 *If A commutes with a hypercyclic operator and is compalent with a normal operator then the following equalities hold modulo a countable set of points.*

$$(6) \quad \sigma(A^*A) = |\sigma(A)|^2 = \sigma(AA^*)$$

Proof. By the previous theorem we may chose N normal such that N be compalent with A and $\sigma(N) = \sigma(A)$. Then the selfadjoint operators N^*N and A^*A are compalent. By [14, Proposition 2.4], $\lambda \in \sigma(A^*A) - \sigma(N^*N)$ implies λ is an

eigenvalue of finite multiplicity of A^*A . A selfadjoint operator on a separable Hilbert space can have countably many eigenvalues. Consequently $\sigma(A^*A) = \sigma(N^*N)$ modulo some countable set. By the spectral mapping theorem for normal operators $\sigma(N^*N) = |\sigma(N)|^2 = |\sigma(A)|^2$. The first equality in (6) is now obvious. Replace A with A^* and N with N^* in the argument above to get the second. \square

Suppose ϕ is an inner function and denote T_ϕ the analytic Toeplitz operator induced by ϕ , that is the operator

$$(7) \quad T_\phi = \phi f \quad f \in H^2$$

Here is an application of the results above.

Application 1. T_ϕ is compalent with a normal operator if and only if ϕ is constant.

Proof. For $\varphi(z) = z$, $z \in D$ denote $T_\varphi = S$ since it is a unilateral shift of multiplicity 1. For λ in \mathbf{C} , $|\lambda| > 1$ λS^* is hypercyclic and obviously T_ϕ^* commutes with λS^* . Since $T_\phi^* T_\phi$ is the identity it follows by (6) that $|\sigma(T_\phi)|$ is countable. This happens if and only if ϕ is constant, since it is simple to see that $\sigma(T_\phi)$ is the closure of $\phi(D)$, [8, Problem 247]. \square

Here is another application.

Proposition 5.4 *If T is a hypercyclic operator compalent with a normal operator and if for some $c \geq \|T\|^2$, $c - T^*T$ is a finite-rank operator then $\sigma(T) \subseteq \partial D$.*

Proof. Denote $A = T/\sqrt{c}$. Then $\|A\| \leq 1$, $\sqrt{c}A$ is hypercyclic so by **Theorem 1.3**, A is a C_0 -contraction. $\text{rank}(1 - A^*A) = \text{rank}(c - T^*T) = n$ is finite. By [17, page 5], we may chose a shift S having multiplicity n and a subspace M in $\text{Lat} S^*$ such that $S^*|_M$ be unitarily equivalent to A . Consequently $\sigma(A) = \sigma(S^*|_M)$. $\dim \text{Ker} S^* = n$. Denote P_0 the orthogonal projection onto $\text{Ker} S^*$ and P the orthogonal projection onto M . A^*A is unitarily equivalent to

$$(S^*|_M)^*(S^*|_M) = PSS^*|_M = P(1 - P_0)|_M = (P - PP_0)|_M.$$

$P - (P - PP_0) = PP_0$ and PP_0 is compact. By [14, Proposition 2.4] and the fact already mentioned in this paper that selfadjoint operators acting on separable Hilbert spaces may have countably many eigenvalues, we deduce $\sigma(P - PP_0) = \sigma(P)$ modulo a countable set. Since $\sigma(P) \subseteq \{0, 1\}$ we deduce $\sigma(P - PP_0)$ is countable. Hence the resolvent set of $P - PP_0$ contains no bounded connected component. By [15, Theorem 0.8] we deduce $\sigma((P - PP_0)|_M) \subseteq \sigma(P - PP_0)$, so $\sigma(A^*A) = \sigma((P - PP_0)|_M)$ is countable. Then by **Theorem 5.3** and **Theorem 2.1** $\sigma(T) \subseteq \partial D$.

□

Corollary 5.5 *Suppose A is a contraction having hypercyclic scalar multiples. If δ_A is finite, then A is not compalnt with any normal operator.*

Proof. Suppose that $T = \lambda A$ is hypercyclic. Since δ_A is finite, we deduce $\|A\| = 1$. Indeed, $h \in \text{Ker} D_A$ is equivalent to $\|h\| = \|Ah\|$. If $\|A\| < 1$, this happens if and only if $h = 0$. So, in this case $D_A = H$. Denote $c = |\lambda|^2 = \|T\|^2$. $c - T^*T = |\lambda|^2 (1 - A^*A)$ is a finite-rank operator. If A is compalnt with a normal operator, then by **Proposition 5.4**, $\sigma(A) \subseteq \partial D$. This contradicts **Corollary 4.3**.

□

Theorem 5.6 *If A is an operator compalnt with a hypercyclic operator T then the spectrum of A has the following properties.*

- a) $\sigma(A) \cap \partial D \neq \emptyset$
- b) *If $\sigma(A)$ is disconnected and (K_1, K_2) is a Riesz decomposition of $\sigma(A)$ into infinite subsets (provided such a decomposition exist) then $K_i \cap \partial D \neq \emptyset$, $i = 1, 2$.*
- c) *If $\sigma(A)$ is a countable set then the cluster points of $\sigma(A)$ (if any) are in ∂D .*

Proof. a) Obviously $\omega(A) = \omega(T) = \sigma(T)$, hence $\sigma(T) \subseteq \sigma(A)$. If we suppose $\sigma(A) \cap \partial D = \emptyset$ we deduce $\sigma(T) \cap \partial D = \emptyset$ which is absurd by **Theorem 1.2**.

b) First we show that for each $i = 1, 2$ we have that $K_i \cap \omega(A) \neq \emptyset$. Suppose that for some fixed i the contrary happens. Then $K_i \subseteq (\sigma(A) - \omega(A))$. Suppose $H \subseteq (\mathbf{C} - \sigma_e(A))$ is any hole of index 0. Suppose also that $H \subseteq \sigma(A)$, then $\bar{H} \subseteq \sigma(A)$ and \bar{H} (the closure of H) is a compact connected set so either $\bar{H} \subseteq K_1$ or $\bar{H} \subseteq K_2$. If $\bar{H} \subseteq K_i \subseteq (\sigma(A) - \omega(A))$ we deduce $\partial H \subseteq K_i$, hence $K_i \cap \sigma_e(A) \neq \emptyset$ and consequently $K_i \cap \omega(A) \neq \emptyset$ which is absurd. We proved that if $K_i \subseteq (\sigma(A) - \omega(A))$ then for each hole H having index 0, $\bar{H} \cap K_i = \emptyset$. According to [14, Proposition 1.27] we deduce K_i is a countable set of isolated points. Since K_i is compact we deduce K_i is finite which is absurd. We must admit that for each $i = 1, 2$ $K_i \cap \omega(A) \neq \emptyset$. Like in the proof of a) $\sigma(T) \subseteq \sigma(A)$. Set $K_i' = K_i \cap \omega(A) = K_i \cap \omega(T)$, $i = 1, 2$. Since $\omega(T) = \sigma(T)$ (K_1', K_2') is a Riesz decomposition of $\sigma(T)$, consequently $K_i' \cap \partial D \neq \emptyset$, $i = 1, 2$ and $K_i' \subseteq K_i$ so $K_i \cap \partial D \neq \emptyset$ for $i = 1, 2$.

c) If $\sigma(A)$ is countable then clearly $\sigma(T)$ is countable and hence $\sigma(T) \subseteq \partial D$ by **Proposition 2.1** that is $\omega(A) \subseteq \partial D$. We deduce $\sigma(A) - \omega(A) \supseteq \sigma(A) - \partial D$. By [14, Proposition 1.27], $(\sigma(A) - \omega(A))$ is a countable set of isolated points which means that if $\sigma(A)$ has cluster points then necessarily they are in ∂D .

□

Corollary 5.7 *If the operator A is compalnt with the hypercyclic operator T , $\sigma(A)$ is disconnected and if (K_1, K_2) is a Riesz decomposition of $\sigma(A)$ such that $K_1 \cap \partial D = \emptyset$ then necessarily both A and A^* have nonvoid point spectrum.*

Proof. By the previous theorem K_1 is in that case finite and nonvoid. Chose λ in K_1 . Obviously λ is isolated. If $\lambda \in \sigma(T)$ set $K_1' = \{\lambda\}$, $K_2' = (\sigma(T) - \{\lambda\})$ to get a Riesz decomposition of $\sigma(T)$ with $K_1' \cap \partial D = \emptyset$. This is absurd since T is hypercyclic. Hence $\lambda \in (\sigma(A) - \sigma(T))$. By [14, Proposition 2.4] λ is an eigenvalue of A having finite multiplicity. Since A^* and T^* are also compalent we can use a similar argument to show the pointspectrum of A^* is nonvoid. \square

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