3-10-2021

Analytical and numerical convexity results for discrete fractional sequential differences with negative lower bound

Christopher S. Goodrich
University of Nebraska at Omaha, cgoodrich@unomaha.edu

Benjamin Lyons
Creighton Preparatory School

Andrea Scapellato
Università di Catani

Mihaela T. Velcsov
University of Nebraska at Omaha, dvelcsov@unomaha.edu

Recommended Citation

Christopher S. Goodrich, Benjamin Lyons, Andrea Scapellato & Mihaela T. Velcsov (2021) Analytical and numerical convexity results for discrete fractional sequential differences with negative lower bound, Journal of Difference Equations and Applications, 27:3, 317-341, DOI: https://doi.org/10.1080/10236198.2021.1894142

Analytical and numerical convexity results for discrete fractional sequential differences with negative lower bound

Christopher S. Goodrich[a], Benjamin Lyons[b], Andrea Scapellato[c] and Mihaela T. Velcsov[d]

[a]School of Mathematics and Statistics, UNSW Sydney, Sydney, NSW, Australia; [b]Department of Mathematics, Creighton Preparatory School, Omaha, NE, USA; [c]Dipartimento di Matematica e Informatica, Università di Catania, Catania, Italy; [d]Department of Mathematics, University of Nebraska-Omaha, Omaha, NE, USA

We investigate relationships between the sign of the discrete fractional sequential difference $\left(\Delta_{1+a-\mu}^{\nu}\Delta_a^{\mu}f\right)(t)$ and the convexity of the function $t \mapsto f(t)$. In particular, we consider the case in which the bound

$$\left(\Delta_{1+a-\mu}^{\nu}\Delta_a^{\mu}f\right)(t) \geq \varepsilon f(a),$$

for some $\varepsilon > 0$ and where $f(a) < 0$, is satisfied. Thus, we allow for the case in which the sequential difference may be negative, and we show that even though the fractional difference can be negative, the convexity of the function f can be implied by the above inequality nonetheless. This demonstrates a significant dissimilarity between the fractional and non-fractional cases. We use a combination of both hard analysis and numerical simulation.

KEYWORDS
Convexity; discrete fractional calculus; Gamma function; sequential difference; numerical approximation

2010 MATHEMATICS SUBJECT CLASSIFICATIONS Primary: 26A51; 33F05; 39A12; 65D15; 65Q20; Secondary: 39A99; 39B62

1. Introduction

Denote by $\mathbb{N}_a$, for any $a \in \mathbb{R}$, the set $\mathbb{N}_a := \{a, a+1, a+2, \dots\}$. Then recall that for a function $f : \mathbb{N}_a \to \mathbb{R}$ the first-order forward difference (or first-order delta difference) is defined by

$$(\Delta f)(n) := f(n+1) - f(n).$$

By composing this operator one obtains the second-order forward difference, $\Delta^2 f$, which is defined by

$$\left(\Delta^2 f\right)(n) := f(n+2) - 2f(n+1) + f(n).$$

It is well known that there is a strong connection between the signs of these functions and, respectively, the monotonicity and convexity of f. That is, f is monotone on $\mathbb{N}_a$ if and only if $(\Delta f)(n) \geq 0$ for $n \in \mathbb{N}_a$, and f is convex on $\mathbb{N}_a$ if and only if $(\Delta^2 f)(n) \geq 0$ for $n \in \mathbb{N}_a$ – see Atici and Yaldiz [13] for additional results as concerns the convexity or concavity of discrete time maps.

At the same time in the past 10 years much interest has been cultivated in understanding a fractional-order analogue of the nth-order forward difference operator Δ^n. One such definition, initially investigated in great detail by Atici and Eloe [7–11] and then later developed in many different directions by Abdeljawad, Al-Mdallal, and Hajji [2], Abdeljawad and Atici [3], Agarwal *et al.* [5], Anastassiou [6], Ferreira [21,22], Goodrich [25], Jonnalagadda [44,45], Lizama [48], Lizama and Murillo-Arcila [49], and Wu and Baleanu [51], among others, is defined (see Section 2 for additional details) by

$$\left(\Delta_a^\nu f\right)(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-s-1)^{\underline{-\nu-1}} f(s),$$

where $t \in \mathbb{N}_{a-\nu+N}$, $N-1 < \nu \leq N$, $a \in \mathbb{R}$, and the function $t \mapsto t^{\underline{\alpha}}$ is defined by

$$t^{\underline{\alpha}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}.$$

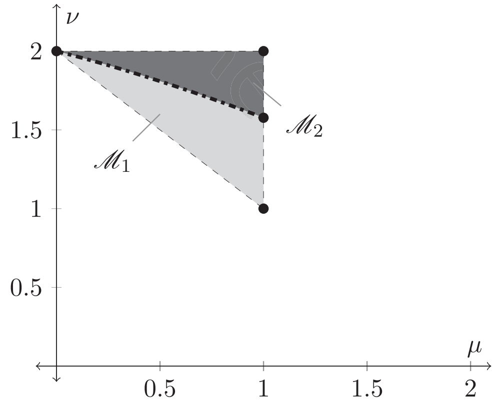
Note that owing to the fact that $(\Delta_a^\nu f)(t)$ is defined in terms of a linear combination of values of $f(n)$ at *previous* time points, the operator $\Delta_a^\nu f$ is inherently *non-local*. As a consequence of this non-local structure the relationship between the sign of $(\Delta_a^\nu f)(t)$ and the monotonicity or convexity of $t \mapsto f(t)$ is both subtle and complex. For example, in case $2 < \nu < 3$ it is known that $(\Delta_a^\nu f)(t) > 0$ does *not* necessarily imply the convexity of f – see Jia *et al.* [41].

It is also the case that compositions of fractional differences have been studied extensively in recent years. A composition of fractional differences of the form

$$\Delta_{1+a-\mu}^\nu \circ \Delta_a^\mu$$

is known as a *sequential fractional difference* since the fractional differences are composed in a particular sequence – to the best of our knowledge this type of fractional difference operators was first considered by Goodrich [23] in the context of the analysis of certain discrete fractional boundary value problems. As with a single fractional difference, there is a very subtle and complex relationship between the sign of a sequential difference and whether the associated function is, say, monotone or convex.

For example, with $0 < \mu < 1$, $1 < \nu < 2$, and $2 < \mu + \nu < 3$, in Goodrich [28] and later in Dahal and Goodrich [18] it was shown that there is a sharp dichotomy between the region of the (μ, ν) parameter space on which there is a connection between the sign of the sequential fractional difference $(\Delta_{1+a-\mu}^\nu \Delta_a^\mu f)(t)$ and the convexity of f, and the region of the parameter space on which such a connection fails to exist. This is illustrated by the drawing shown below.



The dark grey region, $\mathscr{M}_2$, is the subset of the admissible parameter space on which there exists a strong connection between the sign of $(\Delta^{\nu}_{1+a-\mu}\Delta^{\mu}_{a}f)(t)$ and the sign of $(\Delta f)(t)$. By contrast, the light grey region, $\mathscr{M}_1$, is the subset of the admissible parameter space on which no such connection exists. Thus, whether one can produce a convexity-type result is related in a complex way to the values of μ and ν.

In both Dahal and Goodrich [18] and Goodrich [28] it was assumed that

$$\left(\Delta^{\nu}_{1+a-\mu}\Delta^{\mu}_{a}f\right)(t) > 0, \tag{1}$$

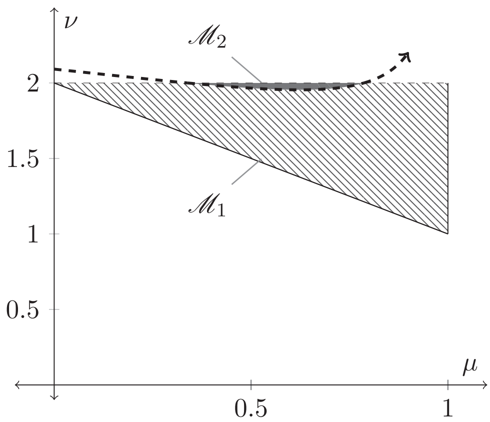
for each $t \in \mathbb{N}_{3+a-\mu-\nu}$. Our goal in this paper is to explore what happens when the zero lower bound in inequality (1) is replaced by a negative lower bound – specifically,

$$\left(\Delta^{\nu}_{1+a-\mu}\Delta^{\mu}_{a}f\right)(t) > \varepsilon f(a) \tag{2}$$

for some $\varepsilon > 0$ and in case $0 < \mu < 1$, $1 < \nu < 2$, and $2 < \mu + \nu < 3$. Since we will assume (as in [18,28]) that $f(a) < 0$, it follows that $\varepsilon f(a) < 0$. We will show that one can still obtain convexity-type results even if the sequential fractional difference is negative and, in particular, satisfies inequality (2) instead of the stronger inequality (1). However, we note that as $\varepsilon f(a)$ decreases (i.e. as the negative lower bound becomes 'more strongly negative') it occurs (as we will show later in Section 4) that

$$\left|\mathscr{M}_2\right| \to 0.$$

In particular (see Section 4 for additional details and other relevant figures), the drawing below shows the effect of choosing $\varepsilon = \frac{1}{20}$ in inequality (2).



As the drawing above suggests, for this value of ε, which by Lemma 3.9 is actually relatively large, the 'good region', which is to say the region on which a convexity-type result can be deduced, has shrunk considerably from the natural configuration shown earlier when $\varepsilon = 0$. This is what we mean when we assert that as the negative lower bound becomes 'more strongly negative' it follows that we can deduce a convexity-type result on a smaller and smaller subregion of the admissible parameter space.

Notice that this type of result is highly aberrant with respect to the integer-order difference calculus. Indeed, it absolutely cannot happen that $(\Delta^2 f)(n) < 0$ and yet f is convex, for if $(\Delta^2 f)(n) < 0$, then $(\Delta f)(n+1) < (\Delta f)(n)$, which at once means that f is concave at the point n rather than convex. Therefore, this illustrates a significant and, we believe, interesting dissimilarity between the integer- and fractional-order difference calculus. We would like to mention that this sort of 'negative lower bound' type of result was considered previously in a very recent paper by Goodrich *et al.* [34]. However, in that paper, only monotonicity-type results were considered – not convexity-type results. As it turns out, while the results are of a similar flavor, new arguments are needed – cf., the proof of Lemma 3.2 in this paper with [34, Lemma 3.2], for example.

To conclude this section we would like to mention the broader literature inasmuch as discrete fractional calculus is concerned, particularly with regard to monotonicity- and convexity-type results. As already suggested earlier in this section there has been much progress in characterizing the precise relationship between the sign of the fractional difference of f and the qualitative properties (e.g. monotonicity and convexity) of the function f. These investigations include papers in the non-sequential case by Abdeljawad and Abdalla [1], Abdeljawad and Baleanu [4], Atici and Uyanik [12], Jia *et al.* [43], Dahal and Goodrich [15], Du *et al.* [19], Erbe *et al.* [20], Goodrich [24,26], Goodrich *et al.* [31], Goodrich and Lizama [33], Jia *et al.* [38–42], and Liu *et al.* [47]; and in the sequential case by Dahal and Goodrich [16,17], Goodrich [28–30], Goodrich and Lizama [32], and Goodrich and Muellner [35]. But other than the previously mentioned paper [34], none considers the implications of a negative lower bound. So, we hope that the results of this paper help to spur on a new direction in which to extend these types of monotonicity and convexity results.

Let us conclude this section with a brief explanation of the organization of the paper. In Section 2 we briefly recall some basic definitions in discrete fractional calculus. In Section 3 we use hard analysis to deduce some convexity-type results in the case where the sequential difference has a negative lower bound. Finally, in Section 4 we provide a *numerical* investigation of these same relationships. Using numerical simulations allow us to more broadly understand the relationships we investigate in this paper.

2. Preliminaries

In this brief section we mention some basic results in discrete fractional calculus. A wealth of additional results may be found in the recent textbook by Goodrich and Peterson [36], and we direct the reader to this resource for a more substantial presentation of the fundamental ideas in discrete fractional calculus. We begin with the definition of the falling factorial function.

Definition 2 .1: Weput

$$t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$$

for any t and ν for which neither $t+1$ and $t+1-\nu$ is a pole of the Gamma function. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} := 0$.

Next we state the definitions of the discrete fractional difference and sum of Riemann-Liouville type. We also state the definition of the fractional Taylor monomial of order ν. Observe that the falling factorial function from Definition 2.1 is the kernel of the summation operators in Definition 2.2. Definitions 2.2 and 2.3 can be found in [36, Definition 2.25 and Theorem 2.33] and [36, Definition 2.24], respectively.

Definition 2.2: The ***ν-th fractional sum***, $\nu > 0$, of a function $f : \mathbb{N}_a \to \mathbb{R}$, where $a \in \mathbb{R}$, is

$$\left(\Delta_a^{-\nu} f\right)(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s),$$

for $t \in \mathbb{N}_{a+\nu}$. The ***ν-th fractional difference*** of f, for $\nu > 0$, by

$$\left(\Delta_a^{\nu} f\right)(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-s-1)^{\underline{-\nu-1}} f(s),$$

where $t \in \mathbb{N}_{a-\nu+N}$ and $N \in \mathbb{N}_1$ is the unique number satisfying $N-1 < \nu \leq N$.

Definition 2.3: The ***ν-th fractional Taylor monomial based at s*** is the map $(t,s) \mapsto h_\nu(t,s)$ defined by

$$h_\nu(t,s) := \frac{(t-s)^{\underline{\nu}}}{\Gamma(\nu+1)},$$

whenever the right-hand side is defined.

The next lemma will be useful in Section 3.

Lemma 2.4: *Assume that* $0 < \mu < 1$, $1 < \nu < 2$, *and* $2 < \mu + \nu < 3$. *Suppose that* $(\Delta_{1+a-\mu}^{\nu}\Delta_a^{\mu}f)(t) \geq \varepsilon f(a)$, *for each* $t \in \mathbb{N}_{3+a-\mu-\nu}$. *Then it holds that*

$$\begin{aligned}\Delta^2 f(a+k+1) \geq & -h_{-\mu-\nu}(a+3-\mu-\nu+k,a)f(a)\\ & - h_{-\mu-\nu+1}(a+3-\mu-\nu+k,a)\Delta f(a)\\ & - \sum_{s=a}^{a+k} h_{-\mu-\nu+1}(a+3-\mu-\nu+k,s+1)\Delta^2 f(s)\\ & + h_{-\nu-1}(a+2-\nu+k,a)f(a)\\ & + \varepsilon f(a),\end{aligned}$$

for each $k \in \mathbb{N}_0$.

Proof: This lemma was proved in [27, Lemma 2.12] in case $\varepsilon = 0$. In case $\varepsilon > 0$ a straightforward modification of that proof yields the statement of this theorem. Therefore, we eliminate the details of this argument. ■

We finally state the following result due to Holm [37]. The key fact expressed by this result is that the fractional difference operator is, in general, a non-commutative operator, a fact which is of considerable importance when considering discrete fractional sequential operators.

Theorem 2.5: *Let* $f : \mathbb{N}_a \to \mathbb{R}$ *be given and suppose* $\nu, \mu > 0$, *with* $N - 1 < \nu \leq N$ *and* $M - 1 < \mu \leq M$, *where* $M, N \in \mathbb{N}_1$. *Then for* $t \in \mathbb{N}_{a+M-\mu+N-\nu}$ *it holds that*

$$\begin{aligned}&\Delta_{a+M-\mu}^{\nu}\Delta_a^{\mu}f(t)\\ &\quad = \Delta_a^{\nu+\mu}f(t) - \sum_{j=0}^{M-1} h_{-\nu-M+j}(t-M+\mu,a)\Delta_a^{j-M+\mu}f(a+M-\mu),\end{aligned} \tag{3}$$

where $N - 1 < \nu < N$. *If* $\nu = N$, *then* (3) *simplifies to*

$$\Delta_{a+M-\mu}^{\nu}\Delta_a^{\mu}f(t) = \Delta_a^{\nu+\mu}f(t),$$

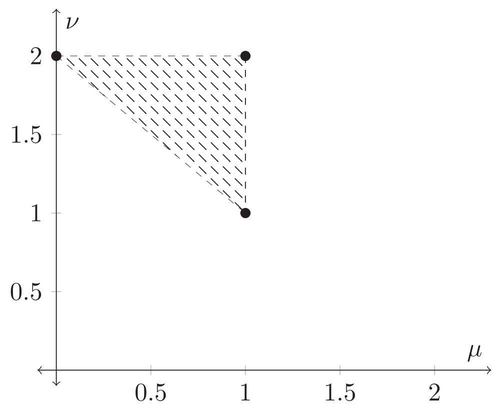
where $t \in \mathbb{N}_{a+M-\mu}$.

3. Analytical results

In this section we focus on what we can prove using hard analysis; in Section 4, by contrast, we will provide a numerical analysis of the problem. Throughout this section and the next we will denote by $\mathscr{M} \subseteq \mathbb{R}^2$ the following set.

$$\mathscr{M} := \left\{(\mu,\nu) \in \mathbb{R}^2 \;:\; 0 < \mu < 1,\; 1 < \nu < 2, \text{ and } 2 < \mu+\nu < 3\right\}$$

Geometrically this set is represented by the hatched region in the following drawing.



Thus, the set $\mathscr{M}$ is the admissible parameter space for the parameter pair (μ, ν).

Our first result provides a sufficient condition for a function $f : \mathbb{N}_a \to \mathbb{R}$ to be convex for at least one time step. Keep in mind that since $f(a) < 0$, it follows that

$$\varepsilon f(a) < 0,$$

recalling that $\varepsilon > 0$.

Theorem 3.1: *Let $(\mu, \nu) \in \mathscr{M}$ and assume that $f : \mathbb{N}_a \to \mathbb{R}$ satisfies each of the following:*

(1) $f(a) \leq 0$;
(2) $(\Delta f)(a) \geq 0$;
(3) $(\Delta^2 f)(a) \geq 0$;
(4) $(\Delta_{1+a-\mu}^{\nu} \Delta_a^{\mu} f)(3 - \mu - \nu + a) \geq \varepsilon f(a)$; *and*
(5) $\nu > -\frac{1}{2}(\mu - 3) + \dfrac{\sqrt{-3(\mu-3)(\mu-1)^2(\mu+1) + 24\varepsilon(3-3\mu)}}{2(3-3\mu)}$.

Then $(\Delta^2 f)(a+1) \geq 0$.

Proof: Similar to the beginning of the proof of [28, Theorem 2.5] and with the help of Lemma 2.4 we begin by writing

$$\begin{aligned}
\left(\Delta^2\right) f(a+1) \geq & -h_{-\mu-\nu}(a+3-\mu-\nu, a) f(a) \\
& - h_{-\mu-\nu+1}(a+3-\mu-\nu, a) \Delta f(a) \\
& \underbrace{- \sum_{s=a}^{a} h_{-\mu-\nu+1}(a+3-\mu-\nu, s+1) \Delta^2 f(s)}_{\geq 0 \text{ since } \left(\Delta^2 f\right)(a) \geq 0} \\
& + h_{-\nu-1}(a+2-\nu, a) f(a) \\
& + \varepsilon f(a)
\end{aligned}$$

$$
\begin{aligned}
&\geq -h_{-\mu-\nu}(a+3-\mu-\nu,a)f(a)\underbrace{-h_{-\mu-\nu+1}(a+3-\mu-\nu,a)\Delta f(a)}_{\geq 0}\\
&\quad + h_{-\nu-1}(a+2-\nu,a)f(a)+\varepsilon f(a)\\
&\geq \big[-h_{-\mu-\nu}(a+3-\mu-\nu,a)+h_{-\nu-1}(a+2-\nu,a)+\varepsilon\big]f(a), \qquad (4)
\end{aligned}
$$

where we have used both the fact that

$$
-h_{-\mu-\nu+1}(a+3-\mu-\nu,a+1)>0
$$

and that

$$
-h_{-\mu-\nu+1}(a+3-\mu-\nu,a)\Delta f(a)\geq 0.
$$

Now, recall that $f(a)<0$. Then note from (4) that

$$
\big(\Delta^2 f\big)(a+1)\geq 0
$$

provided that

$$
\begin{aligned}
&-\frac{(3-\mu-\nu)^{\underline{-\mu-\nu}}}{\Gamma(-\mu-\nu+1)}+\frac{(2-\nu)^{\underline{-\nu-1}}}{\Gamma(-\nu)}+\varepsilon\\
&=\frac{1}{6}(2-\nu)(1-\nu)(-\nu)-\frac{1}{6}(3-\mu-\nu)(2-\mu-\nu)(1-\mu-\nu)+\varepsilon\leq 0. \qquad (5)
\end{aligned}
$$

Note that inequality (5) is equivalent to

$$
(-3\mu+3)\nu^2+\big(-3\mu^2+12\mu-9\big)\nu+\big(-\mu^3+6\mu^2-11\mu+6-6\varepsilon\big)\geq 0. \qquad (6)
$$

The left-hand side of inequality (6) is equal to zero only if

$$
\nu=-\frac{1}{2}(\mu-3)\pm\frac{\sqrt{-3(\mu-3)(\mu-1)^2(\mu+1)+24\varepsilon(3-3\mu)}}{2(3-3\mu)}.
$$

Considering that $3-3\mu>0$ since $\mu\in(0,1)$, it follows that

$$
\begin{aligned}
&-\frac{1}{2}(\mu-3)+\frac{\sqrt{-3(\mu-3)(\mu-1)^2(\mu+1)+24\varepsilon(3-3\mu)}}{2(3-3\mu)}\\
&\quad>-\frac{1}{2}(\mu-3)-\frac{\sqrt{-3(\mu-3)(\mu-1)^2(\mu+1)+24\varepsilon(3-3\mu)}}{2(3-3\mu)}.
\end{aligned}
$$

In addition, we note that

$$-\frac{1}{2}(\mu-3)-\frac{\sqrt{-3(\mu-3)(\mu-1)^2(\mu+1)+24\varepsilon(3-3\mu)}}{2(3-3\mu)}<1,$$

for all $\varepsilon>0$ and $0<\mu<1$. Therefore, keeping in mind that we require that $1<\nu<2$, we conclude that inequality (6) holds for each $(\mu,\nu)\in\mathscr{M}$ such that

$$\nu>-\frac{1}{2}(\mu-3)+\frac{\sqrt{-3(\mu-3)(\mu-1)^2(\mu+1)+24\varepsilon(3-3\mu)}}{2(3-3\mu)}.$$

But since this latter inequality was assumed in (5) in the statement of the theorem, it follows that

$$\left(\Delta^2 f\right)(a+1)\geq 0,$$

as claimed. ■

In what follows it will be convenient to introduce some notation. Therefore, for each $k\in\mathbb{N}_3$ and $\varepsilon\geq 0$ define the set $\mathscr{F}_{k,\varepsilon}\subseteq\mathscr{M}$ by

$$\mathscr{F}_{k,\varepsilon}:=\left\{(\mu,\nu)\in\mathscr{M}\ :\ \frac{1}{k!}\prod_{j=1}^{k}(j-1-\nu)-\frac{1}{k!}\prod_{j=1}^{k}(j-\mu-\nu)\leq-\varepsilon\right\}.$$

We next state and prove a lemma, which shows that on a proper subset of the admissible parameter space the collection $\{\mathscr{F}_{k,\varepsilon}\}_{k=3}^{\infty}$ forms a decreasing collection of sets.

Lemma 3.2: *Let $(\mu,\nu)\in\mathscr{M}$. Also assume that*

$$\nu>-\frac{1}{2}(\mu-3)+\frac{\sqrt{-3(\mu-3)(\mu-1)^2(\mu+1)+24\varepsilon(3-3\mu)}}{2(3-3\mu)}. \tag{7}$$

Then

$$\mathscr{F}_{k,\varepsilon}\supseteq\mathscr{F}_{k+1,\varepsilon} \tag{8}$$

for all $k\geq 3$. Additionally,

$$\bigcap_{k=3}^{\infty}\mathscr{F}_{k,\varepsilon}=\lim_{k\to\infty}\mathscr{F}_{k,\varepsilon}=\varnothing. \tag{9}$$

Proof: Note that the definition of $\mathscr{F}_{k,\varepsilon}$ can be rewritten as

$$\mathscr{F}_{k,\varepsilon}=\left\{(\mu,\nu)\in\mathscr{M}\ :\ \frac{1}{k!}\prod_{j=1}^{k}(j-\mu-\nu)-\frac{1}{k!}\prod_{j=1}^{k}(j-1-\nu)\geq\varepsilon\right\}. \tag{10}$$

So, let $(\mu_0,\nu_0)\in\mathscr{M}$ and $\varepsilon_0>0$ be arbitrary but fixed while satisfying (7). We will establish the desired inclusion (8) by contradiction. In particular, we will show that for all k the conjunction

$$(\mu_0, \nu_0) \notin \mathscr{F}_{k,\varepsilon_0} \wedge (\mu_0, \nu_0) \in \mathscr{F}_{k+1,\varepsilon_0} \tag{11}$$

is false. This conjunction can be rewritten as

$$\begin{aligned} &\frac{1}{k!}\prod_{j=1}^{k}(j-\mu_0-\nu_0) - \frac{1}{k!}\prod_{j=1}^{k}(j-1-\nu_0) \\ &\quad < \varepsilon_0 \leq \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-\mu_0-\nu_0) - \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-1-\nu_0), \end{aligned} \tag{12}$$

based on (10). To help us disprove (12) and thus (11) we define the sets $\mathscr{W}_1$ and $\mathscr{W}_2$ by

$$\begin{aligned} \mathscr{W}_1 := \Bigg\{ k \in \mathbb{N}_3 \ : \ &\frac{1}{k!}\prod_{j=1}^{k}(j-\mu_0-\nu_0) - \frac{1}{k!}\prod_{j=1}^{k}(j-1-\nu_0) \\ &\geq \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-\mu_0-\nu_0) - \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-1-\nu_0) \Bigg\} \end{aligned}$$

and

$$\begin{aligned} \mathscr{W}_2 := \Bigg\{ k \in \mathbb{N}_3 \ : \ &\frac{1}{k!}\prod_{j=1}^{k}(j-\mu_0-\nu_0) - \frac{1}{k!}\prod_{j=1}^{k}(j-1-\nu_0) \\ &< \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-\mu_0-\nu_0) - \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-1-\nu_0) \Bigg\}. \end{aligned}$$

Note that for all $k \in \mathbb{N}_3$, either $k \in \mathscr{W}_1$ or $k \in \mathscr{W}_2$. We will now state and prove two properties of $\mathscr{W}_1$ and $\mathscr{W}_2$.

(1) If $k_0 \in \mathscr{W}_1$, then for each $k \in \mathbb{N}_{k_0}$ it follows that $k \in \mathscr{W}_1$.

(2) If $k_0 \in \mathscr{W}_2$, then $(\mu_0, \nu_0) \in \displaystyle\bigcap_{k=3}^{k_0} \mathscr{F}_{k,\varepsilon_0}$.

We first prove property (1). To start we will rewrite the inequality in the definition of $\mathscr{W}_1$. So, notice that

$$\begin{aligned} &\frac{1}{k!}\prod_{j=1}^{k}(j-\mu_0-\nu_0) - \frac{1}{k!}\prod_{j=1}^{k}(j-1-\nu_0) \\ &\quad \geq \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-\mu_0-\nu_0) - \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-1-\nu_0) \end{aligned}$$

holds if and only if

$$(k+1)\prod_{j=1}^{k}(j-\mu_0-\nu_0)-(k+1)\prod_{j=1}^{k}(j-1-\nu_0)$$
$$\geq \prod_{j=1}^{k+1}(j-\mu_0-\nu_0)-\prod_{j=1}^{k+1}(j-1-\nu_0)$$

if and only if

$$(k+1)\prod_{j=1}^{k}(j-\mu_0-\nu_0)-\prod_{j=1}^{k+1}(j-\mu_0-\nu_0)$$
$$\geq (k+1)\prod_{j=1}^{k}(j-1-\nu_0)-\prod_{j=1}^{k+1}(j-1-\nu_0)$$

if and only if

$$\left(\prod_{j=1}^{k}(j-\mu_0-\nu_0)\right)\big((k+1)-(k+1-\mu_0-\nu_0)\big)$$
$$\geq \left(\prod_{j=1}^{k}(j-1-\nu_0)\right)\big((k+1)-(k+1-1-\nu_0)\big)$$

if and only if

$$-\prod_{j=0}^{k}(j-\mu_0-\nu_0)\geq -\prod_{j=0}^{k}(j-1-\nu_0).$$

Thus, we can redefine $\mathscr{W}_1$ as

$$\mathscr{W}_1=\left\{k\in\mathbb{N}_3\ :\ -\prod_{j=0}^{k}(j-\mu_0-\nu_0)\geq -\prod_{j=0}^{k}(j-1-\nu_0)\right\}. \tag{13}$$

Also note that

$$-\prod_{j=0}^{k}(j-\mu_0-\nu_0)>0$$

and

$$-\prod_{j=0}^{k}(j-1-\nu_0)>0.$$

We will now use an induction argument to prove property (1). By the statement of case (1) there exists some $k_0\in\mathbb{N}_3$ such that $k_0\in\mathscr{W}_1$. Thus, the base case is trivially established.

Now we assume that for some $k \geq k_0$, $k \in \mathscr{W}_1$, and we then show that $k+1 \in \mathscr{W}_1$. First notice that

$$k + 1 - \mu_0 - \nu_0 > k - \nu_0 > 0, \tag{14}$$

which follows from the fact that $\mu_0 + \nu_0 < 3$ and $k \geq 3$. Now, since we assumed that

$$0 < -\prod_{j=0}^{k}(j - 1 - \nu_0) \leq -\prod_{j=0}^{k}(j - \mu_0 - \nu_0) \tag{15}$$

in our induction hypothesis, we may use (14) to rewrite inequality (15) as

$$-(k + 1 - \mu_0 - \nu_0)\prod_{j=0}^{k}(j - \mu_0 - \nu_0) > -(k - \nu_0)\prod_{j=0}^{k}(j - 1 - \nu_0). \tag{16}$$

But (16) is exactly

$$-\prod_{j=0}^{k+1}(j - \mu_0 - \nu_0) > -\prod_{j=0}^{k+1}(j - 1 - \nu_0),$$

which implies, together with (13), that $k + 1 \in \mathscr{W}_1$, as desired. Thus we have proven property (1) by induction.

Next we prove property (2). So, assume that $k_0 \in \mathscr{W}_2$. As a preliminary observation we will show that $(\mu_0, \nu_0) \in \mathscr{F}_{3,\varepsilon}$. Consider the fact that (7) is just (6) for $\mu = \mu_0$, $\nu = \nu_0$, and $\varepsilon = \varepsilon_0$. Due to the reversibility of the steps, this is also (5) for $\mu = \mu_0$, $\nu = \nu_0$, and $\varepsilon = \varepsilon_0$. But (5) clearly implies that $(\mu_0, \nu_0) \in \mathscr{F}_{3,\varepsilon}$, as desired. Now we will prove the desired implication – namely, that since $k_0 \in \mathscr{W}_2$ it follows that $(\mu_0, \nu_0) \in \bigcap_{k=3}^{k_0} \mathscr{F}_{k,\varepsilon_0}$. To see that this must be true, suppose that for some $\widehat{k} < k_0$ it held that $\widehat{k} \in \mathscr{W}_1$. But then property (1), which we have already proven, would imply that $k_0 \in \mathscr{W}_1$, a contradiction. Consequently, recalling that for all $k \in \mathbb{N}_3$ either $k \in \mathscr{W}_1$ or $k \in \mathscr{W}_2$, we get that $\mathbb{N}_3^{k_0} \subseteq \mathscr{W}_2$, where $\mathbb{N}_3^{k_0} := \{3, 4, 5, \ldots\ k_0 - 1, k_0\}$. Therefore,

$$\begin{aligned}\varepsilon_0 &\leq \frac{1}{3!}\prod_{j=1}^{3}(j - \mu_0 - \nu_0) - \frac{1}{3!}\prod_{j=1}^{3}(j - 1 - \nu_0) \\ &< \ldots < \frac{1}{k_0!}\prod_{j=1}^{k_0}(j - \mu_0 - \nu_0) - \frac{1}{k_0!}\prod_{j=1}^{k_0}(j - 1 - \nu_0)\end{aligned}$$

Thus, for all $k \in \mathbb{N}_3^{k_0}$,

$$\frac{1}{k!}\prod_{j=1}^{k}(j - \mu_0 - \nu_0) - \frac{1}{k!}\prod_{j=1}^{k}(j - 1 - \nu_0) \geq \varepsilon_0,$$

implying

$$(\mu_0, \nu_0) \in \bigcap_{k=3}^{k_0} \mathscr{F}_{k,\varepsilon_0},$$

as desired. Thus, we conclude that property (2) is true.

Finally, based on the two properties we have just proven we will show that for each $k \in \mathbb{N}_3$ and each $\varepsilon_0 > 0$ inequality (12) cannot be true, which will establish the desired claim – that is, we show that it cannot occur that $(\mu_0, \nu_0) \in \mathscr{F}_{k_0+1,\varepsilon_0}$ and yet $(\mu_0, \nu_0) \notin \mathscr{F}_{k_0,\varepsilon_0}$. Solely based upon properties (1)–(2), three different cases exist:

(A) $\mathscr{W}_1 = \mathbb{N}_3$ and $\mathscr{W}_2 = \varnothing$;
(B) $\mathscr{W}_1 = \varnothing$ and $\mathscr{W}_2 = \mathbb{N}_3$; and
(C) For some $\overline{k} \in \mathbb{N}_3$, $\mathscr{W}_1 = \mathbb{N}_{\overline{k}+1}$ and $\mathscr{W}_2 = \mathbb{N}_3^{\overline{k}}$.

We consider these cases in turn. Note that the case $\mathscr{W}_1 = \mathbb{N}_3^{\overline{k}}$ and $\mathscr{W}_2 = \mathbb{N}_{\overline{k}+1}$, for some $\overline{k} \in \mathbb{N}_3$, is excluded because of property (1). Indeed, by property (1) if $\mathscr{W}_1 = \mathbb{N}_3^{\overline{k}}$, then $\mathscr{W}_1 = \mathbb{N}_3$ in contradiction of the claim that $\mathscr{W}_2 \neq \varnothing$.

We will first consider case (A). Note that (12) implies

$$\begin{aligned} &\frac{1}{k!}\prod_{j=1}^{k}(j-\mu_0-\nu_0) - \frac{1}{k!}\prod_{j=1}^{k}(j-1-\nu_0) \\ &\quad < \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-\mu_0-\nu_0) - \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-1-\nu_0). \end{aligned} \tag{17}$$

But this is clearly impossible if for all $k \in \mathbb{N}_3$, $k \in \mathscr{W}_1$, based on the definition of $\mathscr{W}_1$. So, (A) cannot be true if (12) is true.

Next, we consider case (B). Note that (12) requires that

$$\frac{1}{k!}\prod_{j=1}^{k}(j-\mu_0-\nu_0) - \frac{1}{k!}\prod_{j=1}^{k}(j-1-\nu_0) < \varepsilon_0. \tag{18}$$

That is, $(\mu_0, \nu_0) \notin \mathscr{F}_{k,\varepsilon}$. But based on the definition of case (B) as well as property (2) of $\mathscr{W}_1$ and $\mathscr{W}_2$, this is impossible as well – that is, in case (B) by property (2) we have that $(\mu_0, \nu_0) \in \bigcap_{j=3}^{k_0} \mathscr{F}_{j,\varepsilon_0}$ for each $k_0 \in \mathbb{N}_3$ so that, in particular, $(\mu_0, \nu_0) \in \mathscr{F}_{k,\varepsilon_0}$. So, (B) also cannot be true if (12) is true.

Lastly, we consider case (C). First we will prove that (12) cannot hold for $k \in \mathbb{N}_3^{\overline{k}}$. This is so because of (18) being incompatible with property (2) of $\mathscr{W}_1$ and $\mathscr{W}_2$ – exactly as in the previous paragraph. On the other hand, (12) cannot hold for $k \in \mathbb{N}_{\overline{k}+1}$. And this is so because (17) is incompatible with property (1) of $\mathscr{W}_1$ and $\mathscr{W}_2$. So, we conclude that (C) cannot be true if (12) is true.

In summary, none of (A), (B), and (C) can be true if (12) is true. Since exactly one of these must be true, we arrive at the desired contradiction. Thus, (12) and therefore (11) is disproven. Since (μ_0, ν_0) and ε_0 were arbitrary, we may conclude that for all $(\mu, \nu) \in \mathscr{M}$ and $\varepsilon \geq 0$ satisfying the condition stated in the lemma,

$$(\mu, \nu) \notin \mathscr{F}_{k,\varepsilon} \wedge (\mu, \nu) \in \mathscr{F}_{k+1,\varepsilon}$$

is impossible. Thus, (8) is proven true, as desired.

At the same time, to prove (9) define the sequences $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ by

$$A_k := \frac{1}{k!} \prod_{j=1}^{k} (j - 1 - \nu)$$

and

$$B_k := \frac{1}{k!} \prod_{j=1}^{k} (j - \mu - \nu).$$

That $A_k, B_k \to 0$ as $k \to \infty$ was actually proved in [34, Lemma 3.2], and the proof in this case is essentially identical, but we, nonetheless, mention the brief argument here for the sake of completeness. So, recalling that (see, for example, Carlson [14, Theorem 3.4-1], Kilbas, et al. [46, (1.5.15)], or Wong and Beals [50, Proposition 2.1.3])

$$\lim_{k \to \infty} \frac{\Gamma(k - \nu)}{\Gamma(k) k^{-\nu}} = 1$$

it follows that

$$\begin{aligned}
\lim_{k \to \infty} A_k = \lim_{k \to \infty} \frac{1}{k!} \prod_{j=1}^{k} (j - 1 - \nu) &= \lim_{k \to \infty} \left[\frac{\Gamma(k - \nu)}{\Gamma(-\nu)\Gamma(k + 1)} \right] \\
&= \lim_{k \to \infty} \left[\underbrace{\frac{\Gamma(k - \nu)}{\Gamma(k + 1) k^{-\nu-1}}}_{\longrightarrow 1} \cdot \frac{k^{-\nu-1}}{\Gamma(-\nu - 1)} \right] \\
&= \lim_{k \to \infty} \frac{k^{-\nu-1}}{\Gamma(-\nu)} \\
&= 0.
\end{aligned}$$

In a completely similar manner we deduce that

$$\lim_{k \to \infty} B_k = 0.$$

Therefore,

$$\lim_{k \to \infty} \frac{1}{k!} \left[\prod_{j=1}^{k} (j - \mu - \nu) - \prod_{j=1}^{k} (j - 1 - \nu) \right] = \lim_{k \to \infty} \left(B_k - A_k \right) = 0.$$

Consequently, since $\varepsilon > 0$, it follows that

$$k_0 := \sup \left\{ k \in \mathbb{N} \ : \ B_k - A_k \geq \varepsilon \right\} < +\infty,$$

which implies that $\mathscr{F}_{k,\varepsilon} = \varnothing$ for all $k \geq k_0$. And this implies that $\bigcap_{k=3}^{\infty} \mathscr{F}_{k,\varepsilon} = \varnothing$, as claimed. ■

Remark 3.3: Note that, in fact, (9) proves that case (B) in Lemma 3.2 is impossible, as otherwise (μ_0, ν_0) would be a counterexample sufficient to show that $\bigcap_{k=3}^{\infty} \mathscr{F}_{k,\varepsilon} \neq \varnothing$.

Remark 3.4: The first claim of Lemma 3.2 may fail if (7) is not satisfied. For example, let $\mu := 0.9$, $\nu := 1.6$, and $\varepsilon := 0.01$. Note that this doesn't satisfy (7), as

$$-\frac{1}{2}(0.9-3) + \frac{\sqrt{-3(0.9-3)(0.9-1)^2(0.9+1) + 24(0.01)(3-3(0.9))}}{2(3-3(0.9))} > 1.77 > \nu.$$

Then $(1-0.9-1.6)(2-0.9-1.6)(3-0.9-1.6) - (-1.6)(-0.6)(0.4) = -0.009 < \varepsilon$, so $(0.9, 1.6) \notin \mathscr{F}_{3,\varepsilon}$, while $(1-0.9-1.6)(2-0.9-1.6)(3-0.9-1.6)(4-0.9-1.6) - (-1.6)(-0.6)(0.4)(1.4) = 0.0249 > \varepsilon$, so $(0.9, 1.6) \in \mathscr{F}_{4,\varepsilon}$. Thus, in this case $\mathscr{F}_{3,\varepsilon} \not\supset \mathscr{F}_{4,\varepsilon}$.

We now present a corollary of Theorem 3.1 and Lemma 3.2.

Corollary 3.5: *Let* $(\mu, \nu) \in \mathscr{M}$ *and* $\varepsilon \geq 0$ *satisfy condition* (7). *Also assume that each of the following is true for f and some* $k_0 \in \mathbb{N}_3$*:*

(1) $f(a) \leq 0$*;*
(2) $(\Delta f)(a) \geq 0$*;*
(3) $(\Delta^2 f)(a) \geq 0$*; and*
(4) $(\Delta^{\nu}_{1+a-\mu} \Delta^{\mu}_{a} f)(t) \geq \varepsilon f(a)$ *for all* $t \in \mathbb{N}^{k_0-\mu-\nu+a}_{3-\mu-\nu+a}$.

If $(\mu, \nu) \in \mathscr{F}_{k_0,\varepsilon}$*, then* $(\Delta^2 f)(t) \geq 0$ *for each* $t \in \mathbb{N}_a^{a+k_0-1}$.

Proof: From the proof of Theorem 3.1 we know that

$$(\Delta^2 f)(a + k_0 - 1) \geq [A_k - B_k + \varepsilon] f(a) \geq 0 \tag{19}$$

holds for all $2 \leq k \leq k_0$ if and only if

$$(\mu, \nu) \in \bigcap_{j=3}^{k_0} \mathscr{F}_{j,\varepsilon}. \tag{20}$$

But by Lemma 3.2,

$$\bigcap_{j=3}^{k_0} \mathscr{F}_{j,\varepsilon} = \mathscr{F}_{k_0,\varepsilon}.$$

We have assumed $(\mu, \nu) \in \mathscr{F}_{k_0,\varepsilon}$ to be true. Therefore, both (20) and (19) are true. (19) and the proof of Theorem 3.1 imply that $(\Delta^2 f)(t) \geq 0$ for each $t \in \mathbb{N}_a^{a+k_0-1}$, and so, the proof is complete. ■

The following example demonstrates that the set of functions

$$\mathscr{D} := \Big\{ f : \mathbb{N}_0 \to \mathbb{R} : f(0) \le 0, (\Delta f)(0) \ge 0, (\Delta^2 f)(0) \ge 0,$$
$$(\exists(\mu,\nu) \in \mathscr{M}) \wedge (\exists \varepsilon \ge 0) \ni \left(\Delta^{\nu}_{1+a-\mu} \Delta^{\mu}_{a} f\right)(3-\mu-\nu+a) \ge \varepsilon f(a)$$
$$\wedge\, \nu > -\frac{1}{2}(\mu-3) + \frac{\sqrt{-3(\mu-3)(\mu-1)^2(\mu+1) + 24\varepsilon(3-3\mu)}}{2(3-3\mu)} \Big\}$$

is non-empty.

Example 3.6: Select $\mu := 0.5$, $\nu := 1.99$, and $\varepsilon := \dfrac{1}{20}$. Also, let $f : \mathbb{N}_0 \to \mathbb{R}$ be a function that satisfies $f(0) = -60$, $f(1) = -58$, $f(2) = -55$, and $f(3) = -50$. This clearly satisfies condition (1) of Theorem 3.1. Also note that $(\Delta f)(0) = 2 \ge 0$ and $(\Delta^2 f)(0) = 1 \ge 0$, so conditions (2) and (3) are satisfied as well. To prove that condition (5) holds, calculate that

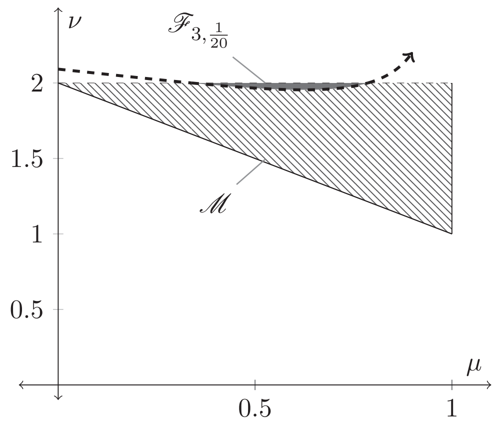
$$-\frac{1}{2}(\mu-3) + \frac{\sqrt{-3(\mu-3)(\mu-1)^2(\mu+1) + 24\varepsilon(3-3\mu)}}{2(3-3\mu)} = 1.96589 < \nu.$$

Lastly, consider condition (4). Note that

$$\begin{aligned}
\left(\Delta^{\nu}_{1+a-\mu} \Delta^{\mu}_{a} f\right)(3-\mu-\nu+a) &= \left(\Delta^{1.99}_{0.5} \Delta^{0.5}_{0} f\right)(0.51) \\
&= (\Delta^{2.49}_{0} f)(0.51) - h_{-2.99}(0.01, 0)\left(\Delta^{-0.5}_{0} f\right)(0.5). \\
&= -\frac{(0.49)(1.49)(2.49) + (0.01)(0.99)(1.99)}{6} f(0) \\
&\quad + \frac{(1.49)(2.49)}{2} f(1) - 2.49 f(2) + f(3) \\
&= -2.2664 \\
&\ge -3 \\
&= \varepsilon f(0).
\end{aligned}$$

Thus condition (4) is satisfied. Since we have proven that all of the conditions of Theorem 3.1 are satisfied, Theorem 3.1 implies that $(\Delta^2 f)(1) \ge 0$. And this is indeed true, as $(\Delta^2 f)(1) = 2$.

Remark 3.7: It turns out that in Example 3.6, ε is close to the supremum value for which $\mathscr{F}_{3,\varepsilon}$ can be non-empty. We provide a graph to show the size of $\mathscr{F}_{3,\frac{1}{20}}$. We will explore this idea further in Lemma 3.9 and Remark 3.10.



Our next theorem demonstrates that, a less restrictive version of condition (4) in Theorem 3.1 exists that still guarantees the validity of the convexity result.

Theorem 3.8: *Let* $(\mu, \nu) \in \mathscr{M}$ *and assume that* $f : \mathbb{N}_a \to \mathbb{R}$ *satisfies each of the following:*

(1) $f(a) < 0$;
(2) $(\Delta f)(a) \geq 0$;
(3) $(\Delta^2 f)(a) \geq 0$;
(4) $(\Delta_{1+a-\mu}^{\nu} \Delta_a^{\mu} f)(3 - \mu - \nu + a) \geq -\varepsilon$ *for some* $\varepsilon \geq 0$; *and*
(5) $\nu > -\frac{1}{2}(\mu - 3) + \frac{\sqrt{-3(\mu-3)(\mu-1)^2(\mu+1)-24\left(\frac{\varepsilon}{f(a)}\right)(3-3\mu)}}{2(3-3\mu)}$.

Then $(\Delta^2 f)(a + 1) \geq 0$.

Proof: From condition (4), we can obtain an equation analogous to (5). First note that

$$\begin{aligned}
&(\Delta^2 f)(a+1) \\
&\quad \geq \big[- h_{-\mu-\nu}(a + 3 - \mu - \nu, a) + h_{-\nu-1}(a + 2 - \nu, a)\big] f(a) - \varepsilon \\
&\quad = \left[\frac{1}{6}(2 - \nu)(1 - \nu)(-\nu) - \frac{1}{6}(3 - \mu - \nu)(2 - \mu - \nu)(1 - \mu - \nu) \right] f(a) - \varepsilon. \quad (21)
\end{aligned}$$

We will show that the right-hand side of (21) is non-negative. Keeping in mind that by condition (1) it holds that $f(a) \neq 0$, note that (21) is true if and only if

$$\frac{1}{6}(2 - \nu)(1 - \nu)(-\nu) - \frac{1}{6}(3 - \mu - \nu)(2 - \mu - \nu)(1 - \mu - \nu) - \frac{\varepsilon}{f(a)} \leq 0. \quad (22)$$

Now make the substitution $\overline{\varepsilon} := -\frac{\varepsilon}{f(a)}$. Then we may use the exact same procedure as in the proof of Theorem 3.1 to rewrite (22) as

$$\nu > -\frac{1}{2}(\mu - 3) + \frac{\sqrt{-3(\mu - 3)(\mu - 1)^2(\mu + 1) + 24\overline{\varepsilon}(3 - 3\mu)}}{2(3 - 3\mu)}. \quad (23)$$

But based on the definition of $\overline{\varepsilon}$, inequality (23) is exactly the same as condition (5) in the statement of the theorem. Therefore,

$$\begin{aligned}&(\Delta^2 f)(a+1)\\ &\geq \left[\frac{1}{6}(2-\nu)(1-\nu)(-\nu)-\frac{1}{6}(3-\mu-\nu)(2-\mu-\nu)(1-\mu-\nu)\right]f(a)-\varepsilon \geq 0.\end{aligned}$$

This completes the proof of the theorem. ∎

Our final analytical result, Lemma 3.9, gives a necessary condition for $\mathscr{F}_{3,\varepsilon}$ to be non-empty.

Lemma 3.9: *Suppose that $\mathscr{F}_{3,\varepsilon}$ is non-empty. Then $\varepsilon < \frac{\sqrt{3}}{27}$.*

Proof: We will show that $\mathscr{F}_{3,\varepsilon}$ can only ever be non-empty if $\varepsilon < \frac{\sqrt{3}}{27}$. From (7) it follows that

$$\nu > -\frac{1}{2}(\mu-3)+\frac{\sqrt{-3(\mu-3)(\mu-1)^2(\mu+1)+24\varepsilon(3-3\mu)}}{2(3-3\mu)}.$$

Therefore, since $1 < \nu < 2$, it follows that

$$2 > -\frac{1}{2}(\mu-3)+\frac{\sqrt{-3(\mu-3)(\mu-1)^2(\mu+1)+24\varepsilon(3-3\mu)}}{2(3-3\mu)}. \tag{24}$$

We will now manipulate this inequality. So, using that $3-3\mu > 0$, we note that (24) is true if and only if

$$4(3-3\mu) > (3-\mu)(3-3\mu)+\sqrt{-3(\mu-3)(\mu-1)^2(\mu+1)+24\varepsilon(3-3\mu)}$$

if and only if

$$3(1-\mu)(1+\mu) > \sqrt{-3(\mu-3)(\mu-1)^2(\mu+1)+24\varepsilon(3-3\mu)}$$

if and only if

$$9(1-\mu)(1+\mu)^2 > 3(3-\mu)(1-\mu)(1+\mu)+72\varepsilon.$$

So, we conclude that (24) is true if and only if

$$\varepsilon < -\frac{1}{6}\mu\left(\mu^2-1\right).$$

Finally, noting that

$$\sup_{\mu\in(0,1)}\left(-\frac{1}{6}\mu\left(\mu^2-1\right)\right)=\frac{\sqrt{3}}{27},$$

we deduce that $\varepsilon < \frac{\sqrt{3}}{27}$. ∎

Remark 3.10: Since we determined in Lemma 3.2 that for $k \in \mathbb{N}_4$ it holds that $\mathscr{F}_{k,\varepsilon} \subseteq \mathscr{F}_{3,\varepsilon}$, it follows from Lemma 3.9 that if $\varepsilon \geq \frac{\sqrt{3}}{27}$, then $\bigcup_{k=3}^{\infty} \mathscr{F}_{k,\varepsilon} = \varnothing$. And this demonstrates the sufficiency of the condition.

Figure 1. Graphical representation of the set $\mathscr{F}_{k,0.01}$ for $k \leq 97$.

Figure 2. Graphical representation of the set $\mathscr{F}_{k,0.001}$ for $k \leq 97$.

4. Numerical simulations

In this section we investigate numerically the properties of the $\mathscr{F}_{k,\varepsilon}$ sets, which were studied analytically in Section 3. These numerical investigations will provide some additional insight into their properties. Recall from Section 3 that the $\mathscr{F}_{k,\varepsilon}$ sets are the key to understanding the extent of the (μ, ν) parameter space to which the convexity results may apply. Each of the drawings presented in this section was produced with the assistance of Matlab.

We consider first Figures 1–5. These five figures show the set $\mathscr{F}_{k,\varepsilon}$ for $\varepsilon = 10^{-m}$ where $m \in \mathbb{N}_2^6$ – in each case for each $k \in \{3, 5, 7, \ldots, 95, 97\}$; note that the set $\mathscr{F}_{k,\varepsilon}$ is shown in black, whereas the entire admissible parameter space is shown in light grey. Furthermore,

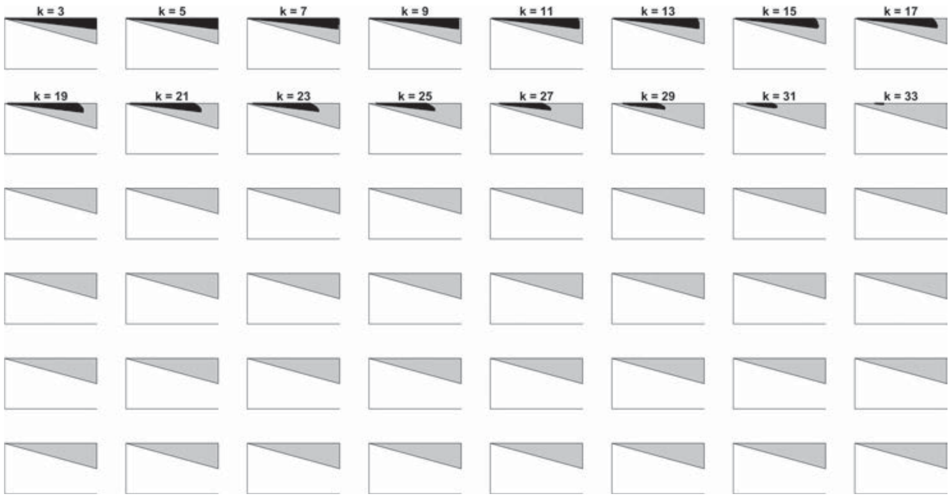


Figure 3. Graphical representation of the set $\mathscr{F}_{k,0.0001}$ for $k \leq 97$.

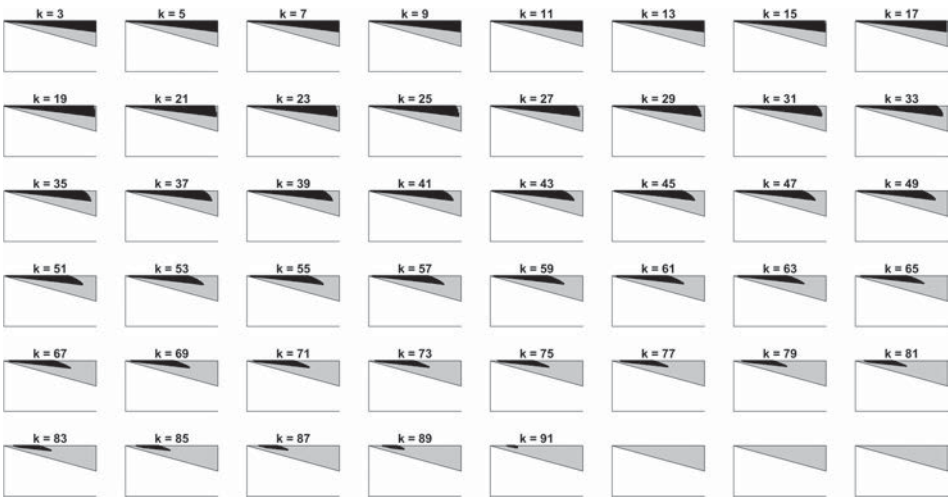


Figure 4. Graphical representation of the set $\mathscr{F}_{k,0.00001}$ for $k \leq 97$.

the k value associated to the plot is labelled only if $\mathscr{F}_{k,\varepsilon} \neq \varnothing$. So, for example, in Figure 1 we note that $\mathscr{F}_{k,0.01} = \varnothing$ for $k > 5$, depicted in Figure 1 as plots with no black region.

Notice that as k increases the measure of the $\mathscr{F}_{k,\varepsilon}$ set decreases. However, we also see that for smaller ε, the measure of the set stays larger (in measure) for more values of k. This is congruent with the analytical observations made in Section 3. In addition, we see that the set $\mathscr{F}_{k,\varepsilon}$ tends to 'shrink away' from the boundary $\mu = 1$ as k increases. In other words, as k increases we see that within the admissible parameter space the set $\mathscr{F}_{k,\varepsilon}$ becomes ever more confined to the upper-left region of the admissible parameter space – see especially Figures 4 and 5, in which this phenomenon is most clearly observable. Interestingly, this is not something which is not so easily discernible from the analytical results of Section 3.

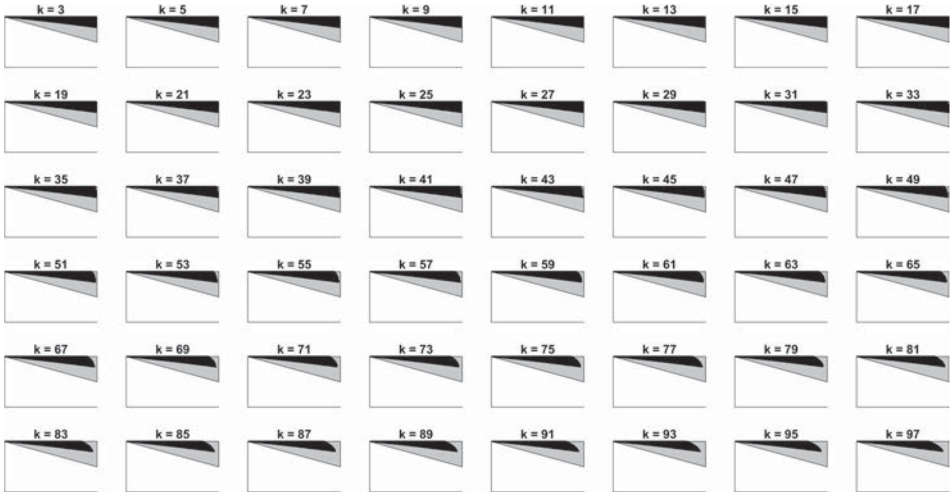


Figure 5. Graphical representation of the set $\mathscr{F}_{k,0.000001}$ for $k \leq 97$.

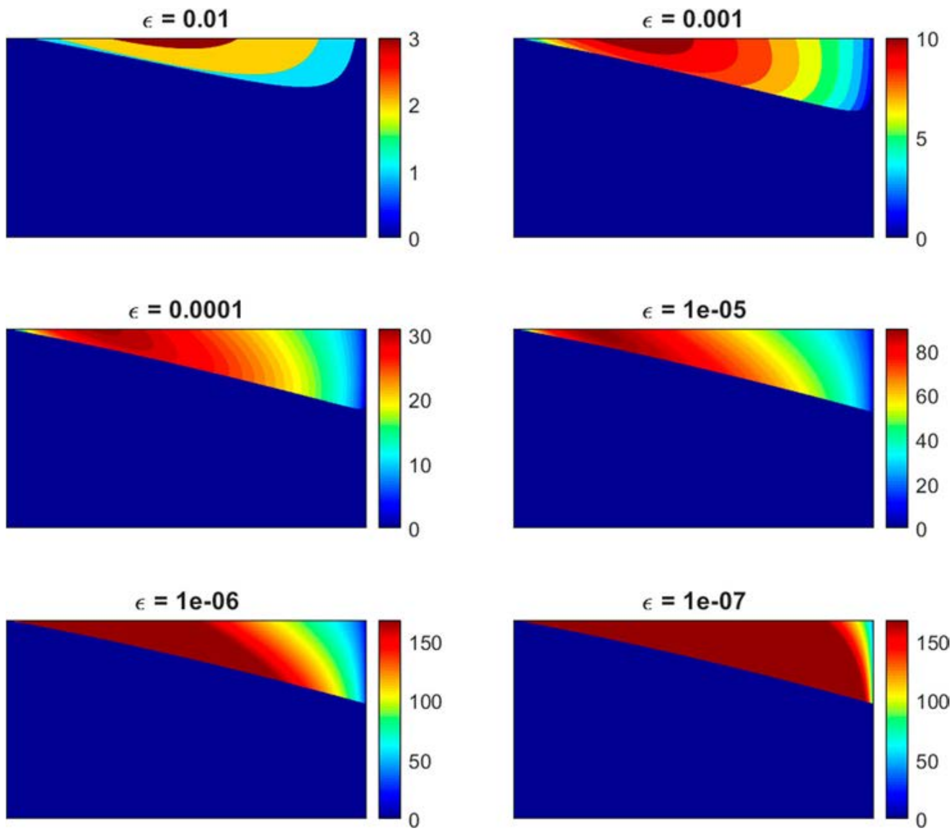


Figure 6. Heat maps generated by the cardinality of the set $\{k : (\mu, \nu) \in \mathscr{F}_{k,\varepsilon}\}$ for $\varepsilon = 0.01$, 0.001, 0.0001, 0.00001, 0.000001, 0.0000001. The cardinality increases from small (lighter shades) to large (darker shades) and the actual cardinalities are shown along the sidebar of each subplot.

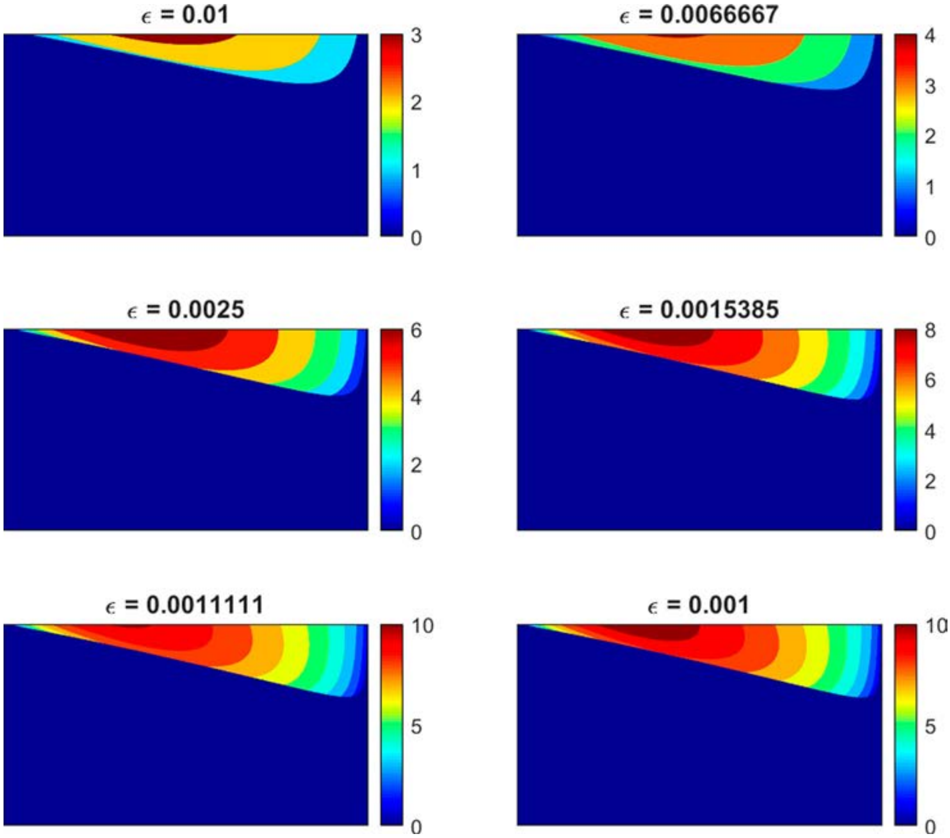


Figure 7. Heat maps generated by the cardinality of the set $\{k : (\mu, \nu) \in \mathscr{F}_{k,\varepsilon}\}$ for $\varepsilon = 1/100$, 1/150, 1/400, 1/650, 1/900, 1/1000. These six plots correspond to the interval of ε reflected in the top two plots in Figure 6.

On the other hand, Figures 6 and 7 provide heat maps, which are generated by the cardinality of the set $\{k : (\mu, \nu) \in \mathscr{F}_{k,\varepsilon}\}$ for a given $\varepsilon \leq \dfrac{\sqrt{3}}{27}$. In other words, in Figures 6 and 7 we are visualizing the number of time steps to which the convexity-type results can, in theory, apply. The warmer the colour the more time steps for which a given point remains in the set $\mathscr{F}_{k,\varepsilon}$, whereas the cooler the colour the fewer time steps for which a given point remains in the set $\mathscr{F}_{k,\varepsilon}$.

From these figures we immediately see that, just as we would expect from Figures 1–5 the warmer colours tend to be concentrated in, roughly speaking, the upper left part of the admissible parameter space, and, correspondingly, the cooler colours tend to be concentrated near the right-boundary $\mu = 1$. Figure 7 particularly illustrates this change since the six plots in this figure are given over the smaller range of ε-values from $\varepsilon = 0.01$ to $\varepsilon = 0.001$, whereas the six plots in Figure 6 range over the, relatively speaking, much larger ε-value range from $\varepsilon = 0.01$ to $\varepsilon = 0.0000001$.

Notice, furthermore, that when ε is, relatively speaking, small (e.g. the two plots in the upper row of Figure 7), few points are in $\mathscr{F}_{k,\varepsilon}$ for more than two or three values of k. This means that the convexity results of Section 3 are only, in theory, applicable for a very limited number of time steps when ε is close to its maximal value of $\frac{\sqrt{3}}{27}$. As ε approaches zero from above, however, more and more points remain in $\mathscr{F}_{k,\varepsilon}$ for a greater number of time steps. And this means that in such cases the analytical results of Section 3 are, in principle, applicable for a much greater number of time steps.

All in all, these observations confirm and enlighten the analytical observations made in Section 3. In particular, our observations seem to be consistent with the following assertions.

- For each fixed ε it holds that

$$\lim_{k\to\infty} \left|\mathscr{F}_{k,\varepsilon}\right| = 0.$$

  That is, for each $\varepsilon > 0$ fixed as k increases the measure of the set $\mathscr{F}_{k,\varepsilon}$ shrinks to zero.
- For each fixed $k \geq 3$ it holds that

$$\lim_{\varepsilon\to 0^+} \left|\mathscr{F}_{k,\varepsilon}\right| = \left|\mathscr{M}\right|.$$

  That is, for each $k \geq 3$ fixed as ε tends to 0 from above it follows that the set $\mathscr{F}_{k.\varepsilon}$ approaches the set $\mathscr{M}$ in the sense that $|\mathscr{M} \setminus \mathscr{F}_{k,\varepsilon}| \to 0$.
- The greatest 'concentration' of the $\mathscr{F}_{k,\varepsilon}$ sets tend to be deflected away from the right boundary $\mu = 1$ and, in fact, are concentrated in the upper right-hand region of the admissible parameter space $\mathscr{M}$. Mathematically, we do not have a precise explanation for this apparent phenomenon. In particular, it is not *a priori* evident to us why the $\mathscr{F}_{k,\varepsilon}$ sets should concentrate in this particular subregion of the admissible parameter space $\mathscr{M}$.

Acknowledgement

The third author would like to acknowledge that this work has been partially supported by 'Piano di incentivi per la ricerca di Ateneo 2020/2022 (Pia.ce.ri)' – Università degli Studi di Catania.

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

Christopher S. Goodrich http://orcid.org/0000-0003-2058-216X

References

[1] T. Abdeljawad and B. Abdalla, *Monotonicity results for delta and nabla Caputo and Riemann fractional differences via dual identities*, Filomat 31 (2017), pp. 3671–3683.
[2] T. Abdeljawad, Q.M. Al-Mdallal, and M.A. Hajji, *Arbitrary order fractional difference operators with discrete exponential kernels and applications*, Discrete Dyn. Nat. Soc. 2017 (2017), Art. ID 4149320, 8 pp.
[3] T. Abdeljawad and F.M. Atici, *On the definitions of nabla fractional operators*, Abstr. Appl. Anal. 2012 (2012), Art. ID 406757. 13 pp.

[4] T. Abdeljawad and D. Baleanu, *Monotonicity results for fractional difference operators with discrete exponential kernels*, Adv. Difference Equ. (2017). Paper No. 78. 9 pp.
[5] R.P. Agarwal, D. Baleanu, S. Rezapour, and S. Salehi, *The existence of solutions for some fractional finite difference equations via sum boundary conditions*, Adv. Difference Equ. 2014 (2014), pp. 282. 16 pp.
[6] G.A. Anastassiou, *Nabla discrete fractional calculus and nabla inequalities*, Math. Comput. Model. 51 (2010), pp. 562–571.
[7] F.M. Atici and P.W. Eloe, *A transform method in discrete fractional calculus*, Int. J. Dif f er. Equ. 2 (2007), pp. 165–176.
[8] F.M. Atici, P.W. Eloe, *Discrete fractional calculus with the nabla operator*, Electron. J. Qual. Theory Dif f er. Equ. (2009). Special Edition I, No. 3, 12 pp.
[9] F.M. Atici and P.W. Eloe, *Initial value problems in discrete fractional calculus*, Proc. Amer. Math. Soc. 137 (2009), pp. 981–989.
[10] F.M. Atici and P.W. Eloe, *Two-point boundary value problems for finite fractional difference equations*, J. Difference Equ. Appl. 17 (2011), pp. 445–456.
[11] F.M. Atici and P.W. Eloe, *Linear systems of fractional nabla difference equations*, Rocky Mountain J. Math. 41 (2011), pp. 353–370.
[12] F.M. Atici and M. Uyanik, *Analysis of discrete fractional operators*, Appl. Anal. Discrete Math. 9 (2015), pp. 139–149.
[13] F.M. Atici and H. Yaldiz, *Convex functions on discrete time domains*, Canad. Math. Bull. 59 (2016), pp. 225–233.
[14] B.C. Carlson, Special Functions of Applied Mathematics, Academic Press, New York, 1977.
[15] R. Dahal and C.S. Goodrich, *A monotonicity result for discrete fractional difference operators*, Arch. Math. (Basel) 102 (2014), pp. 293–299.
[16] R. Dahal and C.S. Goodrich, *An almost sharp monotonicity result for discrete sequential fractional delta differences*, J. Dif f . Equ. Appl. 23 (2017), pp. 1190–1203.
[17] R. Dahal and C.S. Goodrich, *Mixed order monotonicity results for sequential fractional nabla differences*, J. Dif f . Equ. Appl. 25 (2019), pp. 837–854.
[18] R. Dahal and C.S. Goodrich, *A uniformly sharp convexity result for discrete fractional sequential differences*, Rocky Mountain J. Math. 49 (2019), pp. 2571–2586.
[19] F. Du, B. Jia, L. Erbe, and A. Peterson, *Monotonicity and convexity for nabla fractional (q, h)-differences*, J. Dif f . Equ. Appl. 22 (2016), pp. 1224–1243.
[20] L. Erbe, C.S. Goodrich, B. Jia, and A. Peterson, *Survey of the qualitative properties of fractional difference operators: Monotonicity, convexity, and asymptotic behavior of solutions*, Adv. Diff. Equ. 2016 (2016), pp. 43. DOI: 10.1186/s13662-016-0760-3
[21] R.A.C. Ferreira, *A discrete fractional Gronwall inequality*, Proc. Amer. Math. Soc. 140 (2012), pp. 1605–1612.
[22] R.A.C. Ferreira, *Existence and uniqueness of solution to some discrete fractional boundary value problems of order less than one*, J. Dif f . Equ. Appl. 19 (2013), pp. 712–718.
[23] C.S. Goodrich, *On discrete sequential fractional boundary value problems*, J. Math. Anal. Appl. 385 (2012), pp. 111–124.
[24] C.S. Goodrich, *A convexity result for fractional differences*, Appl. Math. Lett. 35 (2014), pp. 58–62.
[25] C.S. Goodrich, *Systems of discrete fractional boundary value problems with nonlinearities satisfying no growth conditions*, J. Dif f . Equ. Appl. 21 (2015), pp. 437–453.
[26] C.S. Goodrich, *A note on convexity, concavity, and growth conditions in discrete fractional calculus with delta difference*, Math. Inequal. Appl. 19 (2016), pp. 769–779.
[27] C.S. Goodrich, *The relationship between sequential fractional differences and convexity*, Appl. Anal. Discrete Math. 10 (2016), pp. 345–365.
[28] C.S. Goodrich, *A sharp convexity result for sequential fractional delta differences*, J. Diff. Equ. Appl. 23 (2017), pp. 1986–2003.
[29] C.S. Goodrich, *A uniformly sharp monotonicity result for discrete fractional sequential differences*, Arch. Math. (Basel) 110 (2018), pp. 145–154.

[30] C.S. Goodrich, *Sharp monotonicity results for fractional nabla sequential differences*, J. Diff. Equ. Appl. 25 (2019), pp. 801–814.
[31] C.S. Goodrich, J.M. Jonnalagadda, B. Lyons, *Convexity, monotonicity and positivity results for sequential fractional nabla difference operators with discrete exponential kernels*, Math. Methods Appl. Sci., doi:10.1002/mma.7247.
[32] C.S. Goodrich and C. Lizama, *A transference principle for nonlocal operators using a convolutional approach: Fractional monotonicity and convexity*, Israel J. Math. 236 (2020), pp. 533–589.
[33] C.S. Goodrich and C. Lizama, *Positivity, monotonicity, and convexity for convolution operators*, Discrete Contin. Dyn. Syst. 40 (2020), pp. 4961–4983.
[34] C.S. Goodrich, B. Lyons, and M.T. Velcsov, *Analytical and numerical monotonicity results for discrete fractional sequential differences with negative lower bound*, Commun. Pure Appl. Anal. 20 (2021), pp. 339–358.
[35] C.S. Goodrich and M. Muellner, *An analysis of the sharpness of monotonicity results via homotopy for sequential fractional operators*, Appl. Math. Lett. 98 (2019), pp. 446–452.
[36] C. Goodrich and A.C. Peterson, Discrete Fractional Calculus, Springer, New York, 2015.
[37] M. Holm, *Sum and difference compositions in discrete fractional calculus*, Cubo 13 (2011), pp. 153–184.
[38] B. Jia, L. Erbe, C.S. Goodrich, and A. Peterson, *The relation between nabla fractional differences and nabla integer differences*, Filomat 31 (2017), pp. 1741–1753.
[39] B. Jia, L. Erbe, C.S. Goodrich, and A. Peterson, *Monotonicity results for delta fractional differences revisited*, Math. Slovaca 67 (2017), pp. 895–906.
[40] B. Jia, L. Erbe, and A. Peterson, *Two monotonicity results for nabla and delta fractional differences*, Arch. Math. (Basel) 104 (2015), pp. 589–597.
[41] B. Jia, L. Erbe, and A. Peterson, *Convexity for nabla and delta fractional differences*, J. Dif f. Equ. Appl. 21 (2015), pp. 360–373.
[42] B. Jia, L. Erbe, and A. Peterson, *Some relations between the Caputo fractional difference operators and integer order differences*, Electron. J. Dif f er. Equ. 163 (2015), pp. 1–7.
[43] B. Jia, L. Erbe, and A. Peterson, *Monotonicity and convexity for nabla fractional q-differences*, Dynam. Syst. Appl. 25 (2016), pp. 47–60.
[44] J.M. Jonnalagadda, *Asymptotic stability for* 2×2 *fractional nabla difference systems*, Int. J. Dyn. Control 7 (2019), pp. 326–329.
[45] J.M. Jonnalagadda, *An ordering on Green's function and a Lyapunov-type inequality for a family of nabla fractional boundary value problems*, Fract. Differ. Calc. 9 (2019), pp. 109–124.
[46] A. Kilbas, H.M. Srivastava, and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland, New York, 2006.
[47] X. Liu, F. Du, D. Anderson, and B. Jia, *Monotonicity results for nabla fractional h-difference operators*, Math. Methods Appl. Sci. 44 (2021), pp. 1207–1218.
[48] C. Lizama, *The poisson distribution, abstract fractional difference equations, and stability*, Proc. Amer. Math. Soc. 145 (2017), pp. 3809–3827.
[49] C. Lizama and M. Murillo-Arcila, *Well posedness for semidiscrete fractional Cauchy problems with finite delay*, J. Comput. Appl. Math. 339 (2018), pp. 356–366.
[50] R. Wong and R. Beals, Special Functions: A Graduate Text, Cambridge University Press, New York, 2010.
[51] G. Wu and D. Baleanu, *Discrete fractional logistic map and its chaos*, Nonlinear Dyn. 75 (2014), pp. 283–287.