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# ANALYTICAL AND NUMERICAL MONOTONICITY RESULTS FOR DISCRETE FRACTIONAL SEQUENTIAL DIFFERENCES WITH NEGATIVE LOWER BOUND

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Abstract. We investigate the relationship between the sign of the discrete fractional sequential difference $\left(\Delta_{1+a-\mu}^{\nu}\Delta_a^{\mu}f\right)(t)$ and the monotonicity of the function $t \mapsto f(t)$. More precisely, we consider the special case in which this fractional difference can be negative and satisfies the lower bound

$$\left(\Delta_{1+a-\mu}^{\nu}\Delta_a^{\mu}f\right)(t) \geq -\varepsilon f(a),$$

for some $\varepsilon > 0$. We prove that even though the fractional difference can be negative, the monotonicity of the function $f$, nonetheless, is still implied by the above inequality. This demonstrates a significant dissimilarity between the fractional and non-fractional cases. Because of the challenges of a purely analytical approach, our analysis includes numerical simulation.

1. **Introduction.** In this paper for a function $f : \mathbb{N}_a \to \mathbb{R}$ we consider the sequential fractional difference $\left(\Delta_{1+a-\mu}^{\nu}\Delta_a^{\mu}f\right)(t)$; here and throughout we use the notation $\mathbb{N}_a := \{a, a+1, a+2, \dots\}$ for any $a \in \mathbb{R}$. A composition of fractional differences of the form

$$\Delta_{1+a-\mu}^{\nu} \circ \Delta_a^{\mu}$$

is known as a *sequential fractional difference* since the fractional differences are composed in a particular sequence – such fractional difference operators were first considered by Goodrich [15] in the context of boundary value problems.

The fractional difference and sum is an inherently nonlocal operator, the properties of which were initially investigated by Atici and Eloe [4–6] and then later by Anastassiou [3], Ferreira [14], Jonnalagadda [31], Lizama [33], and Lizama and

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Murillo-Arcila [34], among others. For example, the definition (see Section 2 for more details) of the fractional difference used in this paper is

$$\left(\Delta_a^{\nu} f\right)(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-s-1)^{\underline{-\nu-1}} f(s),$$

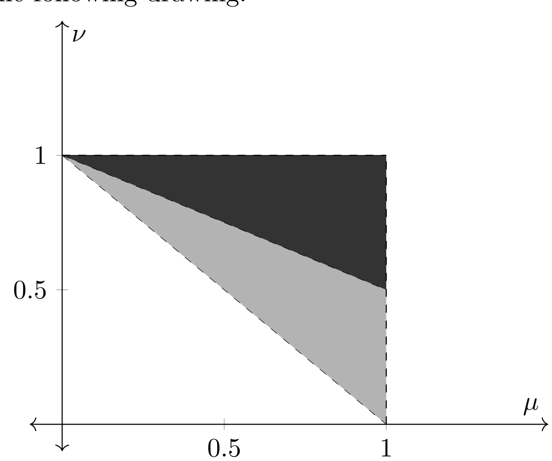
where $t \in \mathbb{N}_{a-\nu+N}$ and $N-1 < \nu \leq N$. As a consequence of this nonlocal structure the relationship between the sign of $\left(\Delta_a^{\nu} f\right)(t)$ and the monotonicity or convexity of $t \mapsto f(t)$ is very complicated. This is quite different than the integer-order setting, in which there is a trivial connection – namely, for example, if $(\Delta f)(t) := f(t+1) - f(t) \geq 0$ for some $t$, then $f$ is increasing at $t$ in the sense that $f(t+1) \geq f(t)$.

Recently there has been much progress in characterizing the precise relationship between the sign of the fractional difference of $f$ and the qualitative properties (e.g., monotonicity and convexity) of the function $f$. These investigations include papers

- in the non-sequential case by Abdeljawad and Abdalla [1], Abdeljawad and Baleanu [2], Atici and Uyanik [7], Dahal and Goodrich [9], Du, Jia, Erbe, and Peterson [12], Erbe, Goodrich, Jia, and Peterson [13], Goodrich [16], Jia, Erbe, Goodrich, and Peterson [26], and Jia, Erbe, and Peterson [27–30]; and
- in the sequential case by Dahal and Goodrich [10, 11], Goodrich [17–19], Goodrich and Lizama [20, 21], Goodrich and Lyons [22], and Goodrich and Muellner [23].

The sequential case has proved to be especially interesting inasmuch as it has been shown that there is a complicated relationship between the range in which $(\mu, \nu)$ lives and whether there is a relationship between the monotonicity of $f$ and an appropriate sequential fractional difference.

For example, with $0 < \mu < 1$, $0 < \nu < 1$, and $1 < \mu + \nu < 2$, in [18, Theorem 2.5] it was shown that there is a sharp dichotomy between the region of the $(\mu, \nu)$ parameter space on which there is a connection between the sign of the sequential fractional difference $\left(\Delta_{1+a-\mu}^{\nu} \Delta_a^{\mu} f\right)(t)$ and the monotonicity of $f$, and the region of the parameter space on which such a connection fails to exist – see also [23]. This is illustrated by the following drawing.




The dark grey region is the subset of the admissible parameter space on which there exists a strong connection between the sign of $\left(\Delta_{1+a-\mu}^{\nu} \Delta_a^{\mu} f\right)(t)$ and the sign of $(\Delta f)(t)$. By contrast, the light grey region is the subset of the admissible parameter space on which no such connection exists. Thus, there is a subtle interplay between the values of $\mu$ and $\nu$.

In spite of the increasingly significant literature on these problems, as far as we are aware there has been no investigation of the possibility of a relationship between a *negative* value of the sequential fractional difference of $f$ and the monotonicity of $f$. At first glance, such a relationship ought not to exist because in the integer-order case it is straightforwardly the case that

$$(\Delta f)(t) < 0 \Longleftrightarrow f(t+1) < f(t).$$

So a function with a negative first-order difference at a point $t$ cannot ever be increasing at $t$. On the other hand, it has been known for several years that a function can decrease in spite of a *positive* fractional difference – see Jia, Erbe, and Peterson [27]. So, with this in mind, one might wonder whether one can have

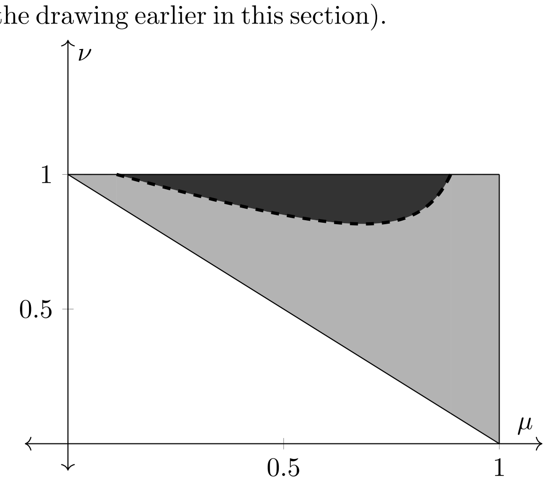
$$\left(\Delta_{1+a-\mu}^{\nu}\Delta_{a}^{\mu}f\right)(t) < 0 \Longrightarrow (\Delta f)(t) > 0. \tag{1.1}$$

In this paper we show that a relationship such as (1.1) does hold. More precisely, we show that so long as $\left(\Delta_{1+a-\mu}^{\nu}\Delta_{a}^{\mu}f\right)(t)$ is not "too negative", the function $f$ can still increase *at least for a time*. How long the monotonicity can be maintained is a complicated interplay of the parameter pair $(\mu, \nu)$ and the precise quantity of "negativity" that $\left(\Delta_{1+a-\mu}^{\nu}\Delta_{a}^{\mu}f\right)(t)$ possesses. In most of our results we characterize this negativity by means of the inequality

$$\left(\Delta_{1+a-\mu}^{\nu}\Delta_{a}^{\mu}f\right)(t) \geq -\varepsilon f(a), \tag{1.2}$$

where $\varepsilon \geq 0$ and $f(a) \geq 0$.

In particular and recalling the drawing from earlier in this section, we note that with the negative lower bound on the fractional difference as given by (1.2) the dark grey region is significantly smaller. For example, when $\varepsilon = 0.05$ in (1.2), the dark grey region (i.e., the region on which we can say something about the monotonicity of $f$) is as shown in the following drawing; notice how much smaller it is than in the case where $\varepsilon = 0$ (i.e., the drawing earlier in this section).




In both Example 3.5 and the entirety of Section 4 we expand on this observation significantly.

We conclude with a brief overview of the presentation in the remainder of this paper. In Section 2 we recall some basic definitions in discrete fractional calculus. In Section 3 we provide an *analytical* investigation of the relationships described in section. Finally, in Section 4 we provide a *numerical* investigation of these relationships. It turns out that an analytical investigation is very difficult, and so, by using numerical simulation we are able to provide a much more complete analysis of this problem.

## 2. **Preliminaries.**

We collect here some basic results in discrete fractional calculus. Considerable additional background may be found in the textbook by Goodrich and Peterson [24]. We begin with the definition of the falling factorial function, which acts as the kernel in the summation operator that defines the fractional difference and sum (see Definition 2.2).

**Definition 2.1.** We put

$$t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$$

for any $t$ and $\nu$ for which neither $t+1$ and $t+1-\nu$ is a pole of the Gamma function. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} := 0$.

Next we recall the definitions of the discrete fractional difference and sum of Riemann-Liouville type. We also recall the definition of the fractional Taylor monomial of order $\nu$.

**Definition 2.2.** The ***ν-th fractional sum***, $\nu > 0$, of a function $f : \mathbb{N}_a \to \mathbb{R}$, where $a \in \mathbb{R}$, is

$$\left(\Delta_a^{-\nu} f\right)(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s),$$

for $t \in \mathbb{N}_{a+\nu}$. The ***ν-th fractional difference*** of $f$, for $\nu > 0$, by

$$\left(\Delta_a^{\nu} f\right)(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-s-1)^{\underline{-\nu-1}} f(s),$$

where $t \in \mathbb{N}_{a-\nu+N}$ and $N \in \mathbb{N}_1$ is the unique number satisfying $N-1 < \nu \leq N$.

**Definition 2.3.** The ***ν-th fractional Taylor monomial*** based at $s$ is the map $(t,s) \mapsto h_\nu(t,s)$ defined by

$$h_\nu(t,s) := \frac{(t-s)^{\underline{\nu}}}{\Gamma(\nu+1)},$$

whenever the right-hand side is defined.

We finally recall the following result due to Holm [25]. Note that this result is, in fact, the basis for the mathematical interest of discrete sequential fractional operators since the result asserts that, in general, the fractional delta difference is a non-commutative operator.

**Theorem 2.4.** *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and suppose $\nu, \mu > 0$, with $N-1 < \nu \leq N$ and $M-1 < \mu \leq M$, where $M$, $N \in \mathbb{N}_1$. Then for $t \in \mathbb{N}_{a+M-\mu+N-\nu}$ it holds that*

$$\begin{aligned} &\Delta_{a+M-\mu}^{\nu} \Delta_a^{\mu} f(t) \\ =&\Delta_a^{\nu+\mu} f(t) - \sum_{j=0}^{M-1} h_{-\nu-M+j}(t-M+\mu, a) \Delta_a^{j-M+\mu} f(a+M-\mu), \end{aligned} \tag{2.1}$$

*where $N-1 < \nu < N$. If $\nu = N$, then (2.1) simplifies to* $\Delta_a^{\nu}$

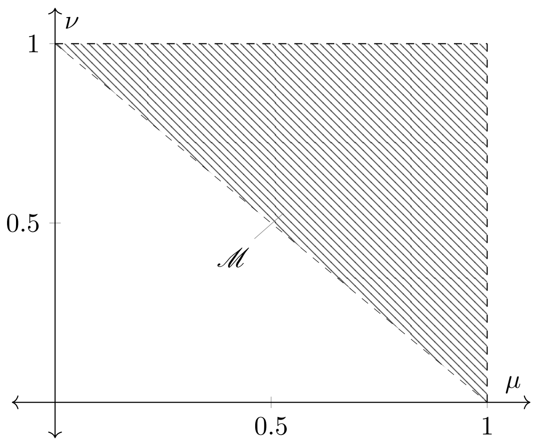
$$_{+M-\mu}\Delta_a^{\mu} f(t) = \Delta_a^{\nu+\mu} f(t),$$

*where* $t \in \mathbb{N}_{a+M-\mu}$.

3. **Analytical Results.** In this section we focus on what we can prove analytically. Throughout this section and the next we will denote by $\mathscr{M} \subseteq \mathbb{R}^2$ the following set.

$$\mathscr{M} := \big\{(\mu,\nu) \in \mathbb{R}^2 \ : \ 0<\mu<1,\, 0<\nu<1, \text{ and } 1<\mu+\nu<2\big\}$$

Geometrically this set is represented by the hatched region in the following drawing.




Thus, the set $\mathscr{M}$ is the admissible parameter space for the parameter pair $(\mu,\nu)$.

Our first result provides a sufficient condition for a function $f : \mathbb{N}_a \to \mathbb{R}$ to be increasing for at least one time step.

**Theorem 3.1.** *Let* $(\mu,\nu) \in \mathscr{M}$ *and assume that* $f : \mathbb{N}_a \to \mathbb{R}$ *satisfies each of the following.*

1. $f(a) \geq 0$
2. $\big(\Delta f\big)(a) \geq 0$
3. $\big(\Delta_{1+a-\mu}^{\nu}\Delta_a^{\mu} f\big)(2-\mu-\nu+a) \geq -\varepsilon f(a)$
4. $\nu \geq \dfrac{1}{2}(2-\mu) - \dfrac{\varepsilon}{\mu-1}$

*Then* $(\Delta f)(a+1) \geq 0$.

*Proof.* Similar to the proof of [10, Theorem 2.5] and with the help of Theorem 2.4 we begin by writing

$$\begin{aligned}\big(\Delta_{1+a-\mu}^{\nu}\Delta_a^{\mu} f\big)(t) =& \Delta_a^{\mu+\nu} f(t) - h_{-\nu-1}(t-1+\mu,a)\Delta_a^{\mu-1} f(1+a-\mu) && (3.1)\\ =& \frac{1}{\Gamma(-\mu-\nu)}\int_a^{t+\mu+\nu+1} (t-s-1)^{\underline{-\mu-\nu-1}} f(s)\ \Delta s && (3.2)\\ & - \frac{(t-1-a+\mu)^{\underline{-\nu-1}}}{\Gamma(-\nu)}\Delta_a^{\mu-1} f(1+a-\mu) && (3.3)\\ =& h_{-\mu-\nu}(t,a)f(a) + \underbrace{h_{-\mu-\nu}(t,t+\mu+\nu)}_{\equiv 1}\Delta f(t+\mu+\nu-1) && (3.4)\\ & + \sum_{s=a}^{t+\mu+\nu-2} h_{-\mu-\nu}(t,s+1)\Delta f(s) && (3.5)\\ & - \frac{(t-1-a+\mu)^{\underline{-\nu-1}}}{\Gamma(-\nu)}\underbrace{\Delta_a^{\mu-1} f(1+a-\mu)}_{=f(a)}. && (3.6)\end{aligned}$$

Now, letting $t = 2 - \mu - \nu + a$ in (3.1) and recalling that

$$\left(\Delta_{1+a-\mu}^{\nu}\Delta_a^{\mu}f\right)(2-\mu-\nu+a) \geq -\varepsilon f(a),$$

it follows that

$$\begin{aligned}\left(\Delta f\right)(1+a) &\geq \left[\frac{\Gamma(2-\nu)}{\Gamma(3)\Gamma(-\nu)} - \frac{\Gamma(3-\mu-\nu)}{\Gamma(3)\Gamma(1-\mu-\nu)} - \varepsilon\right] f(a) - \underbrace{\sum_{s=a}^{a} h_{-\mu-\nu}(t, s+1)\left(\Delta f\right)(s)}_{\geq 0} \\ &\geq \left[\frac{\Gamma(2-\nu)}{\Gamma(3)\Gamma(-\nu)} - \frac{\Gamma(3-\mu-\nu)}{\Gamma(3)\Gamma(1-\mu-\nu)} - \varepsilon\right] f(a). \qquad (3.7)\end{aligned}$$

We study the quantity

$$\frac{\Gamma(2-\nu)}{\Gamma(3)\Gamma(-\nu)} - \frac{\Gamma(3-\mu-\nu)}{\Gamma(3)\Gamma(1-\mu-\nu)} - \varepsilon, \qquad (3.8)$$

which appears in (3.7).

Since $f(a) \geq 0$ by assumption, we need that quantity (3.8) is nonnegative. The nonnegativity of (3.8) is equivalent to

$$\frac{1}{2}(1-\nu)(-\nu) - \frac{1}{2}(2-\mu-\nu)(1-\mu-\nu) - \varepsilon \geq 0. \qquad (3.9)$$

Note that (3.9) is itself equivalent to

$$-\mu^2 - 2\mu\nu + 3\mu + 2\nu - 2 - 2\varepsilon \geq 0,$$

which is equivalent to

$$\nu \geq \frac{1}{2}(2-\mu) + \frac{\varepsilon}{1-\mu}. \qquad (3.10)$$

So, we see that if (3.10) holds, which it does by assumption, then (3.8) will be nonnegative. And from inequality (3.7) this means that conditions (1)–(4) imply that $(\Delta f)(a+1) \geq 0$, as desired. □

In what follows it will be convenient to introduce some notation. Therefore, for each $k \in \mathbb{N}_2$ and $\varepsilon \geq 0$ define the set $\mathscr{E}_{k,\varepsilon} \subseteq \mathscr{M}$ by

$$\mathscr{E}_{k,\varepsilon} := \left\{(\mu,\nu) \in \mathscr{M} : \frac{1}{k!}\prod_{j=1}^{k}(j-1-\nu) - \frac{1}{k!}\prod_{j=1}^{k}(j-\mu-\nu) \geq \varepsilon\right\}.$$

We next state and prove a lemma, which shows that on a proper subset of the admissible parameter space the collection $\left\{\mathscr{E}_{k,\varepsilon}\right\}_{k=2}^{\infty}$ forms a decreasing collection of sets. This fact will be useful in the proof of Corollary 3.4.

**Lemma 3.2.** *If it holds that*

$$(-\mu-\nu)(1-\mu-\nu)(2-\mu-\nu) - (-1-\nu)(-\nu)(1-\nu) \geq 0, \qquad (3.11)$$

*then for each* $k \in \mathbb{N}_2$ *and* $\varepsilon \geq 0$ *it holds that*

$$\mathscr{E}_{k,\varepsilon} \supseteq \mathscr{E}_{k+1,\varepsilon}.$$

*Furthermore, if* $\varepsilon > 0$*, then under condition* (3.11) *it holds that*

$$\bigcap_{k=2}^{\infty} \mathscr{E}_{k,\varepsilon} = \lim_{k\to\infty} \mathscr{E}_{k,\varepsilon} = \varnothing.$$

*Proof.* We intend to show that for all $k \in \mathbb{N}_2$ and $(\mu, \nu) \in \mathscr{M}$,

$$\frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-1-\nu) - \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-\mu-\nu) \geq \varepsilon$$

implies that

$$\frac{1}{k!}\prod_{j=1}^{k}(j-1-\nu) - \frac{1}{k!}\prod_{j=1}^{k}(j-\mu-\nu) \geq \varepsilon,$$

provided that auxiliary condition (3.11) holds. The above implication will establish that

$$\mathscr{E}_{k,\varepsilon} \supseteq \mathscr{E}_{k+1,\varepsilon}.$$

To do this, we will show that

$$\frac{1}{k!}\prod_{j=1}^{k}(j-1-\nu) - \frac{1}{k!}\prod_{j=1}^{k}(j-\mu-\nu) \geq \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-1-\nu) - \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-\mu-\nu). \quad (3.12)$$

Note that (3.12) is a sufficient condition for the desired inclusion to hold since if (3.12) holds and $(\mu, \nu) \in \mathscr{E}_{k+1,\varepsilon}$, then it will follow that

$$\frac{1}{k!}\prod_{j=1}^{k}(j-1-\nu) - \frac{1}{k!}\prod_{j=1}^{k}(j-\mu-\nu) \geq \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-1-\nu) - \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-\mu-\nu) \geq \varepsilon,$$

from which it follows at once that $(\mu, \nu) \in \mathscr{E}_{k,\varepsilon}$.

Now, inequality (3.12) is equivalent to the inequality

$$\frac{1}{k!}\prod_{j=1}^{k}(j-1-\nu) - \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-1-\nu) \geq \frac{1}{k!}\prod_{j=1}^{k}(j-\mu-\nu) - \frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-\mu-\nu). \quad (3.13)$$

And inequality (3.13) is itself equivalent to the inequality

$$\left(\prod_{j=1}^{k}(j-1-\nu)\right)\underbrace{\left(\frac{1}{k!} - \frac{k-\nu}{(k+1)!}\right)}_{=-\frac{(-1-\nu)}{(k+1)!}} \geq \left(\prod_{j=1}^{k}(j-\mu-\nu)\right)\underbrace{\left(\frac{1}{k!} - \frac{k+1-\mu-\nu}{(k+1)!}\right)}_{=-\frac{(-\mu-\nu)}{(k+1)!}}. \quad (3.14)$$

Note that we can absorb $-1-\nu$ into the left-hand product of (3.14) and $-\mu-\nu$ into the right-hand product of (3.14) to yield

$$\frac{-1}{(k+1)!}\prod_{j=0}^{k}(j-1-\nu) \geq \frac{-1}{(k+1)!}\prod_{j=0}^{k}(j-\mu-\nu) \quad (3.15)$$

Then inequality (3.15) is equivalent to

$$\prod_{j=0}^{k}(j-\mu-\nu) - \prod_{j=0}^{k}(j-1-\nu) \geq 0. \quad (3.16)$$

In particular, inequality (3.12) is true if and only if inequality (3.16) is true. So, we now prove (3.16) by induction, using $k = 2$ as a base case since $k \in \mathbb{N}_2$. Note, however, that in case $k = 2$ inequality (3.16) is true by virtue of assumption (3.11) in the statement of the lemma.

So, to establish the induction step, we now assume (3.16) to be true and show that

$$\prod_{j=0}^{k+1}(j-\mu-\nu) - \prod_{j=0}^{k+1}(j-1-\nu) \geq 0. \tag{3.17}$$

To prove (3.17) we will show that

$$\prod_{j=0}^{k+1}(j-\mu-\nu) - \prod_{j=0}^{k+1}(j-1-\nu) \geq \underbrace{\prod_{j=0}^{k}(j-\mu-\nu) - \prod_{j=0}^{k}(j-1-\nu)}_{\geq 0}. \tag{3.18}$$

We can rearrange (3.18) by writing

$$\prod_{j=0}^{k+1}(j-\mu-\nu) - \prod_{j=0}^{k}(j-\mu-\nu) \geq \prod_{j=0}^{k+1}(j-1-\nu) - \prod_{j=0}^{k}(j-1-\nu). \tag{3.19}$$

Then we notice that inequality (3.19) is equivalent to

$$\left(\prod_{j=0}^{k}(j-\mu-\nu)\right)(k-\mu-\nu) \geq \left(\prod_{j=0}^{k}(j-1-\nu)\right)(k-1-\nu), \tag{3.20}$$

which we will show to be true by showing that $0 < \prod_{j=0}^{k}(j-1-\nu) \leq \prod_{j=0}^{k}(j-\mu-\nu)$ and $0 < k-1-\nu \leq k-\mu-\nu$. Therefore, inequality (3.17) is true if inequality (3.20) is true.

We can split this task into considering four inequalities.

$$0 < \prod_{j=0}^{k}(j-1-\nu) \tag{3.21}$$

$$\prod_{j=0}^{k}(j-1-\nu) \leq \prod_{j=0}^{k}(j-\mu-\nu) \tag{3.22}$$

$$0 < k-1-\nu \tag{3.23}$$

$$k-1-\nu \leq k-\mu-\nu \tag{3.24}$$

Now, inequality (3.21) is true since $j-1-\nu < 0$ for $j=0$ and $j=1$, and $j-1-\nu > 0$ for $2 \leq j \leq k$. (3.22) is true since it is a rearrangement of (3.16), which we have assumed to be true. (3.23) is true since $k \in \mathbb{N}_2$. And (3.24) is true because $\mu < 1$. Thus, we have proven (3.18), and so, inequality (3.17) is true, and this establishes the induction step. Combining this step with the basis for induction completed earlier in the proof, we conclude that the desired inclusion holds for each $k \in \mathbb{N}_2$, provided that $(\mu, \nu)$ is chosen such that inequality (3.11) is true. And this concludes the proof of the first claim.

At the same time, to prove the second claim in the statement of the lemma define the sequences $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ by

$$A_k := \frac{1}{k!}\prod_{j=1}^{k}(j-1-\nu)$$

and

$$B_k := \frac{1}{k!}\prod_{j=1}^{k}(j-\mu-\nu).$$

Then note that

$$\lim_{k\to\infty} A_k = \lim_{k\to\infty} \frac{\Gamma(k-\nu)}{\Gamma(-\nu)\Gamma(k+1)}$$
$$= \lim_{k\to\infty} \left[\frac{\Gamma(k-\nu)}{\Gamma(k+1)k^{-\nu-1}} \cdot \frac{k^{-\nu-1}}{\Gamma(-\nu)}\right] = \lim_{k\to\infty} \frac{k^{-\nu-1}}{\Gamma(-\nu)} = 0,$$

using the fact that $\lim_{k\to\infty} \frac{\Gamma(k-\nu)}{\Gamma(k+1)k^{-\nu-1}} = \lim_{k\to\infty} \frac{\Gamma(k-\nu)}{\Gamma(k)k^{-\nu}} = 1$, where one may consult, for example, [8, Theorem 3.4-1], [32, (1.5.15)], and [35, Proposition 2.1.3] for a proof of the second equality. In a similar manner we deduce that

$$\lim_{k\to\infty} B_k = \lim_{k\to\infty} \frac{\Gamma(k+1-\mu-\nu)}{\Gamma(1-\mu-\nu)\Gamma(k+1)} \tag{3.25}$$
$$= \lim_{k\to\infty} \left[\frac{\Gamma(k+1-\mu-\nu)}{\Gamma(k+1)k^{-\mu-\nu}} \cdot \frac{k^{-\mu-\nu}}{\Gamma(1-\mu-\nu)}\right] = \lim_{k\to\infty} \frac{k^{-\mu-\nu}}{\Gamma(1-\mu-\nu)} = 0.$$

Therefore,

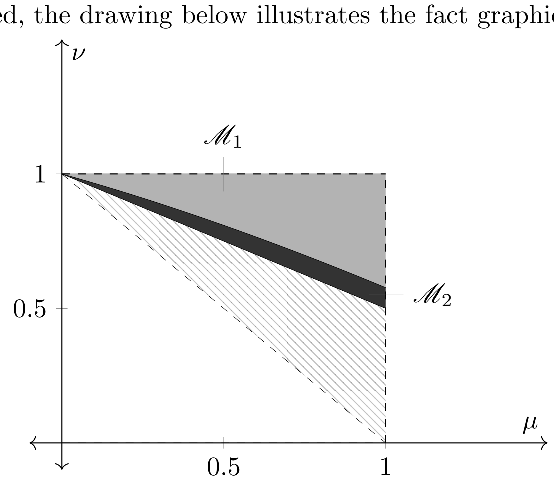
$$\lim_{k\to\infty} \frac{1}{k!}\left[\prod_{j=1}^{k}(j-1-\nu) - \prod_{j=1}^{k}(j-\mu-\nu)\right] = \lim_{k\to\infty} \left(A_k - B_k\right) = 0.$$

Consequently, since $\varepsilon > 0$, it follows that

$$k_0 := \sup\left\{k \in \mathbb{N} : A_k - B_k \geq \varepsilon\right\} < +\infty,$$

which implies that $\mathscr{E}_{k,\varepsilon} = \varnothing$ for all $k \geq k_0$. And this implies that $\bigcap_{k=2}^{\infty} \mathscr{E}_{k,\varepsilon} = \varnothing$, as claimed. □

*Remark* 3.3. We note that the auxiliary condition (3.11) is not particularly restric-tive. Indeed, the drawing below illustrates the fact graphically.




Note that the set $\mathscr{M}_1 \subseteq \mathscr{M}$ is the set on which condition (3.11) is satisfied, whereas the set $\mathscr{M}_2 \subseteq \mathscr{M}$ is the seton which condition (3.11) fails. Therefore, from the above drawing we see that $|\mathscr{M}_2| \ll |\mathscr{M}_1|$, where by $|\cdot|$ we mean the usual Lebesuge measure on $\mathbb{R}^2$. Consequently, we conclude that "most" points $(\mu, \nu) \in \mathscr{M}$ satisfy condition (3.11). Finally, observe that $\partial\mathscr{M}_2$ does not coincide with the segment $\mu + \nu = 1$ on account of condition (4) in Theorem 3.1.

Our next result shows that if $(\mu, \nu) \in \mathscr{E}_{k_0,\varepsilon}$, for some $k_0 \in \mathbb{N}$, then we can bootstrap the estimate (3.7) in the proof of Theorem 3.1 in order to deduce that

$$(\Delta f)(a) \geq 0 \Longrightarrow (\Delta f)(a+1) \geq 0 \Longrightarrow (\Delta f)(a+2) \Longrightarrow \cdots \Longrightarrow (\Delta f)\left(a+k_0-1\right) \geq 0$$

so that, in particular, $(\Delta f)\left(a+k_0-1\right) \geq 0$. We note that Corollary 3.4 holds on all of $\mathscr{M}_1 \cup \mathscr{M}_2$, the entirety of the admissible parameter space subject to condition (4) in Corollary 3.4.

**Corollary 3.4.** *Let $(\mu, \nu) \in \mathscr{M}$ such that condition (3.11) holds. Assume that $f : \mathbb{N}_a \to \mathbb{R}$ satisfies each of the following for some $k_0 \in \mathbb{N}_2$.*

1. $f(a) \geq 0$
2. $\big(\Delta f\big)(a) \geq 0$
3. $\big(\Delta^{\nu}_{1+a-\mu}\Delta^{\mu}_{a} f\big)(t) \geq -\varepsilon f(a)$ *for each* $t \in \mathbb{N}^{k_0+a-\mu-\nu}_{2+a-\mu-\nu}$
4. $\nu \geq \dfrac{1}{2}(2-\mu) - \dfrac{\varepsilon}{\mu-1}$

*If $(\mu, \nu) \in \mathscr{E}_{k_0,\varepsilon}$, then $(\Delta f)(t) \geq 0$ for each $t \in \mathbb{N}_a^{a+k_0-1}$.*

*Proof.* Fix a number $k_0 \in \mathbb{N}$. Using the notation introduced in the proof of Lemma 3.2 we know from the proof of Theorem 3.1 that

$$(\Delta f)\big(a+k_0-1\big) \quad \big[A_k - B_k - \varepsilon\,\big] f(a) \geq 0 \tag{3.26}$$

will hold for each $2 \leq k \leq k_0$ if and only if

$$(\mu, \nu) \in \bigcap_{j=2}^{k_0} \mathscr{E}_{j,\varepsilon}. \tag{3.27}$$

We first show that this is true on the set $\mathscr{M}_1$ as defined in Remark 3.3. So for $(\mu, \nu) \in \mathscr{M}_1$ note that by Lemma 3.2 we note that

$$\bigcap_{j=2}^{k_0} \mathscr{E}_{j,\varepsilon} = \mathscr{E}_{k_0,\varepsilon}.$$

Consequently, since $(\mu, \nu) \in \mathscr{E}_{k_0,\varepsilon}$ by assumption it follows that (3.27) holds and, thus, (3.26) holds. Therefore, by (3.26) together with the proof of Theorem 3.1 it follows that

$$\big(\Delta f\big)(t) \geq 0,$$

for each $t \in \mathbb{N}_a^{a+k_0-1}$, as claimed.

On the other hand, suppose next that $(\mu, \nu) \in \mathscr{M}_2$. Then (3.16) fails in case $k = 2$, which thus means that (3.12) fails in case $k = 2$. In addition, the proof of Theorem 3.1 demonstrates that whenever condition (4) in the statement of the corollary holds, it follows that $(\mu, \nu) \in \mathscr{E}_{2,\varepsilon}$. Therefore, it holds that

$$\begin{aligned} \varepsilon &\leq \frac{1}{2}(1-\nu)(-\nu) - \frac{1}{2}(2-\mu-\nu)(1-\mu-\nu) \\ &< \frac{1}{6}(2-\nu)(1-\nu)(-\nu) - \frac{1}{6}(3-\mu-\nu)(2-\mu-\nu)(1-\mu-\nu) \end{aligned} \tag{3.28}$$

Now, one of two cases occurs.

On the one hand, it may occur that (3.12) continues to fail for each integer $k > 2$. Thus,

$$\frac{1}{k!}\prod_{j=1}^{k}(j-1-\nu)-\frac{1}{k!}\prod_{j=1}^{k}(j-\mu-\nu)<\frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-1-\nu)-\frac{1}{(k+1)!}\prod_{j=1}^{k+1}(j-\mu-\nu). \tag{3.29}$$

Then combining inequalities (3.28) and (3.29) we conclude that

$$(\mu, \nu) \in \bigcap_{k=2}^{\infty} \mathscr{E}_{k,\varepsilon}.$$

But recalling from Lemma 3.2 that $\cap_{k=2}^{\infty}\mathscr{E}_{k,\varepsilon} = \varnothing$, the above computation shows that this case is void.

On the other hand, it may occur that (3.12) fails for each $k \in \mathbb{N}_2^{k_0}$, for some $k_0 \geq 3$, but holds in case $k = k_0 + 1$. Then from the preceding paragraph we know that

$$(\mu, \nu) \in \bigcap_{k=2}^{k_0} \mathscr{E}_{k,\varepsilon},$$

which means that by repeating the reasoning in the first two paragraphs of the proof of this corollary we deduce that $(\Delta f)(t) \geq 0$ for each $t \in \mathbb{N}_a^{a+k_0-1}$. And this means that, once again, the conclusion of the corollary holds.

Since the preceding cases are exhaustive, we deduce that the conclusion of the corollary is true for each $(\mu, \nu) \in \mathscr{M}$. And this completes the proof. □

We next provide an example to demonstrate that the conditions in Theorem 3.1 are non-void. In particular, one may reasonably wonder whether there exists a function $f$ such that not only conditions (1)–(4) are satisfied, but, in particular, it holds that

$$0 > \left(\Delta_{1+a-\mu}^{\nu}\Delta_a^{\mu} f\right)(t) \geq -\varepsilon f(a).$$

The following example demonstrates explicitly that this is possible.

**Example 3.5.** Set $\mu := 0.5$ and $\nu := 0.97$. Also set $\varepsilon := \frac{1}{10}$. Let $f : \mathbb{N}_0 \to \mathbb{R}$ such that $f(0) = 0.99$, $f(1) = 1$, and $f(2) = 1.02$. Note that $f(0) \geq 0$ and $(\Delta f)(0) \geq 0$. So, conditions (1)–(2) of Theorem 3.1 are satisfied. In addition, notice that

$$\frac{1}{2}(2-0.5)-\frac{\frac{1}{10}}{0.5-1}=\frac{3}{4}+\frac{1}{5}=\frac{19}{20}<\nu.$$

So, condition (4) of Theorem 3.1 is also satisfied. Finally, observe that

$$\left(\Delta_{0.5}^{0.97}\Delta_0^{0.5} f\right)(0.53) = -0.0936 < 0,$$

where we have used the fact that

$$\begin{aligned}
&\left(\Delta_{1-\mu}^{\nu}\Delta_0^{\mu} f\right)(2-\mu-\nu) \qquad (3.30)\\
=&\Delta_0^{\mu+\nu} f(2-\mu-\nu) - h_{-\nu-1}(1-\nu, 0) f(0)\\
=&\frac{1}{\Gamma(-\mu-\nu)}\sum_{s=0}^{2}(2-\mu-\nu-s-1)^{\overline{-\mu-\nu-1}} f(s) - \frac{(1-\nu)^{\overline{-\nu-1}}}{\Gamma(-\nu)} f(0)\\
=&\frac{1}{\Gamma(-\mu-\nu)}\Big[(1-\mu-\nu)^{\overline{-\mu-\nu-1}} f(0) + (-\mu-\nu)^{\overline{-\mu-\nu-1}} f(1)\\
&+(-1-\mu-\nu)^{\overline{-\mu-\nu-1}} f(2)\Big] - \frac{\Gamma(2-\nu)}{\Gamma(3)\Gamma(-\nu)} f(0)
\end{aligned}$$

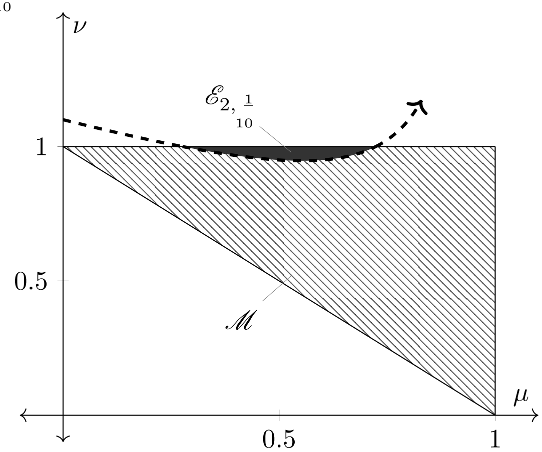
$$= \left[\frac{1}{2}(1-\mu-\nu)(-\mu-\nu)f(0) + (-\mu-\nu)f(1) + f(2)\right] - \frac{1}{2}(1-\nu)(-\nu)f(0),$$

which follows from (3.1) and Definition 2.2. Noticing that

$$\left(\Delta_{0.5}^{0.97}\Delta_0^{0.5}f\right)(0.53) = -0.0936 \geq -0.099 = -\frac{1}{10}\cdot 0.99 = -\varepsilon f(0),$$

we see that conditions (1)–(4) are satisfied. In particular, we see that $(\Delta f)(1) > 0$ in spite of the fact that $\Delta\left(\Delta_{0.5}^{0.97}\Delta_0^{0.5}f\right)(0.53) < 0$.

We conclude the example by noting that for the parameter pair $(0.5, 0.97) =: (\mu, \nu)$, the set $\mathscr{E}_{2,\frac{1}{10}}$ is represented in the following drawing.




Recall that

$$\mathscr{E}_{2,\frac{1}{10}} := \left\{(\mu,\nu) \in \mathscr{M} : \quad \frac{1}{2}(1-\nu)(-\nu) - \frac{1}{2}(2-\mu-\nu)(1-\mu-\nu) \geq \frac{1}{10}\right\}.$$

In particular, the dashed curve appearing in the drawing above is the graph of the function

$$\nu(\mu) := \frac{1}{2}(2-\mu) - \frac{1}{10(\mu-1)}, \quad 0 \leq \mu < 1.$$

We note that the intersections of the graph of $\nu \mapsto \nu(\mu)$ with the horizontal line $\nu = 1$ occur (to six decimal places of accuracy) when $\mu \approx 0.276393$ and $\mu \approx 0.723607$.

Now, Lemma 3.9 demonstrates it must hold that $\varepsilon \leq \frac{1}{8}$. Since the chosen value of $\varepsilon$ in this problem is very close to this maximum value, this is the reason why the set $\mathscr{E}_{2,\varepsilon}$ is observed to be very small in measure.

*Remark* 3.6. Note that the choice the pair $(\mu, \nu)$ in Example 3.5 does satisfy condition (3.11), for with $\mu = 0.5$ and $\nu = 0.97$ we directly calculate

$$(-\mu-\nu)(1-\mu-\nu)(2-\mu-\nu) - (-1-\nu)(-\nu)(1-\nu) = 0.30855 > 0.$$

Theorem 3.1 can be reformulated in a slightly different way if one wishes. In this alternative formulation we decouple the (negative) lower bound on $\left(\Delta_{1+a-\mu}^{\nu}\Delta_a^{\mu}f\right)(2-\mu-\nu+a)$ from the number $f(a)$. This is our next result.

**Theorem 3.7.** *Let* $(\mu,\nu) \in \mathscr{M}$ *and assume that* $f : \mathbb{N}_a \to \mathbb{R}$ *satisfies each of the following.*

1. $f(a) > 0$

2. $\left(\Delta f\right)(a) \geq 0$

3. $\left(\Delta^{\nu}_{1+a-\mu}\Delta^{\mu}_{a} f\right)(2-\mu-\nu+a) \geq -\varepsilon$

4. $\nu \geq \frac{1}{2}(2-\mu) - \frac{\varepsilon}{f(a)(\mu-1)}$, *where* $f(a) \neq 0$

*Then* $(\Delta f)(a+1) \geq 0$.

*Proof.* Since estimate (3.1) does not change except for replacing $-\varepsilon f(a)$ by $-\varepsilon$, we immediately obtain an analogue inequality (3.7) – namely,

$$(\Delta f)(1+a) \geq \left[\frac{\Gamma(2-\nu)}{\Gamma(3)\Gamma(-\nu)} - \frac{\Gamma(3-\mu-\nu)}{\Gamma(3)\Gamma(1-\mu-\nu)}\right] f(a) - \varepsilon.$$

As in the proof of Theorem 3.1 we need the quantity on the right-hand side of the inequality to be nonnegative. From the calculations in the proof of Theorem 3.1 we see that this is equivalent to requiring that

$$\frac{1}{2}(1-\nu)(-\nu) - \frac{1}{2}(2-\mu-\nu)(1-\mu-\nu) \geq \frac{\varepsilon}{f(a)},$$

keeping in mind that $f(a) \neq 0$. Then we see that this is equivalent to

$$-\mu^2 - 2\mu\nu + 3\mu + 2\nu - \frac{2\varepsilon}{f(a)} \geq 0,$$

which implies that

$$\nu \geq \frac{1}{2}(2-\mu) - \frac{\varepsilon}{f(a)(\mu-1)},$$

which proves the desired claim. ☐

*Remark* 3.8. Regarding Theorem 3.7 notice that if $\varepsilon \ll f(a)$, then condition (4) in Theorem 3.7 is less restrictive than the corresponding condition (4) in Theorem 3.1. In fact, notice that

$$\lim_{\varepsilon(f(a))^{-1}\to 0^+} \left[\frac{1}{2}(2-\mu) - \frac{\varepsilon}{f(a)(\mu-1)}\right] = \frac{1}{2}(2-\mu),$$

which means that for $\frac{\varepsilon}{f(a)} \approx 0$ condition (4) returns almost to the "natural" condition $\nu \geq \frac{1}{2}(2-\mu)$ discovered by Dahal and Goodrich [10, Theorem 2.5].

As our final analytical result we state a further property of the set $\mathscr{E}_{k,\varepsilon}$. In particular, the property we state provides an upper bound on the number $\varepsilon$ – namely $\varepsilon \leq \frac{1}{8}$ This places a restriction on just how much "negativity" the lower bound for the sequential difference can possess relative to the number $f(a)$. Especially, it provides a necessary condition for $\mathscr{E}_{k,\varepsilon}$ to be non-void, provided that $(\mu,\nu) \in \mathscr{M}_1$.

**Lemma 3.9.** *Assume that* $(\mu,\nu) \in \mathscr{M}_1$. *For any* $k \in \mathbb{N}$ *it follows that if* $\mathscr{E}_{k,\varepsilon} \neq \varnothing$, *then* $\varepsilon \leq \frac{1}{8}$.

*Proof.* Notice that

$$\mathscr{E}_{2,\varepsilon} = \left\{(\mu,\nu) \in \mathscr{M} : \frac{1}{2}(1-\nu)(-\nu) - \frac{1}{2}(2-\mu-\nu)(1-\mu-\nu) \geq \varepsilon\right\}.$$

From (3.10) we know that

$$\mathscr{E}_{2,\varepsilon} = \left\{(\mu,\nu) \in \mathscr{M} : \nu \geq \frac{1}{2}(2-\mu) - \frac{\varepsilon}{\mu-1}\right\}.$$

Now notice that

$$\frac{1}{2}(2-\mu) - \frac{\varepsilon}{\mu - 1} \leq 1$$

is a necessary condition for $\mathscr{E}_{2,\varepsilon}$ to be non-void; this is a consequence of the restriction $0 \leq \nu \leq 1$. Inequality (3) is equivalent to

$$\frac{1}{2}(2-\mu)(\mu-1) - \varepsilon \geq \mu - 1,$$

keeping in mind that $\mu - 1 < 0$. Note that

$$\frac{1}{2}(2-\mu)(\mu-1) - \varepsilon \geq \mu - 1$$

if and only if

$$\mu \in \left[\frac{1}{2}\left(1 - \sqrt{1-8\varepsilon}\right), \frac{1}{2}\left(1 + \sqrt{1-8\varepsilon}\right)\right]$$

But then we immediately see that $\mu \in \mathbb{R}$ only if $\varepsilon \leq \frac{1}{8}$. So, this means that $\mathscr{E}_{2,\varepsilon} \neq \varnothing$ if and only if $\varepsilon \leq \frac{1}{8}$.

Finally, since $(\mu, \nu) \in \mathscr{M}_1$ and recalling Lemma 3.2 it follows that if for some $k \in \mathbb{N}$ it holds that $\mathscr{E}_{k,\varepsilon} \neq \varnothing$, then $\mathscr{E}_{j,\varepsilon} \neq \varnothing$ for each $j \leq k$. In particular, this must hold when $j = 2$. And from this observation the desired claim follows at once. □

4. **Numerical simulations.** As discussed in Section 3 the $\mathscr{E}_{k,\varepsilon}$ sets are the key to understanding for which $(\mu, \nu) \in \mathscr{M}$ the monotonicity result holds and for how many time steps it holds. Unfortunately, analytically analyzing the set $\mathscr{E}_{k,\varepsilon}$ for a given $\varepsilon > 0$ is very complicated even for small $k$. Therefore, in this final section we provide some numerical results so as to estimate the structure of the set $\mathscr{E}_{k,\varepsilon}$ by means of numerical approximation.

First, we give a graphical representation of the sets $\mathscr{E}_{k,\varepsilon}$ for multiple $\varepsilon \leq \frac{1}{8}$ values and various $k$ values as specified in Figures 1–5. For simplicity we show only values of $\varepsilon$ that are powers of $\frac{1}{10}$. Each figure contains 36 subplots with the same values of $k \in \{2n : n = 1, 2, \ldots, 36\}$ for a consistent comparison of the effect of both $\varepsilon$ and $k$ on the shape of the set $\mathscr{E}_{k,\varepsilon}$. Due to the computational limitations of the Matlab software, we could not consider values of $k$ larger than 170. Under this constraint, all sets reach an apparently final format for $k < 72$ regardless of $\varepsilon$. However, notice that larger values of $\varepsilon$ are paired with a clear reduction of the area of the sets $\mathscr{E}_{k,\varepsilon}$ for $k < 72$, which leads to some empty subplots. That may be an indication that not only does the area of the sets $\mathscr{E}_{k,\varepsilon}$ decrease with increased $k$ values, but also that the rate of area reduction decreases as $\varepsilon$ decreases.

In Figure 1 we have let $\varepsilon = 0.1$. We see that the admissible parameter space is very small even when $k = 2$ (i.e., the initial case). It subsequently becomes empty very quickly as $k$ increases. In light of Lemma 3.9 this is due to the fact that 0.1 is very close to the maximum value of $\varepsilon$, namely 0.125.

In Figure 2 we have let $\varepsilon = 0.01$, and we see that now the admissible parameter space appears nonempty until $k > 10$. Moreover, the admissible parameter space is much larger for small $k$ than was observed in Figure 1, thus suggesting that when $\varepsilon$ is smaller, not only is larger $k$ permitted but also are more $(\mu, \nu)$ pairs admissible even for small $k$. We also notice that the admissible parameter regions appear to be nested as $k$ increases – just as deduced in Lemma 3.2.

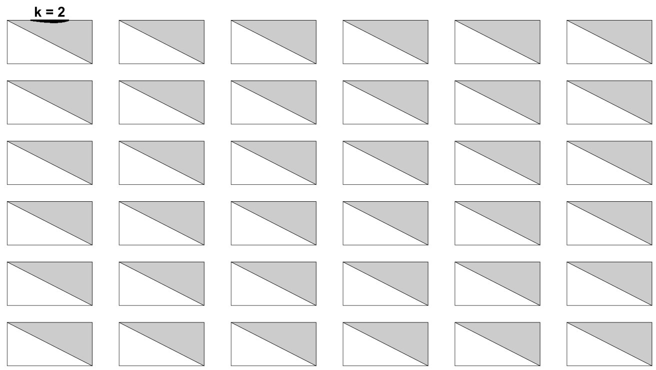



FIGURE 1. Graphical representation of the set $\mathscr{E}_{k,0.1}$ for $k \leq 72$.

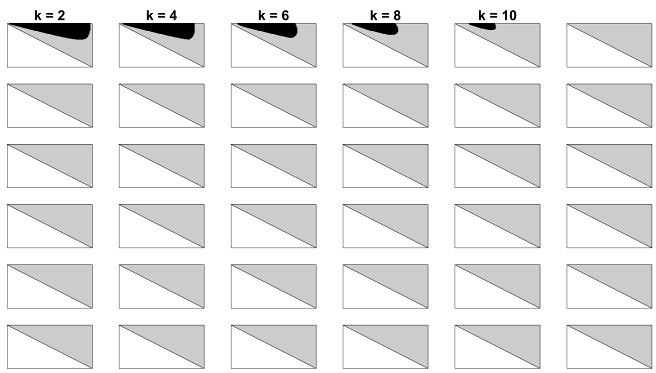



FIGURE 2. Graphical representation of the set $\mathscr{E}_{k,0.01}$ for $k \leq 72$.

In Figure 3 we have now set $\varepsilon = 0.001$. Now the admissible parameter space appears to be nonempty at least for all $2 \leq k \leq 72$. We once again see the nested appearance to the sets as $k$ increases.

Finally, in Figures 4–5, in which we have set $\varepsilon$ to be equal to 0.0001 and 0.00001, respectively, we see a continuation of the trends already noted for the earlier cases. In particular, the sets appear to be nested as $k$ increases and, moreover, as $\varepsilon$ con-tinues to decrease, for a given $k$ the admissible parameter appears to grow. In fact, we notice in Figure 5, in particular, that these most of the admissible parameter sets appear to be very close to the maximum admissible parameter set associated with $\varepsilon = 0$ – i.e., the case considered in [18].

All in all, then, from these numerical experiments we can draw the following conclusions.

- As $\varepsilon \to 0^+$ the admissible parameter space converges to parameter space configuration when $\varepsilon = 0$ – namely,

$$\left\{(\mu, \nu) \in \mathscr{M} : \nu \geq \frac{1}{2}(2 - \mu)\right\},$$

  as identified in [18].

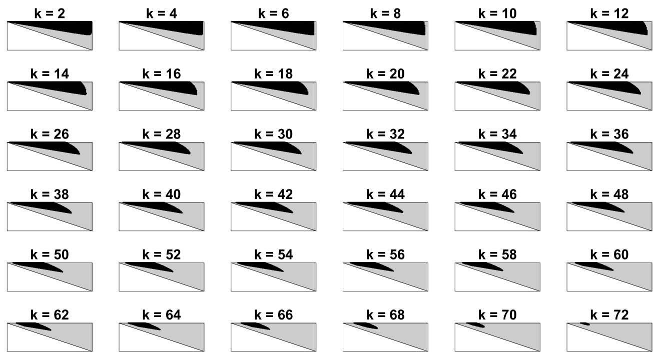



FIGURE 3. Graphical representation of the set $\mathscr{E}_{k,0.001}$ for $k \leq 72$.

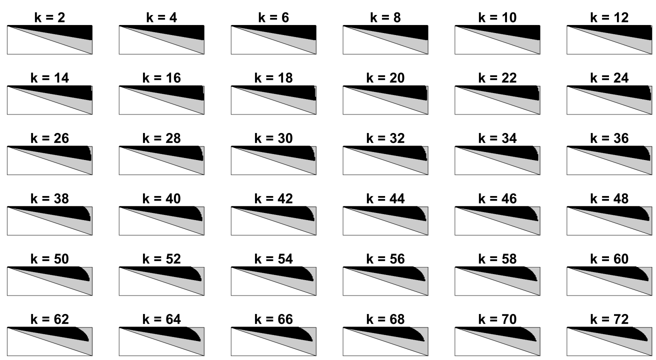



FIGURE 4. Graphical representation of the set $\mathscr{E}_{k,0.0001}$ for $k \leq 72$.

- For any fixed $\varepsilon > 0$ it appears that
$$\lim_{k\to\infty} \left|\mathscr{E}_{k,\varepsilon}\right| = 0,$$
where by $|\cdot|$ we denote the usual Lebesgue measure on $\mathbb{R}^2$.
- For any $k \in \mathbb{N}_2$ and any $\varepsilon > 0$ it appears that
$$\mathscr{E}_{k,\varepsilon} \supseteq \mathscr{E}_{k+1,\varepsilon}$$
so that the collection $\left\{\mathscr{E}_{k,\varepsilon}\right\}_{k=2}^{\infty}$ forms a (decreasing) nested collection of sets.

All of this confirms and is congruent with the analytical results that we were able to deduce in Section 3.

On the other hand, Figures 6–7 provide heat maps indicating the cardinality of the set $\{k : (\mu, \nu) \in \mathscr{E}_{k,\varepsilon}\}$ for a given $\varepsilon \leq \frac{1}{8}$. The cardinality increases from small (dark blue) to large (dark red) and the actual cardinalities are shown along the sidebar of each subplot. In Figure 6 we use values of $\varepsilon$ that are powers of 1 – similar to Figures 1–5. Notice that for very small $\varepsilon$ the entire admissible region is colored in dark red. Figure 7 supplements the numerical analysis with values of $\varepsilon \in \left[\frac{1}{1000}, \frac{1}{100}\right]$; these correspond to the interval of $\varepsilon$ reflected in the top two subplots

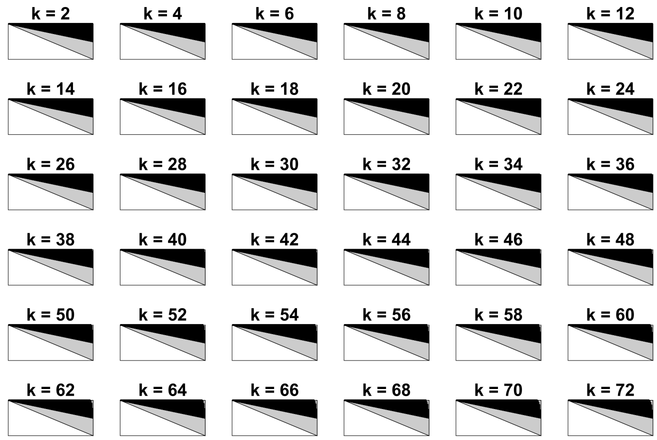



FIGURE 5. Graphical representation of the set $\mathscr{E}_{k,0.00001}$ for $k \leq 72$.

of Figure 6. Notice the change in the sharpness of the boundaries of the color regions of the heat maps due to the increase in the cardinality of $\{k : (\mu, \nu) \in \mathscr{E}_{k,\varepsilon}\}$ as indicated in the sidebars of each subplot.

In particular, it appears that we can draw the following conclusions from the heat maps.

- Setting $\mathscr{F}_\varepsilon := \{k : (\mu, \nu) \in \mathscr{E}_{k,\varepsilon}\}$ it appears that
$$\lim_{\varepsilon \to 0^+} \left|\mathscr{F}_\varepsilon\right| = +\infty.$$
In particular, this seems to have the following implication. Suppose that we have $f : \mathbb{N}_a \to \mathbb{R}$ such that
$$\left(\Delta^{\nu}_{1+a-\mu}\Delta^{\mu}_{a} f\right)(t) \geq -\varepsilon f(a).$$
Then as $\varepsilon \to 0^+$ the number of points at which $\big(\Delta f\big)(t) > 0$ can, in principle, hold tends to $+\infty$ – i.e., $\left|\mathscr{F}_\varepsilon\right| \to +\infty$.
- In light of the preceding point, roughly speaking, then, if the negative lower bound on $\left(\Delta^{\nu}_{1+a-\mu}\Delta^{\mu}_{a} f\right)(t)$ is not "too negative", then just as in Corollary 3.4 the function $f$ may possibly increase for a long time.
- If $\varepsilon$ is very close to zero (e.g., see Figure 7), then for many, many time steps the configuration of the admissible parameter space remains approximately equal to the $\varepsilon = 0$ state – that is,
$$\mathscr{E}_{k,\varepsilon} \approx \left\{(\mu, \nu) \in \mathscr{M} : \nu \geq \frac{1}{2}(2 - \mu)\right\},$$
for $k$ not too large. More precisely, it appears that
$$\left|\mathscr{E}_{k,\varepsilon} \setminus \left\{(\mu, \nu) \in \mathscr{M} : \nu \geq \frac{1}{2}(2 - \mu)\right\}\right| \approx 0,$$
whenever $\varepsilon$ is sufficiently small and $k$ is not too large.

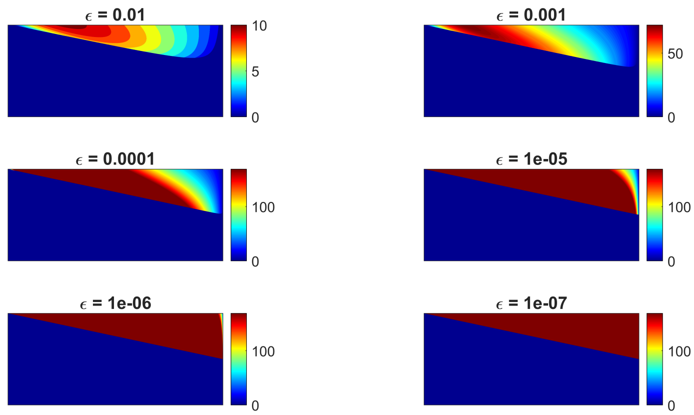



FIGURE 6. Heat maps for the cardinality of the set $\{k : (\mu, \nu) \in \mathscr{E}_{k,\varepsilon}\}$ for $\varepsilon = 0.01, 0.001, 0.0001, 0.00001, 0.000001, 0.0000001$. The cardinality increases from small (dark blue) to large (dark red) and the actual cardinalities are shown along the sidebar of each subplot.

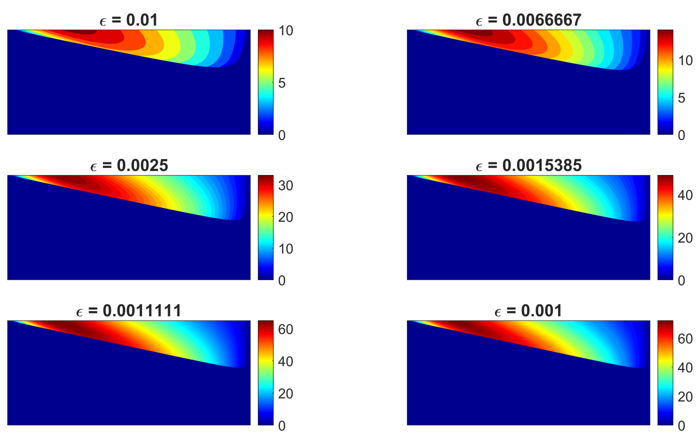



FIGURE 7. Heat maps for the cardinality of the set $\{k : (\mu, \nu) \in \mathscr{E}_{k,\varepsilon}\}$ for $\varepsilon = 1/100, 1/150, 1/400, 1/650, 1/900, 1/1000$. These correspond to the interval of $\varepsilon$ reflected in the top two subplots of Figure 6. Notice the change of cardinality values as $\varepsilon$ decreases.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