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A template for the exploration of chaotic locomotive patterns

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Abstract

Inverted pendulum and spring-mass models have been successfully used to explore the dynamics of the lower extremity for animal and human locomotion. These models have been classified as templates that describe the biomechanics of locomotion. A template is a simple model with all the joint complexities, muscles and neurons of the locomotor system removed. Such templates relate well to the observed locomotive patterns and provide reference points for the development of more elaborate dynamical systems. In this investigation, we explored if a passive dynamic double pendulum walking model, that walks down a slightly sloped surface ($\gamma < 0.0189$ rad), can be used as a template for exploring chaotic locomotion. Simulations of the model indicated that as $\gamma$ was increased, a cascade of bifurcations were present in the model's locomotive pattern that lead to a chaotic attractor. Positive Lyapunov exponents were present from $0.01839$ rad < $\gamma$ < $0.0189$ rad (Lyapunov exponent range = $+0.002$ to $+0.158$). Hurst exponents for the respective c confirmed the presence of chaos in the model's locomotive pattern. These results provide evidence that a passive dynamic double pendulum walking model can be used as a template for exploring the biomechanical control parameters responsible for chaos in human locomotion.

Introduction

Typically, we think of human locomotion as exhibiting a periodic movement pattern. For example, it can be readily observed that the legs oscillate to-and-fro with a limit cycle behavior that is similar to the pendulum motions of a clock [1]. Any variations from this periodic pattern have traditionally been considered as “noise” within the system [2]. However, recent investigations of human locomotion have observed that variations from one step to the next may not be noise. Rather these variations have a chaotic structure [2–7]. Several authors have postulated that the chaotic structure present in human locomotion may be a window to the control mechanisms of locomotion [2–4].

Examination of the chaotic processes in physiological systems has come under increasingly closer examination by mathematicians, physicists and recently biomechanists. Much of the insights on understanding the origin of such chaotic dynamics have come from the analysis of simplified mathematical models that are sufficiently close to the behavior of the real system [8]. Full and
Koditschek [9] referred to such simple models as templates. A template for locomotion has all the joint complexities, muscles and neurons of the locomotor system removed [9]. The advantage of

![Diagram of a passive dynamic walking model](image)

Fig. 1. The passive dynamic walking model where $\phi$ is the angle of the swing leg, $\theta$ is the angle of the stance leg, $\gamma$ is the angle of inclination of the supporting surface, and $g$ is gravity. Both legs are of length $\ell$.

using a template is that the basic principles that govern the dynamics of the system can be revealed and future more elaborate models can be developed based on the basic principles of the template [9].

The mechanics of the walking human leg during ground contact have been successfully modeled with an inverted pendulum template [9–14]. This template consists of a point mass representing the torso, set on top a stiff rod representing the leg during ground contact. During locomotion, the inverted pendulum passes through a given sweep angle that is dependent on the initial conditions and the Newtonian forces acting on the system. Although such a model is relatively simplistic compared to the complexities found in the human body, it has been used successfully to study locomotion mechanics [9–14]. This template has inspired the development of a class of passive dynamic bipedal robots that walk down a slightly sloped surface [15–18]. These walking models are based on the same principles as the inverted pendulum template. The legs are composed of an inverted double pendulum system where one leg is in contact with the ground and the other leg swings freely with the trajectory of the system’s center of mass (Fig. 1). Passive dynamic bipedal walking models have been used to address questions about the biomechanical requisites and energetics of bipedal human locomotion [15–18]. Additionally, Garcia et al. [17] have demonstrated that an irreducibly simple passive dynamic walking model can exhibit a cascade of period doublings in the locomotive strategies. Garcia et al. noted that the distances between consecutive period doublings appear to converged to the Feigenbaum number (4.669201...). This suggested that a passive dynamic walking model may exhibit a chaotic bipedal locomotive pattern [19]. However, Garcia et al. did not examine or prove the presence of chaos per se in the model’s locomotion. Nor did Garcia et al. identify which ramp angle is associated with the onset of a chaotic walking pattern.

We speculate that the passive dynamic walking model may provide a viable template for further investigations of biomechanical control parameters responsible for chaotic locomotion. Therefore, the purpose of this investigation was to rigorously investigate the potential chaotic properties of a bipedal passive dynamic walking model and determine if it can serve as a template. 2.

**Modeling bipedal locomotion**
Our passive dynamic walking model was a simplified mathematical model of the lower extremity based on the work of Garcia et al. [17]. The model consisted of two rigid massless legs connected by a frictionless hinge at the hip (Fig. 1). During locomotion, the stance leg swung like an inverted pendulum until the swing leg made contact with the supporting surface. At heel-contact, the swing leg became the stance leg and the stance leg became the swing leg for the next step. The swing leg was allowed to pass through the supporting surface during midstance and had a plastic collision with the surface. Energy for the locomotive pattern was supplied to the model via a slightly sloped rigid walking surface ($\gamma < 0.0189$ rad).

Two coupled second-order differential equations were used to define the behavior of the legs of the passive dynamic walking model (Eq. (1)). These equations were the equations of motion for a double pendulum where $\beta$ was the mass, $l$ was the leg length, $g$ was gravity, $\Theta$ was the angle of the stance leg, $\dot{\Theta}$ was the angular velocity of the stance leg, $\Theta$ was the angular acceleration of the swing leg, $\phi$ is the angle of the swing leg, $\phi$ is the angular velocity of swing leg and $\phi_2$ was the angular acceleration of the swing leg.

\[
\begin{pmatrix}
1 + 2\beta(1 - \cos \phi) & -\beta(1 - \cos \phi) & -\beta & -\beta \\
\beta(1 - \cos \phi) & -\beta & -\beta & -\beta \\

\frac{(\beta g/l)\sin(\theta - \phi - \gamma) - \sin(\theta - \gamma)}{(\beta g/l)\sin(\theta - \phi - \gamma)} & \frac{g/l\sin(\theta - \gamma)}{(\beta g/l)\sin(\theta - \phi - \gamma)} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\dot{\theta} \\
\dot{\phi} \\
\ddot{\phi} \\
\ddot{\phi}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\quad (1)
\]

The equations of motion were simplified by assuming that the hip had the greatest influence on the motion of the walking system. This assumption allowed the large hip payload to negate the influence of the point foot masses on the system’s dynamics. Therefore, $b$ was set to zero in the first row of Eq. (1), and the row 2 of Eq. (1) was divided by $b$. Non-dimensionalizing the governing equations resulted in the locomotive pattern of the model to be dependent on the initial stance leg conditions ($\Theta, \dot{\Theta}$) and the slope of the rigid walking surface ($\gamma$).

The governing equations were integrated using a modified version of Matlab’s (MathWorks, Natick, MA) ODE45. The ODE45 was modified to integrate the equations of motion with a tolerance of $10^{-12}$ and to stop integrating when the angle of the swing leg angle was twice as large as the stance leg angle (Eq. (2)).

\[\phi - 2\dot{\theta} = 0\quad (2)\]

The swing leg became the stance leg and the former stance leg became the swing leg when the conditions presented in Eq. (2) were satisfied. The model assumed that angular momentum of the modeled system was conserved when the legs switched roles. Therefore, the energy gained in the descent of the walking model was balanced with the energy lost at each heel-strike. Eq. (3) was used to model the conservation of angular momentum at points of transition; where “+” indicated the behavior of the model just after the swing leg made contact with the ground and “−” indicated the behavior of the model just before the swing leg made contact with the ground. In Eq. (3), $\Theta$ was the angle of the stance leg, $\dot{\Theta}$ was the angular velocity of the stance leg, $\phi$ was the angle of the swing leg and $\dot{\phi}$ was the angular velocity of the swing leg.

\[\Theta \quad \dot{\Theta} \quad \phi \quad \dot{\phi}\]
Eq. (3) indicated that the foot of the swing leg was brought to rest instantaneously each time there was a heel-strike. At heel-strike, no torque was generated about the point of contact. Therefore, the angular momentum of the system was the same before and after heel-strike. In Eq. (3), it should also be noted that the swing leg ($\phi$) made no contribution to the behavior of the model during points of transition. Therefore, the behavior of the stance leg and $y$ determine the overall strategies used for each consecutive step of the locomotive pattern.

**Analysis of locomotive chaos**

Despite the simplicity of our passive dynamic walking model, its analysis was quite complicated since its governing equations consist of nonlinear ordinary differential equations and algebraic switching conditions. Therefore, we used systematic numerical simulations to confirm that the locomotive patterns of our model had chaotic dynamics. Analyses of the locomotive patterns of the model were performed from 5000 footfalls with the first 500 footfalls removed to be certain that the model converged to the given attractor.

Bifurcations in the model’s locomotive pattern were noted with Poincare maps composed from the step time interval of the right leg of the model (Eq. (4)).

$$\zeta_{n+1} = f(\zeta_n)$$

where $\zeta_n$ is the step time interval for the $n$th step and $\zeta_{n+1}$ is the step time interval for the proceeding step. The Poincare maps provided a way to simplify the dynamics of the system by viewing the behavior of the system stroboscopically [20]. This involved cutting or sectioning the attractor at regular intervals or events. In this investigation, step time interval from heel-contact to toe-off of the right leg was utilized to construct the Poincare map because it represented the global outcome of the walking dynamics of the model. Additionally, several previous investigations have demonstrated that the step time interval of human locomotion can exhibit chaotic dynamics [2,5–7].

Tools from mathematical chaos theory were used to classify the data that composed the Poincaré map as representing a periodic or chaotic locomotive pattern [19–22]. To use such tools, it was essential to quantify an appropriate time delay and embedding dimension for the investigated data series. These variables allowed for the reconstruction of a state space that defined the dynamics of the system at any point during the locomotive pattern. Investigation of the characteristics of the state space is a powerful tool for examining a dynamic system because it provides information that is not apparent by just observing the data series [20,21].

To reconstruct the state space, a state vector was created from the simulated step time interval data series. This vector was composed of mutually exclusive information about the dynamics of the system (Eq. (5)).

$$y(t) = [x(t), x(t - T_1), x(t - T_2), \ldots]$$
where $y(t)$ was the reconstructed state vector, $x(t)$ was the original data and $x(t-T_i)$ were time delay copies of $x(t)$. The time delay ($T_i$) for creating the state vector was determined by estimating when information about the state of the dynamic system at $x(t)$ was different from the information contained in its time-delayed copy. If the time delay was too small then no additional information about the dynamics of the system would be contained in the state vector. Conversely, if the time delay was too large then information about the dynamics of the system may be lost and can result in random information [20,21]. Selection of the appropriate time delay was performed by using an average mutual information algorithm (Eq. (6); [21]).

$$I_{x(t),x(t+T)} = \sum P(x(t),x(t+T)) \log_2 \left[ \frac{P(x(t),x(t+T))}{P(x(t))P(x(t+T))} \right]$$

where $T$ was the time delay, $x(t)$ was the original data, $x(t+T)$ was the time delay data, $P(x(t),x(t+T))$ was the joint probability for measurement of $x(t)$ and $x(t+T)$; $P(x(t))$ was the probability for measurement of $x(t)$, $P(x(t+T))$ was the probability for measurement of $x(t)$, $P(x(t))$. The probabilities were constructed from the frequency of $(t+T)$ occurring in the step interval time series. Average mutual information was iteratively calculated for various time delays and the selected time delay was at the first local minimum of the iterative process [21,22]. This selection was based on previous investigations that have determined that the time delay at the first local minimum contains sufficient information about the dynamics of the system to reconstruct the state vector [21].

It was additionally necessary to determine the number of embedding dimensions to unfold the dynamics of the system in an appropriate state space. An inappropriate number of embedding dimensions may result in a projection of the dynamics of the system that has orbital crossings in the state space that are due to false neighbors and not the actual dynamics of the system [21]. To unfold the state space we systematically inspected $x(t)$ and its neighbors in various dimensions (e.g. dimension = 1, 2, 3,... etc.). The appropriate embedding dimension occurred when neighbors of the $x(t)$ stopped being un-projected by the addition of further dimensions of the state vector (Eq. (7)).

$$\gamma(t) = [x(t),x(t+T),x(t+2T),\ldots x(t+(d_e-1)T)].$$

where $d_e$ was the number of embedding dimensions, $\gamma(t)$ was the $d_e$-dimensional state vector, $x(t)$ was the original data, and $T$ was the time delay. A global false nearest neighbors algorithm with the time delay determined from the local minimum of the average mutual information was used to determine the number of necessary embedding dimensions to reconstruct the step time interval data series [21]. The calculated embedding dimension indicated the number of governing equations that were necessary to appropriately reconstruct the dynamics of the system [21]. Hence we were expecting the embedding dimensions of the passive dynamic walking model to be two since the dynamics of the system was based on two governing equations. The Tools for Dynamics (Applied Chaos, LLC) software was used to calculate the embedding dimension of the passive dynamic bipedal model.

Lyapunov exponents were calculated to quantify the exponential separation of nearby trajectories in the reconstructed state space of the simulated locomotive pattern at the respective $y$. This information was necessary to classify the stability of the locomotive pattern and determine if the pattern was periodic or chaotic. As nearby points of the state space separate, they diverge rapidly and can produce instability. Lyapunov exponents from a stable system with little to no divergence will be
zero (e.g. sine wave). Alternatively, Lyapunov exponents for an unstable system that has a high amount of divergence will be positive (e.g. random data). A chaotic system will have both positive and negative Lyapunov exponents. Although a positive Lyapunov exponent indicates instability, the sum of the Lyapunov exponents for a chaotic system remains negative and allows the system to maintain stability [20,21]. This notion can be seen by inspecting the largest Lyapunov exponent for a sine wave (0), a chaotic Lorenz attractor (0.100), and random data series (0.469). Hence a chaotic system lies somewhere between a completely periodic system and a completely random system. The Chaos Data Analyzer (American Institute of Physics) was used to numerically calculate the largest Lyapunov exponent for each $\gamma$.

If the passive dynamic walking model had a positive Lyapunov exponent and did not fall down, we further inspected if the walking pattern was in fact chaotic and not random. This analysis was performed by calculating the Hurst exponent for a given simulated step interval data series [23,24]. A Hurst exponent greater than 0.05 indicated that the positive Lyapunov exponent was due to a persistent chaotic locomotive pattern. Conversely, Hurst exponents less than 0.05 indicated that the positive Lyapunov exponent was related to random noise in the model; not chaos. To calculate the Hurst exponent we used a re-scaled range algorithm [23,24]. The step interval data series was iteratively separated into non-overlapping bins of size $n$. The smallest bin contained 10 data points while the largest bin contained $n/2$ data points. Within each bin an integrated time series was constructed. The range within each bin was determined by computing the difference between the maximum and minimum values of the integrated time series. The range was then re-scaled by dividing the range by the standard deviation of the original data series contained in the bin. In the final step, the re-scaled ranges for the respective bins of equal size were averaged. The re-scaled range was related to the number of data points in the respective bins by a power law (Eq. (8); [23,24]).

$$\text{Re-scaled Range} = \left(\frac{a}{n}\right)^H$$

where $a$ was a constant, $n$ was the number of data points in the bin and $H$ was the Hurst exponent. The Hurst exponent was calculated from the slope of the log-log plot of the re-scaled range versus the number of data points in the bin [23,24]. The Chaos Data Analyzer (American Institute of Physics) software was used to calculate the Hurst exponents for each $\gamma$.

**Simulation Results**

Simulations of the passive dynamic walking model indicated that a period one locomotive attractor was present for $\gamma < 0.0169$ rad (Fig. 2A). This indicated that the model selected the same step time interval for every step of the continuous locomotive pattern. At $\gamma = 0.017$ rad, the locomotive pattern bifurcated from period one to period two Poincare map (Fig. 2B). A period two Poincare map indicated that the locomotive pattern alternated between two different step time intervals.

Beyond $\gamma = 0.017$ rad there was a cascade of bifurcations in the step time interval that lead to a strange attractor in the model’s locomotion (Fig. 3). At $\gamma = 0.0180$ rad (Fig. 3A), the model exhibited a period four locomotive pattern. This indicated that the model utilized four different step time intervals for locomotion. As $\gamma$ was increased to 0.01823 rad (Fig. 3B), the locomotive pattern bifurcated from a period four to a period eight locomotive pattern. Hence indicating that the locomotive pattern utilized eight different step time intervals as it traversed down the walking surface. Additional increases in $\gamma$ (Fig.
3C and D) resulted in further bifurcations in the step intervals chosen by the walking model. These bifurcations appeared to lead to a strange locomotive attractor where multiple step time intervals are chosen for a stable walking pattern (Fig. 3D).

Using a time delay of one, the global false nearest neighbors was used to calculate the embedding dimensions of the strange locomotive attractor (Fig. 3D). The algorithm indicated that the number of necessary embedding dimensions for chaotic analysis of the step interval data was two. Additionally, since the data was taken from a clean chaotic system, the percent of false nearest neighbors expectedly dropped from nearly 100% in dimension one to zero at dimension two (Fig. 4).

Sprott and Rowlands [25] suggested that the embedding dimension should be set somewhat higher than the expected dimension of the attractor when numerically calculating the largest Lyapunov exponent of the system. Therefore, we used an embedding dimension of three when calculating the largest Lyapunov exponents from the step interval data time series for each respective $\gamma$. Positive Lyapunov exponents were present from $0.01839 \text{ rad} < \gamma < 0.0189 \text{ rad}$

Fig. 2. Poincaré map of the step interval time for the passive dynamic walking model. (A) period-one locomotion at $\gamma = 0.007 \text{ rad}$, (B) period-two locomotion at $\gamma = 0.017 \text{ rad}$.
(Lyapunov exponent range = +0.002 to +0.158). This suggested the presence of chaotic locomotive patterns at the respective $\gamma$ greater than 0.01839 rad. The Hurst exponents for the integrated data series at these respective $\gamma$ confirmed the presence of chaotic locomotive patterns ($H = 0.98$ for all $\gamma > 0.01839$ rad). No further locomotive patterns were investigated beyond a $\gamma = 0.019$ rad because the model became unstable and would fall down.
Discussion

Several recent investigations have indicated that human locomotive patterns are rarely strictly periodic [2–7]. Rather, locomotive patterns fluctuate overtime with a chaotic structure. The origin of such complex physiological rhythms in locomotion has come under closer examination because it has been suggested that they are linked to the control mechanisms of the neuromuscular system [3–5]. In this investigation, we explored the possibility of using a passive dynamic double pendulum walking model as a template for exploring the principles of chaotic locomotion. The results of our simulations indicated that a passive dynamic walking model was capable of a cascade of bifurcations that lead to a chaotic walking pattern. Tools from mathematical chaos theory indicated that a chaotic walking pattern was present for $0.01839 \, \text{rad} < \gamma < 0.0189 \, \text{rad}$. These results indicate that the bipedal passive dynamic walking model can serve as a viable template where more elaborate models can be built. In the following sections we suggest how the passive dynamic walking model can be used as a template for future investigations of chaotic locomotive patterns.

The model presented in this investigation produced a chaotic walking pattern with no active control. This observation brings to question about how much of the chaotic dynamics observed in human locomotion are actually due to the mechanical properties of the bipedal system. It is possible that the structure of the human body may lead to governing equations that have chaos built into the bipedal walking system. Therefore, chaos may actually underlie the normal dynamics of the musculoskeletal system. Modifications in the structural morphology (i.e. limb configuration, inertia properties, etc.) and Newtonian forces acting on the bipedal walking system may influence the observed chaotic structure. The effect of these mechanical factors on the chaotic locomotive patterns can be modeled by modifying the governing and algebraic switching equations of the bipedal passive dynamic walking model presented in this investigation.

Recent investigations have suggested that the control of locomotion at intermediate and fast locomotive speeds may be governed in part by the natural passive dynamics present in the musculoskeletal system [12,9,26,27]. Based on this notion, it is possible that the passive dynamic feedback found in a chaotic locomotive system may play a role in the control and stability of locomotion. In a sense, the natural flexibility of the chaotic system may allow the musculoskeletal system to overcome perturbations experienced during locomotion without extensive neural control. Future investigations can address the influence of chaotic passive dynamics on the control of stable walking patterns by providing perturbations to the walking model presented in this investigation. The ability of the chaotic passive dynamic walking system to attenuate perturbations may reveal that a passive dynamic chaotic system can provide adaptive control of locomotion not found in a limit cycle system.

The difficulty on relying only on passive dynamics to control locomotion is that such a system lacks the true plasticity that has been observed in locomotive patterns. For example, humans and animals can instantaneously change their locomotive speed or type of movement pattern [28–32], attenuate external perturbations [33], and demonstrate locomotive stability in complex environments that contain obstacles [34–36]. Such complex locomotive dynamics appear to be related to active neural control mechanisms. If chaotic dynamics are inherent to the bipedal walking system, then how the nervous system takes advantage and utilizes the properties of the attractor may be a source of the observed differences in chaotic dynamics of locomotive patterns [3–5]. It has been demonstrated that
well-timed perturbations can lead to transitions to stable orbits present in the rich chaotic attractor [37–39]. Theoretically, the nervous system may select a desired locomotive pattern from among the infinite number of behaviors naturally present in the chaotic attractor. The ability of the nervous system to capitalize on the properties inherent to the chaotic system may allow for a healthy and flexible locomotive system. Currently, we are exploring if well timed lower extremity joint actuations applied to the passive dynamic walking model template can be used to switch between stable locomotive patterns available in the rich chaotic attractor as environmental circumstances change.

Conclusions

Chaotic dynamics are a central characteristic of human locomotion. In this investigation, we have shown that a passive dynamic walking model provides a viable template to examine these dynamics and to address future questions about the requisites for chaotic locomotion. This model can be used to address questions related to how mechanical factors influence the structure of the chaotic locomotive attractor, and how the neuromuscular system can be used as a controller for chaotic locomotion. Such models will enhance our understanding of chaotic locomotion.

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