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Let’s Make Patterns!: Symmetric Rubik’s Cube Permutations

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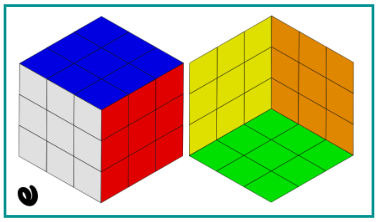
1 What is a Rubik’s Cube?

The Rubik’s cube is a cube that is split into 27 smaller cubes, arranged in a $3 \times 3 \times 3$ grid. These 27 “cubelets” can be manipulated in six different ways: Each of these manipulations rotate the 9 cubelets on one face of the Rubik’s cube a total of 90°, centered on the middle cubelet. These six actions collectively lead to millions of permutations of the cubelets overall: we can collect these permutations into one set, which we will label as $\mathbb{G}$. This set is a group under composition: the action of performing two permutations on a cube in succession. We can let e represent the identity element of $\mathbb{G}$, which is the ”nothing” permutation given by performing none of the six actions possible on the cube.

To track the movements of the cubelets, each of the six sides of the Rubik’s cube are painted with a different color: Red, Blue, White, Orange, Green, and Yellow. (The cubelet faces inside of the Rubik’s cube are usually black, but this inner color is not important for this project.) These colors may vary depending on which cube you have, but my Rubik’s cube has these colors such that red and orange are on opposite sides, as are blue and green and, additionally, white and yellow. Furthermore, the red, blue, and white sides are adjacent and are arranged counterclockwise.

To progress our research, we should begin by picking a “default” way to hold the cube: I chose to keep the red face facing me and the blue face on the top of the Rubik’s cube. Since I like to hold the Rubik’s cube in my left hand, it felt natural to bring the white face into view as well, leading to red, blue, and white always being in view. This causes their opposing colors—orange, green, and yellow, respectively—to be out of view.

This leads to a solution of an upcoming problem: how can I represent all of the cubelets on a two-dimensional piece of paper? I split the six faces into two groups: red, blue, and white, which are always in view; and orange, green, and yellow, which are always out of view. An example of this is shown in the figure to the right. Of course, these colors will vary as we manipulate the Rubik’s cube, so we will need to build a stronger definition; however, this should provide a good place to start.



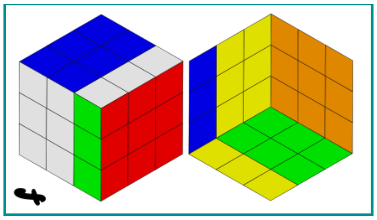
1.1 The Six Rotations

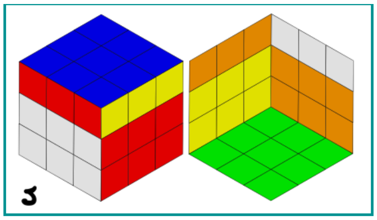
Let’s investigate the six rotations mentioned previously. When rotating one of the faces of the cube, nine cubelets are moved. The four vertices cycle among themselves, and the four edges cycle among themselves. The center cubelet, though, is rotated on itself: it essentially stays in place! This occurs for all six rotations, which means that the center cubelet of each face remains in place regardless of the rotations performed. As a result, the red, blue, white, orange, green, and yellow center pieces will always stay on the same side of the cube—as long as we do not rotate the whole cube out of the standard position. This allows us to identify the red center cubelet as always being in the front, the blue center cubelet on the top, and so forth. Since each face of the Rubik’s cube has only one color, this means that we can identify the six faces of the Rubik’s cube with the six colors painted on the sides, regardless of the number or types of

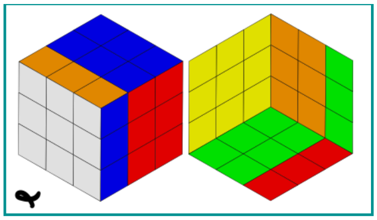
Side of the Cube	Color of Center
Front	Red
Up	Blue
Left	White
Back	Orange
Down	Green
Right	Yellow

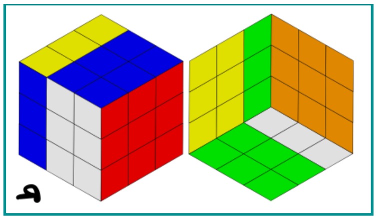
rotations performed on the cube. This association is explicitly listed in the table to the right.

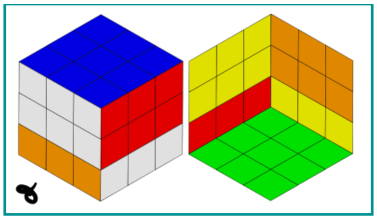
Additionally, we should label all six of the different rotations that we can perform. Since there is one for each side of the Rubik's cube, we can use the first letter for each side name[1]. For example, we can let a fancy "f" represent a rotation of the front side of the cube; we will choose this to be a $90°$ clockwise rotation. Thus, our six right-angle rotations are f, u, ℓ, b, d, and r. These are depicted in the following diagrams:

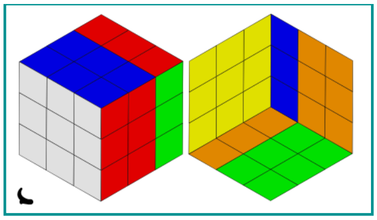




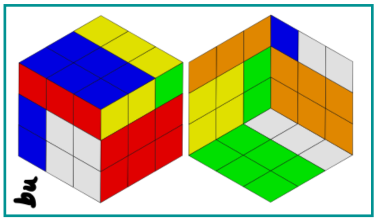








These six rotations generate all of the permutations of the Rubik's cube: everything that can be done on a Rubik's cube is just a combination of these permutations. For example, the action of rotating the horizontal center layer is the same as rotating the Up and Down layers an appropriate distance, which can be done by using u and d respectively. As a result, we can write every permutation of the Rubik's cube as a string of the six letters f, u, ℓ, b, d, and r. While a right-to-left direction would more closely follow function notation, I instead chose to write the letters left-to-right. Hence, the permutation created by rotating the Back layer and then the Up layer is labeled as bu. To help demonstrate this, bu is depicted to the right.



Sometimes we will need to perform a $180°$ rotation of a side of the cube. This can be done by performing two successive $90°$ clockwise rotations: we can thus notate the $180°$ rotation of the front face as ff. As a shorthand, we can label this as f^2. Then the other $180°$ rotations that can be performed are u^2, ℓ^2, b^2, d^2, and r^2.

What about a $90°$ counterclockwise rotation? We can perform a rotation three consecutive times to produce the same result: $f^3, u^3, \ell^3, b^3, d^3$, and r^3. For simplicity, we can notate these with the negative exponent -1, such as $f^{-1} \coloneqq f^3$. This helps reflect the fact that ff^{-1} and $f^{-1}f$ are both the identity permutation e.

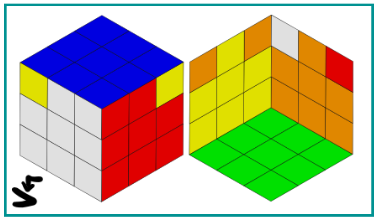
More generally for any $x \in \mathbb{G}$ and any $n \in \mathbb{N}$, we can define x^n to be the product of n copies of x. We can then let x^{-1} denote the inverse permutation (which is given by the fact that $\mathbb{G}$ is a group), and then

[1] This is why I (and the Rubik's cube community) chose to use "Up" and "Down" instead of a more intuitive labeling such as "Top" and "Bottom"—the "B" in "Bottom" would conflict with the "B" in "Back."

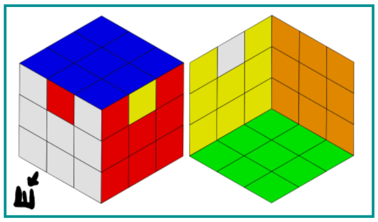
define $x^{-n} \coloneqq (x^{-1})^n$ for all $x \in \mathbb{G}$ and $n \in \mathbb{N}$. Lastly, we define $x^0 \coloneqq e$ to be the identity permutation for all $x \in \mathbb{G}$.

1.2 Special Permutations

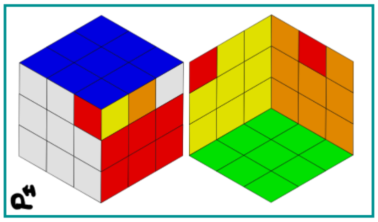
We want to create a list of elements in $\mathbb{G}$ that will help with some of the arguments later in this paper. The first and simplest permutation is the "nothing" permutation e, which was introduced previously. The other more complex permutations are $V_{\circlearrowright}, V_{\Lsh}, E_{\nwarrow}, E_{\curvearrowright}$, and $P_{\text{↥}}$. To assist with explaining how to create $V_{\circlearrowright}$, I also include two other permutations named T and $\bar{T}$.



We shall start with the Vertex Permutor $V_{\Lsh}$ that is defined as $V_{\Lsh} \coloneqq r^{-1}fr^{-1}b^2rf^{-1}r^{-1}b^2r^2$. This permutation is displayed in the diagram to the right. This algorithm is used with the classic solving algorithm, namely when positioning the vertex cubelets with yellow sides into their correct spaces. This algorithm moves around three of the four vertices on the top of the cube: it moves the front right vertex to the back left position, moves the back left vertex to the back right position, and moves the back right vertex to the front right position. This is done while keeping the blue side of all three vertices on the top. Since this cycles three vertices, the order of $V_{\Lsh}$ is $|V_{\Lsh}| = 3$. Thus, $V_{\Lsh}^3 = e$ and $V_{\Lsh}^{-1} = V_{\Lsh}^2$.

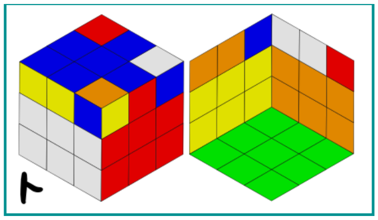


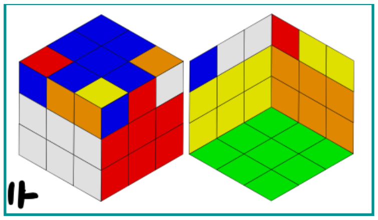
Next, we will introduce the Edge Permutor $E_{\nwarrow} \coloneqq f^2u\ell r^{-1}f^2\ell^{-1}ruf^2$. This permutation is displayed in the diagram to the right. It is usually used in the classic solving algorithm in the final step which moves the edge cubelets with yellow faces to their final positions. Instead of cycling three vertices, this permutation cycles three edge cubelets on the top. In particular, the top front edge is moved to the top left position, the top left edge moves to the top right position, and the top right edge is moved to the top front position. During this manipulation, all these sides keep their blue faces on the top of the cube. Just like with $V_{\Lsh}$, its order is $|E_{\nwarrow}| = 3$, which implies that $E_{\nwarrow}^3 = e$ and $E_{\nwarrow}^{-1} = E_{\nwarrow}^2$.



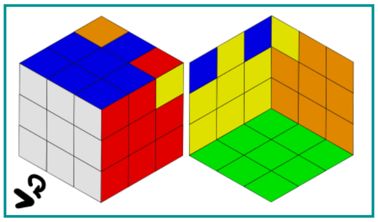
The third cubelet-moving permutation is the Parity Permutor $P_{\text{↥}} \coloneqq V_{\Lsh}E_{\nwarrow}^{-1}u^{-1}E_{\nwarrow}^{-1}$, displayed to the right. While performing this permutation is time-consuming, its explanation is a bit shorter: the two top front vertices swap places, and the front and back edges on the top swap places—all while keeping their blue sides on top. For a change of pace, it has order $|P_{\text{↥}}| = 2$. As a result, $P_{\text{↥}}^2 = e$ and $P_{\text{↥}}^{-1} = P_{\text{↥}}$.

Our final cubelet-moving permutation is the Vertex Transformers $T \coloneqq \ell^{-1}u^{-1}\ell u^{-1}\ell^{-1}u^{-2}\ell$ and $\bar{T} \coloneqq rur^{-1}uru^2r^{-1}$. These are depicted below where T is the left diagram and $\bar{T}$ is the right diagram. The classic solving algorithm provides $\bar{T}$, and T is a reflection of $\bar{T}$ onto the left side of the cube. These two permutations move around the vertices and edges on the top of the Rubik's cube in a complicated way.

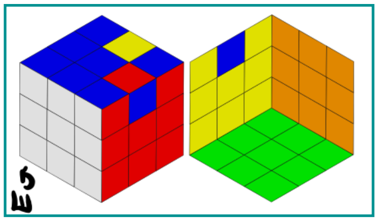




The remaining two permutations to introduce keep all of the cubelets in their original places, but they rotate two particular cubelets in place. Our first one is the Vertex Rotator $V_{\circlearrowright}$, which is built with the Vertex Transformers T and $\overline{T}$; its diagram is provided to the right of this paragraph. It is made by performing $V_{\circlearrowright} \coloneqq T\overline{T}$. This rotates the two top corners on the right side of the cube. The corner at the front is rotated $120°$ clockwise, and the back corner is rotated $120°$ counterclockwise. Effectively, this permuation moves the blue side of both vertices to the yellow side of the cube. Note that $V_{\circlearrowright}$ has order 3, which means that $V_{\circlearrowright}^{3} = e$. Additionally it follows that $V_{\circlearrowright}^{-1} = V_{\circlearrowright}^{2}$.



The final algorithm is used to rotate two edges. The Edge Rotator is $E_{\curvearrowleft} \coloneqq E_{\nwarrow}^{-1} rbu^{-1} E_{\nwarrow} ub^{-1} r$, which will flip the top front edge along and the top right edge. It is depicted to the right. It has order 2, which implies that $E_{\curvearrowleft}^{2} = e$ and $E_{\curvearrowleft}^{-1} = E_{\curvearrowleft}$.



1.3 Reoriented Permutations

Occasionally, we may want to perform one of the permutations above at a different position on the cube; for example, what if we wanted to flip some of the edges on the green face of the cube? For this, we can provide reoriented versions of the permutations previously produced.

Note that all of the previously mentioned permutations were introduced in the standard position, which has the Red side of the Rubik's cube on the front and the Blue side on the top. We create the symbol RB to symbolize this position. We can then place this character in place of the exponent to signify the orientation that the permutation is used at. For example, $E_{\curvearrowleft}$ can be written as $E_{\curvearrowleft}^{\mathrm{RB}}$, although this is more complicated.

We will not use this notation when a permutation is performed while the Rubik's cube is in its standard orientation, but it will be used in all other orientations of the cube. For example, we can flip the edges on the green face of the cube by performing $E_{\curvearrowleft}^{\mathrm{RG}}$ (which flips the front and left edges on the Down side of the Rubik's cube) and then performing $E_{\curvearrowleft}^{\mathrm{OG}}$ (which flips the remaining right and back edges on the Down side of the Rubik's cube).

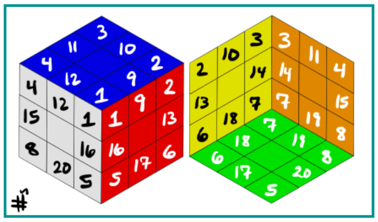
As a final note, we will also give the special notation $P_{\uparrow}^{-\mathrm{WB}} \coloneqq \left(P_{\uparrow}^{\mathrm{WB}}\right)^{-1}$. A similar notation applies for all of the other permutations and orientations possible. Note that this does not extend to other integers of "exponents."

2 It's Just $A_{20} \cap S_{\mathcal{A}}S_{\mathcal{B}}$…

While there are 27 cubelets in the Rubik's cube, not all of them move around when the six rotations are applied. Clearly, the one cubelet in the center of the cube does not move, but we also know that the center cubelet on each face of the Rubik's cube effectively does not move. In effect, there are only 20 cubelets that move around the cube when u, d, f, b, ℓ and r are applied. This naturally induces a homomorphism into the Symmetric group S_{20}:

$$\phi\colon \mathbb{G} \to S_{20}$$

where each number $\{1, 2, \cdots, 20\}$ represents one of the moving cubelets of the Rubik's cube. In particular, we can choose the numbering provided in the diagram to the right. We can restrict this further, as shown in the following sections. Ultimately, we will be able to extend this concept so that we have a useful way of representing $\mathbb{G}$.



2.1 ϕ maps Onto Subgroup $\mathbb{P} \subseteq S_{20}$

Define the sets $\mathcal{A} = \{1, 2, 3 \cdots, 8\}$ and $\mathcal{B} = \{9, 10, 11, \cdots, 20\}$. We can restrict the range of ϕ to be $\mathbb{P} \coloneqq A_{20} \cap S_{\mathcal{A}}S_{\mathcal{B}}$, and this restricted definition of ϕ is onto. (After completing this proof, we will let ϕ represent the restricted homomorphism $\phi|^{\mathbb{P}}\colon \mathbb{G} \to \mathbb{P}$.)

Recall that $\mathbb{G}$ is generated by u, d, f, b, ℓ and r. Under the labeling provided, we have that $\phi(u) = (1, 2, 3, 4)(9, 10, 11, 12)$. This must be in A_{20} since it is an even permutation, and we have that $(1, 2, 3, 4) \in S_{\mathcal{A}}$ and that $(9, 10, 11, 12) \in S_{\mathcal{B}}$. Therefore $\phi(u) \in A_{20} \cap S_{\mathcal{A}}S_{\mathcal{B}}$, and it can be checked that this will also apply for d, f, b, ℓ, and r. Then $\phi(\mathbb{G}) \subseteq A_{20} \cap S_{\mathcal{A}}S_{\mathcal{B}}$ since these six permutations generate $\mathbb{G}$.

If we use $V_{\Lsh}$ at different orientations around the cube, we can generate any of the permutations of vertices that cycle three adjacent vertices, such as $\phi\left(V_{\Lsh}\right) = (243)$. This can be rewritten as $(24)(43)$; we can also create the pair $\phi\left(V_{\Lsh}^{\mathrm{GO}}\right) = (834) = (34)(48)$, and multiplying this to $\phi\left(V_{\Lsh}\right)$ provides $(24)(43) \cdot (34)(48) = (248)$, which is a cycle of three nonadjacent vertices. (While 4 and 8 are adjacent, neither are adjacent to 2.) If we continue this method, we will be able to create any permutation of $S_{\mathcal{A}}$ of cycle type (3)—that is, any cycle of three of the vertices. Since these generate $A_{\mathcal{A}}$, it follows that $A_{\mathcal{A}} \subseteq \phi(\mathbb{G})$.

By a similar argument, we can show that $A_{\mathcal{B}} \subseteq \phi(\mathbb{G})$ by using the $\phi(E_{\nwarrow})$ at different orientations for $E_{\nwarrow}$. Since we have that $A_{\mathcal{A}}$ and $A_{\mathcal{B}}$ are in $\phi(\mathbb{G})$, it will then follow that $A_{\mathcal{A}}A_{\mathcal{B}}$ will be inside of $\phi(\mathbb{G})$.

For our next step, we can note that $\phi(P_{\updownarrow}) = (1, 2)(9, 11)$, which then shows that the element $(1, 2)(9, 11)$ will be in $\phi(\mathbb{G})$. From this and the fact that $A_{\mathcal{A}}A_{\mathcal{B}} \subseteq \phi(\mathbb{G})$, we can conclude that the set $(1, 2)(9, 11) \cdot A_{\mathcal{A}}A_{\mathcal{B}}$ is inside of $\phi(\mathbb{G})$.

Lastly, we want to rewrite our set $A_{20} \cap S_{\mathcal{A}}S_{\mathcal{B}}$. Let $\alpha \in A_{20} \cap S_{\mathcal{A}}S_{\mathcal{B}}$ be arbitrary. Because $\alpha \in S_{\mathcal{A}}S_{\mathcal{B}}$, we can rewrite $\alpha = \alpha_{\mathcal{A}}\alpha_{\mathcal{B}}$ for some $\alpha_{\mathcal{A}} \in S_{\mathcal{A}}$ and for some $\alpha_{\mathcal{B}} \in S_{\mathcal{B}}$. Meanwhile, we can use the fact that $\alpha \in A_{20}$ to note that $\alpha_{\mathcal{A}}\alpha_{\mathcal{B}} \in A_{20}$, which implies that $\alpha_{\mathcal{A}}$ and $\alpha_{\mathcal{B}}$ are either both even permutations or odd permutations. If the former is true, we can write $\alpha_{\mathcal{A}} \in A_{\mathcal{A}}$ and $\alpha_{\mathcal{B}} \in A_{\mathcal{B}}$, which then shows that $\alpha = \alpha_{\mathcal{A}}\alpha_{\mathcal{B}} \in A_{\mathcal{A}}A_{\mathcal{B}}$; using the fact that $A_{\mathcal{A}}A_{\mathcal{B}} \subseteq \phi(\mathbb{G})$, this implies that $\alpha \in \phi(\mathbb{G})$.

Otherwise, $\alpha_{\mathcal{A}}$ and $\alpha_{\mathcal{B}}$ are both odd permutations. We can write the set of odd permutations of $S_{\mathcal{A}}$ as $(1,2)A_{\mathcal{A}}$, so it follows that $\alpha_{\mathcal{A}} \in (1,2)A_{\mathcal{A}}$. Thus, $\alpha_{\mathcal{A}} = (1,2)\alpha'_{\mathcal{A}}$ for some $\alpha'_{\mathcal{A}} \in A_{\mathcal{A}}$. Likewise, we can show that $\alpha_{\mathcal{B}} \in (9,11)A_{\mathcal{B}}$, which then means that there is some $\alpha'_{\mathcal{B}} \in A_{\mathcal{B}}$ such that $\alpha_{\mathcal{B}} = (9,11)\alpha'_{\mathcal{B}}$. We can then write $\alpha = \alpha_{\mathcal{A}}\alpha_{\mathcal{B}}$ as $(1,2)\alpha'_{\mathcal{A}} \cdot (9,11)\alpha'_{\mathcal{B}}$. Because $\alpha'_{\mathcal{A}}$ and $(9,11)$ are disjoint permutations of S_{20} (they do not interact with the same integers), they can commute. This property will allow us to write $\alpha = (1,2)(9,11) \cdot \alpha'_{\mathcal{A}}\alpha'_{\mathcal{B}}$. We know that $\alpha'_{\mathcal{A}} \in A_{\mathcal{A}}$ and $\alpha'_{\mathcal{B}} \in A_{\mathcal{B}}$, so this implies that $\alpha \in (1,2)(9,11) \cdot A_{\mathcal{A}}A_{\mathcal{B}}$. We have previously shown that $(1,2)(9,11) \cdot A_{\mathcal{A}}A_{\mathcal{B}} \subseteq \phi(\mathbb{G})$, so it follows that $\alpha \in \phi(\mathbb{G})$.

In conclusion, any arbitrary $\alpha \in A_{20} \cap S_{\mathcal{A}}S_{\mathcal{B}}$ will be in $\phi(\mathbb{G})$, so we can conclude that $(A_{20} \cap S_{\mathcal{A}}S_{\mathcal{B}}) \subseteq \phi(\mathbb{G})$. We have shown the reverse containment at the beginning of this proof, so it follows that $(A_{20} \cap S_{\mathcal{A}}S_{\mathcal{B}}) = \phi(\mathbb{G})$. ■

2.1.1 Definition of $\mathbb{K} = \ker\phi$ and Consequences

We know that the kernel of any homomorphism will be a normal subgroup of the domain. Hence, we can define $\mathbb{K} = \ker\phi$ and note that $\mathbb{K}$ will be a normal subgroup of $\mathbb{G}$. As a result, $|\mathbb{G}| = |\mathbb{K}| \cdot |\mathbb{G}/\mathbb{K}|$, where $\mathbb{G}/\mathbb{K}$ is the factor group of $\mathbb{G}$ by $\mathbb{K}$.

From the structure of the proof above, the First Isomorphism Theorem provides some isomorphism $\Phi\colon \mathbb{G}/\mathbb{K} \to \mathbb{P}$. This shows that $\mathbb{G}/\mathbb{K}$ is isomorphic to $\mathbb{P} = A_{20} \cap S_{\mathcal{A}}S_{\mathcal{B}}$. We can observe that this can be alternatively written as $A_{\mathcal{A}}A_{\mathcal{B}} \cup (1,2)(9,11)A_{\mathcal{A}}A_{\mathcal{B}}$. Since both of these sets are disjoint and contain $|A_{\mathcal{A}}| \cdot |A_{\mathcal{B}}| = \frac{8!}{2} \cdot \frac{12!}{2}$ elements, it follows that:

$$|\mathbb{G}/\mathbb{K}| = \left(\frac{8!}{2} \cdot \frac{12!}{2}\right) + \left(\frac{8!}{2} \cdot \frac{12!}{2}\right) = \frac{8! \cdot 12!}{2}$$

What are the elements of $\mathbb{K}$? I claim that this is the set generated by some versions of $V_{\circlearrowright}$ and $E_{\curvearrowright}$ done at different orientations of the Rubik's Cube. These can be shown to be in $\mathbb{K}$ because neither move the cubelets out of their original positions, but we will need to prove that they generate $\mathbb{K}$, which will be done in an upcoming section. Before this, we need to create a new set that generates $\mathbb{G}$:

2.2 Alternate Way to Generate $\mathbb{G}$.

Recall that u, d, f, b, ℓ, and r generate $\mathbb{G}$. While these elements are easy to understand, they are not optimal for understanding the limitations of what permutations can be created. Instead, we can introduce a set $\mathcal{P}$ containing a selected set of permutations:

$$\mathcal{P} = \{V_{\Lsh}, E_{\nwarrow}, P_{↥}, V_{\circlearrowright}, E_{\curvearrowright}, \quad V_{\Lsh}^{\mathrm{OB}}, E_{\nwarrow}^{\mathrm{OB}}, \quad V_{\Lsh}^{\mathrm{WR}}, E_{\nwarrow}^{\mathrm{WR}}, \quad V_{\Lsh}^{\mathrm{YO}}, E_{\nwarrow}^{\mathrm{YO}}, \quad E_{\nwarrow}^{\mathrm{RY}}, \quad E_{\nwarrow}^{\mathrm{OW}}\}$$

(The strange spacing is used to help clarify the orientations needed.) This set will generate $\mathbb{G}$.

Firstly, note that since the orders of $V_{\Lsh}$ and $E_{\nwarrow}$ are both 3, it follows that $V_{\Lsh}^{-1} = V_{\Lsh}^{2}$ and $E_{\nwarrow}^{-1} = E_{\nwarrow}^{2}$, which shows that the inverses of $V_{\Lsh}$ and $E_{\nwarrow}$ are generated by $\mathcal{P}$. A similar method shows that any element of finite order—such as the other versions of $V_{\Lsh}$ and $E_{\nwarrow}$ in $\mathcal{P}$—will also be generated by $\mathcal{P}$.

If we can show that these generate all six of the original permutations, it will follow that they can generate $\mathbb{G}$. We can manually compute that:

$$u = P_{↥}V_{\Lsh}E_{\nwarrow}E_{\nwarrow}^{-\mathrm{OB}}$$

$$f = E_{\nwarrow}^{\mathrm{RY}}E_{\nwarrow}^{-\mathrm{OB}}P_{↥}E_{\nwarrow}^{\mathrm{OB}}E_{\curvearrowright}E_{\nwarrow}^{-\mathrm{RY}}E_{\nwarrow}^{\mathrm{WR}}V_{\Lsh}^{-\mathrm{OB}}V_{\Lsh}^{\mathrm{WR}}V_{\Lsh}^{\mathrm{OB}}(u^{-1}V_{\circlearrowright}^{-1}u)$$

$$b = E_{\nwarrow}^{\mathrm{OW}} E_{\nwarrow}^{-1} (u^2 P_{↥} u^2) E_{\nwarrow} (u^2 E_{⤻} u^2) E_{\nwarrow}^{-\mathrm{OW}} E_{\nwarrow}^{\mathrm{YO}} V_{\Lsh}^{-1} V_{\Lsh}^{\mathrm{YO}} V_{\Lsh} (u V_{\circlearrowright}^{-1} u^{-1})$$

$$\ell = E_{\nwarrow}^{-\mathrm{OW}} f b^{-1} P_{↥} V_{\Lsh} b f^{-1} E_{\nwarrow}^{-\mathrm{OW}}$$

$$r = E_{\nwarrow}^{-\mathrm{RY}} b f^{-1} (u^2 P_{↥} u^2) V_{\Lsh}^{\mathrm{OB}} f b^{-1} E_{\nwarrow}^{-\mathrm{RY}}$$

$$d = \ell^2 r^2 E_{\nwarrow}^{-\mathrm{WR}} E_{\nwarrow}^{-\mathrm{YO}} u E_{\nwarrow} E_{\nwarrow}^{\mathrm{OB}} E_{\nwarrow} E_{\nwarrow}^{\mathrm{WR}} E_{\nwarrow}^{\mathrm{YO}} r^2 \ell^2$$

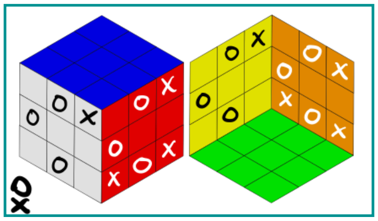
We calculate these from top to bottom: once we show that u is generated by $\mathcal{P}$, we can then use it in later equations since we can substitute for $u = P_{↥} V_{\Lsh}^{\mathrm{RB}} E_{\nwarrow}^{\mathrm{RB}} E_{\nwarrow}^{-\mathrm{OB}}$ as needed. In addition, we can substitute $u^{-1} = u^3$ to show that u^{-1} is generated by $\mathcal{P}$ as well. Together, these facts allow us to conclude that f is generated by $\mathcal{P}$. Similarly, we can use f, b, ℓ and r and their inverses for the equations after showing that each is generated by $\mathcal{P}$.

Note that b and r are similar to f and ℓ, respectively, in their construction: parenthetical values help show where the top layer is rotated to simulate $P_{↥}$, $E_{⤻}$, and $V_{\circlearrowright}$ done in different rotations. Namely, we have that:

$$u^2 P_{↥} u^2 = P_{↥}^{\mathrm{OB}} \qquad u^2 E_{⤻} u^2 = E_{⤻}^{\mathrm{OB}} \qquad u^{-1} V_{\circlearrowright}^{-1} u = V_{\circlearrowright}^{-\mathrm{WB}} \qquad u V_{\circlearrowright}^{-1} u^{-1} = V_{\circlearrowright}^{-\mathrm{YB}}$$

Then altogether, we have that $\mathcal{P}$ generates the six original permutations; because the six permutations generate $\mathbb{G}$, it follows that $\mathcal{P}$ will also generate $\mathbb{G}$. ■

2.2.1 XO Notation and $\mathbb{G}/\mathbb{K} \cong \mathbb{H}$



The elements of $\mathcal{P}$ were chosen to have a specific structure. The copies of $V_{\Lsh}$ cause the vertices of the Rubik's cube to permute around the cube, but only enough to allow each to access any other point on the cube. The figure to the right depicts a cube that helps display this pattern: When permuting the vertices, the $V_{\Lsh}$ map the "X" sides of the vertices to the "X" side of other vertices. The same can be said of $P_{↥}$ as well. This shows that the only way to rotate a vertex with the elements of $\mathcal{P}$ is with the single element $V_{\circlearrowright}$. A similar property applies to the edges: In the diagram, the "O" side of an edge can only be mapped to another "O" edge when using the $E_{\nwarrow}$ and $P_{↥}$.

Out of all the elements of $\mathcal{P}$, only $V_{\circlearrowright}$ and $E_{⤻}$ can move the "X" or "O" of a cubelet out of position. The other elements can be used to generate a subgroup $\mathbb{H}$:

$$\mathbb{H} \coloneqq \langle \mathcal{P} \setminus \{V_{\circlearrowright}, E_{⤻}\} \rangle$$

Informally, we can see that for every valid permutation $\sigma \in \mathbb{P}$ of the edges and vertices, there exists some element $h \in \mathbb{H}$ such that $\phi(h) = \sigma$. In fact, ϕ establishes a bijection between $\mathbb{H}$ and $\mathbb{P}$ when the domain is restricted to $\mathbb{H}$, as shown in the following proof:

First, we should show that $\phi|_{\mathbb{H}} : \mathbb{H} \to \mathbb{P}$ is injective. For this, suppose that there were two elements $h_1, h_2 \in \mathbb{H}$ such that $\phi(h_1) = \phi(h_2)$. It would then follow that $\phi(h_1) \cdot \left(\phi(h_2)\right)^{-1} = \phi(h_1 h_2^{-1}) = (\,)$, the identity element of $\mathbb{P}$. This implies that $h_1 h_2^{-1}$ is an element of $\mathbb{H}$ where every cubelet remains in its original place. But since none of the generators of $\mathbb{H}$ can move a cubelet's "X" or "O" side, none of the elements of $\mathbb{H}$ can do so. Thus $h_1 h_2^{-1}$ is some permutation of the Rubik's cube such that none of the cubelets can be moved out of their original positions (because $\phi(h_1 h_2^{-1}) = (\,)$) and that cannot rotate the cubelets in place (due to the “XO” argument). The only element of $\mathbb{G}$ that satisfies both of these requirements is e, so we have that $h_1 h_2^{-1} = e$. Then we can quickly conclude that $h_1 = h_2$. In summary, since $\phi(h_1) = \phi(h_2)$ implies that $h_1 = h_2$, it follows that $\phi|_{\mathbb{H}}$ is injective.

Now let $\sigma \in \mathbb{P}$ be arbitrary. Since $\phi: \mathbb{G} \to \mathbb{P}$ is surjective, there exists some $g \in \mathbb{G}$ such that $\phi(g) = \sigma$. Since $\mathcal{P}$ generates $\mathbb{G}$, we can write g as the product of elements in $\mathcal{P}$, say $g = \prod_{i=1}^{t} x_i$ for some $t \in \mathbb{N}$ and $x_i \in \mathcal{P}$. If we remove every copy of $E_{\curvearrowright}$ and $V_{\circlearrowright}$ in this product, we get another element $h = \prod_{i=1}^{t} y_i$, where for all $1 \leq i \leq t$ we define $y_i = e$ if $x_i \in \{E_{\curvearrowright}, V_{\circlearrowright}\}$ and $y_i = x_i$ otherwise. We can then observe that $\phi(x_i) = \phi(y_i)$ for all i since $\phi(E_{\curvearrowright}) = (\,) = \phi(e)$ and $\phi(V_{\circlearrowright}) = (\,) = \phi(e)$, so we can conclude that:

$$\sigma = \phi(g) = \phi\left(\prod_{i=1}^{t} x_i\right) = \prod_{i=1}^{t} \phi(x_i) = \prod_{i=1}^{t} \phi(y_i) = \phi\left(\prod_{i=1}^{t} y_i\right) = \phi(h)$$

Therefore $\phi(h) = \sigma$. Meanwhile, $h \in \mathbb{H}$ since $h = \prod_{i=1}^{t} y_i$, where $y_i \in \{e\} \cup \mathcal{P} \setminus \{E_{\curvearrowright}, V_{\circlearrowright}\}$ are all elements of $\mathbb{H}$. In summary, for every $\sigma \in \mathbb{P}$ there exists some $h \in \mathbb{H}$ such that $\phi(h) = \sigma$, so we can conclude that $\phi|_{\mathbb{H}}$ is also surjective.

Since $\phi|_{\mathbb{H}}$ is a homomorphism that is both injective and surjective, it follows that it will produce an isomorphism between $\mathbb{P}$ and $\mathbb{H}$. ■

2.2.2 $\mathbb{H}$ is Set of Representatives of $\mathbb{G}/\mathbb{K}$

Every element of $\mathbb{G}/\mathbb{K}$ can be expressed as $h\mathbb{K}$ for some $h \in \mathbb{H}$. Additionally, if $h_1\mathbb{K} = h_2\mathbb{K}$, then it follows that $h_1 = h_2$. This then means that $\mathbb{H}$ is a set of coset representatives of $\mathbb{G}/\mathbb{K}$.

We showed in section 2.1.1 that $\Phi: \mathbb{G}/\mathbb{K} \to \mathbb{P}$ is an isomorphism defined by $\Phi(g\mathbb{K}) = \phi(g)$. Meanwhile, we have that $\phi|_{\mathbb{H}}: \mathbb{H} \to \mathbb{P}$ is an isomorphism as shown in the prior section. We can combine these together into an isomorphism $\left(\phi|_{\mathbb{H}}^{-1} \circ \Phi\right): \mathbb{G}/\mathbb{K} \to \mathbb{H}$.

Let $g\mathbb{K} \in \mathbb{G}/\mathbb{K}$ be arbitrary. Then $\left(\phi|_{\mathbb{H}}^{-1} \circ \Phi\right)(g\mathbb{K})$ is equal to some $h \in \mathbb{H}$. On the other hand, the definition of Φ shows that $\Phi(h\mathbb{K}) = \phi(h)$, so it follows that $\left(\phi|_{\mathbb{H}}^{-1} \circ \Phi\right)(h\mathbb{K}) = \left(\phi|_{\mathbb{H}}^{-1} \circ \phi\right)(h)$. Replacing $\phi(h)$ with $\phi|_{\mathbb{H}}(h)$, this value simplifies to be $\left(\phi|_{\mathbb{H}}^{-1} \circ \phi|_{\mathbb{H}}\right)(h)$. It then becomes clear that $\left(\phi|_{\mathbb{H}}^{-1} \circ \Phi\right)(h\mathbb{K}) = h$. As a result:

$$\left(\phi|_{\mathbb{H}}^{-1} \circ \Phi\right)(g\mathbb{K}) = h = \left(\phi|_{\mathbb{H}}^{-1} \circ \Phi\right)(h\mathbb{K})$$

But since $\left(\phi|_{\mathbb{H}}^{-1} \circ \Phi\right)$ is a bijection, this implies that $g\mathbb{K} = h\mathbb{K}$. Thus, every coset $g\mathbb{K}$ can be expressed as $h\mathbb{K}$ for some $h \in \mathbb{H}$. ◪

Now suppose that there are two elements $h_1, h_2 \in \mathbb{H}$ where $h_1\mathbb{K} = h_2\mathbb{K}$. Using a similar argument to above, we can find that:

$$\left(\phi|_{\mathbb{H}}^{-1} \circ \Phi\right)(h_1\mathbb{K}) = h_1 \qquad \left(\phi|_{\mathbb{H}}^{-1} \circ \Phi\right)(h_2\mathbb{K}) = h_2$$

Since $h_1\mathbb{K} = h_2\mathbb{K}$, we can then set both of these equal to each other:

$$h_1 = \left(\phi|_{\mathbb{H}}^{-1} \circ \Phi\right)(h_1\mathbb{K}) = \left(\phi|_{\mathbb{H}}^{-1} \circ \Phi\right)(h_2\mathbb{K}) = h_2$$

Thus $h_1 = h_2$. ◪

Due to both of the proofs presented above, it follows that $\mathbb{H}$ will be a set of coset representatives of $\mathbb{G}/\mathbb{K}$ as desired. ■

2.3 Structure of $\mathbb{K}$

We can understand that vertices can be rotated in place, but we need to make a notion of how much a vertex has been rotated—even if the vertex is not in its “solved” position. The XO notation provides an excellent solution to this problem: It is the amount that the “X” side of a vertex has been rotated from the “X” position for the position it currently is in. Thus $V_{\circlearrowright}$, which is the only way to rotate a vertex in $\mathcal{P}$, will rotate the top front right vertex $240°$ and the top back right vertex $120°$—considering all angles to be counterclockwise. Since we can move any two vertices to the two positions that $V_{\circlearrowright}$ rotates, we can rotate any arbitrary pair of vertices.

In other words, we can rotate any pair of vertices such that the sum of the angles is $360°$. If we rotate a second pair of vertices afterwards, the sum of the angles rotated will be $2 \cdot 360° = 720°$. And more generally, any rotation of vertices performed on the vertices must have a sum of angles rotated equal to some multiple of $360°$.

We can build a similar notion for flipped edges: an edge is flipped if its “O” side is not aligned with the “O” position for the spot it currently is in. We can see that $E_{\curvearrowright}$ will flip two edges. When we extend this reasoning over several flipped pairs of edges, the total number of flipped edges across the cube must always be an even number.

If we take any element of $\mathbb{K}$, we know that it must keep all cubelets in their original positions. The only significant difference that k can have is by rotating vertices and flipping edges; as shown above, this can only happen where the sum of the vertices’ rotations must be a multiple of $360°$, and there must be an even number of edges flipped. In fact, $\mathbb{K}$ contains a permutation for every possible way to rotate the vertices and flip edges that has this property, so we can note that these two properties are the key distinction of $\mathbb{K}$. In other words, $\mathbb{K}$ *is* the set of all permutations of the cube where the vertices’ rotations add up to a multiple of $360°$ and where the number of flipped edges is even.

As a result, we can find that $\mathbb{K} \cong \mathbb{Z}_2^{11} \times \mathbb{Z}_3^7$:

We will define the map $\phi\colon \mathbb{K} \to \mathbb{Z}_3^7 \times \mathbb{Z}_2^{11}$ in the following way for any $k \in \mathbb{K}$:

- For each vertex position $1 \leq i \leq 7$, set the i^{th} coordinate of $\phi(k)$ (i.e. the i^{th} coordinate of the $\mathbb{Z}_3^7$ portion) to be 0 if the vertex in that position is not rotated at all, 1 if the vertex in that position is rotated $120°$, or 2 if the vertex in that position is rotated $240°$.
- For each edge position $9 \leq i \leq 19$, set the $(i-1)^{\text{th}}$ coordinate of $\phi(k)$ (i.e. the $(i-8)^{\text{th}}$ coordinate of the $\mathbb{Z}_2^{11}$ portion) to be 1 if the edge in that position is flipped, or 0 otherwise.

Firstly, this map is well-defined since there is only one possible status for all of the cubelets being measured. Additionally, we can determine that ϕ is additive: if k_1 rotates some cubelet i a total of A_i degrees and if k_2 rotates i a total of B_i degrees, then $k_1 k_2$ will rotate i a total of $(A_i + B_i)$ degrees. This applies for each cubelet i, so it follows that:

$$
\begin{aligned}
(\forall k_1, k_2 \in \mathbb{K}) \qquad \phi(k_1) + \phi(k_2) &= (A_1, \cdots, A_7, A_9, \cdots, A_{19}) + (B_1, \cdots, B_7, B_9, \cdots, B_{19}) \\
&= (A_1 + B_1, \cdots, A_7 + B_7, A_9 + B_9, \cdots, A_{19} + B_{19}) \\
&= \phi(k_1 k_2)
\end{aligned}
$$

Then ϕ will be a homomorphism.

Now suppose that k_1 and k_2 are two elements of $\mathbb{K}$ such that $\phi(k_1) = \phi(k_2)$. Then $\phi(k_1 k_2^{-1}) = (0, \cdots, 0)$ is the zero element of $\mathbb{Z}_3^7 \times \mathbb{Z}_2^{11}$. By the definition of ϕ, this then means that $k_1 k_2^{-1}$ does not rotate any of the cubelets except possibly the 8^{th} vertex and the 20^{th} edge (in terms of how the cubelets were numbered at the beginning of this chapter). When considering the vertices, we know that the sum of the angles rotated should be a multiple of $360°$; because seven of the eight are $0°$, the eighth vertex must be rotated a multiple of $360°$, which implies that it is essentially not rotated at all. Similarly, we know that eleven of the twelve edges are not flipped, so the twelfth edge must not be flipped either in order for the number of flipped edges to be even.

Then none of the cubelets for $k_1 k_2^{-1}$ are flipped or rotated; since the elements of $\mathbb{K}$ cannot move vertices around, it follows that $k_1 k_2^{-1}$ cannot move the cubelets around either. The only element of $\mathbb{G}$ that satisfies both of these conditions is e, so $k_1 k_2^{-1} = e$. This implies that $k_1 = k_2$. In summary, $\phi(k_1) = \phi(k_2)$ implies that $k_1 = k_2$ for all $k_1, k_2 \in \mathbb{K}$, so we can conclude that ϕ will be injective.

Lastly, we can see that ϕ will be surjective: for any $\sigma = (a_1, \cdots, a_7, a_9, \cdots, a_{19}) \in \mathbb{Z}_3^7 \times \mathbb{Z}_2^{11}$, we can create an element $k \in \mathbb{K}$ that maps to it in the following manner: For the vertices $1 \leq i \leq 7$, define k_i to be the element of $\mathbb{K}$ that only rotates the i^{th} vertex and the 8^{th} vertex a total of $a_i \cdot 120°$ and $a_i \cdot 120°$ respectively. For the edges $9 \leq i \leq 19$, define k_i to be e if $a_i = 0$, or to be the element of $\mathbb{K}$ that only flips both the i^{th} edge and the 20^{th} edge if $a_i = 1$. Since neither the 8^{th} vertex nor the 20^{th} edge affect the value of $\phi(k_i)$, each $\phi(k_i)$ is equal to $(0, \cdots, 0, a_i, 0, \cdots, 0)$, where the a_i is in the same position as in σ. It then follows that if we define $k \coloneqq k_1 \cdots k_7 k_9 \cdots k_{19}$, then $\phi(k) = \sigma$. Then for every $\sigma \in \mathbb{Z}_3^7 \times \mathbb{Z}_2^{11}$, there exists some $k \in \mathbb{K}$ such that $\phi(k) = \sigma$, so we can conclude that ϕ is surjective.

Overall, we know that ϕ is a bijective homomorphism, so then $\phi: \mathbb{K} \to \mathbb{Z}_3^7 \times \mathbb{Z}_2^{11}$ is an isomorphism. ■

2.3.1 Order of $\mathbb{G}$

We then have that $|\mathbb{G}| = 12! \cdot 8! \cdot 2^{10} \cdot 3^7$. Therefore, the number of permutations of a Rubik's cube is:

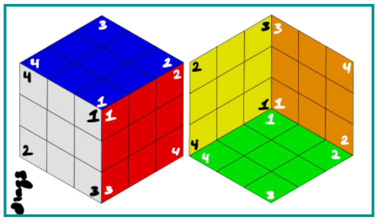
$$43, 252, 003, 274, 489, 856, 000$$

which is approximately 43 quintillion (short scale).

Recall from 2.1.1 that $|\mathbb{G}| = |\mathbb{G}/\mathbb{K}| \cdot |\mathbb{K}|$ and $|\mathbb{G}/\mathbb{K}| = 12! \cdot 8!/2$. Since $|\mathbb{K}| = \left|\mathbb{Z}_2^{11} \times \mathbb{Z}_3^7\right| = 2^{11} \cdot 3^7$, this result comes from multiplying both values together. ■

3 Cubes are Very Symmetric

As you might expect, cubes have many different types of symmetry, but what exactly counts as a symmetry of a cube, and how many are there?

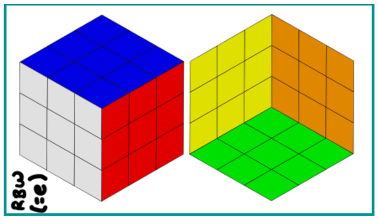


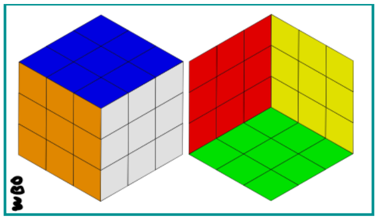
There is a well-known result that the cube has 48 different types of symmetry and that the symmetry group of the cube—which will be denoted $\mathbb{S}$—is isomorphic to $\mathbb{Z}_2 \times S_4$. In particular, an isomorphism $\Sigma: \mathbb{S} \to (\mathbb{Z}_2 \times S_4)$ can be established in the following way:

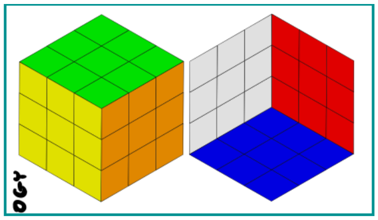
- For $\tau \in \mathbb{S}$, define $z \in \mathbb{Z}_2$ and $\sigma \in S_4$ such that $\Sigma(\tau) = (z, \sigma)$.
- Label the long interior diagonals of the cube with the numbers 1 to 4, such as done in the diagram to the right.
- It follows that τ is equivalent to following these two steps:
    - If $z = 1$, invert each diagonal by moving each vertex to the furthest vertex relative to it. If $z = 0$, do not do anything to the cube in this step.
    - Then the cube is rotated such that the diagonals are permuted according to σ. For example, the first diagonal must be taken to $\sigma(1)$.
        - This second step must be done without reflecting or inverting the cube. (This step is called an *orientation-preserving symmetry*.)

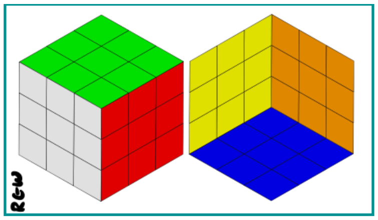
In general, we will write $\tau \sim (z, \sigma)$ to help explain how τ operates. An alternative way of writing τ can be presented with the form XYZ, where X, Y, and Z are selected from the set $\{\mathrm{R}, \mathrm{B}, \mathrm{W}, \mathrm{O}, \mathrm{G}, \mathrm{Y}\}$. What three letters do we replace X, Y, and Z with? If we apply τ onto a blank Rubik's cube in standard position, it will permute the faces of the cube; the colors of the front, up, and right sides of the cube uniquely identify this permutation. We let X represent the color of the front face, Y be the color of the up face, and Z be the color of the right face; we select the first letter of the color from $\{\mathrm{R}, \mathrm{B}, \mathrm{W}, \mathrm{O}, \mathrm{G}, \mathrm{Y}\}$ for each part.

For example, we have that $\mathrm{RBW} \sim \big((\,), 0\big)$ is the identity symmetry (which does not rotate the Rubik's cube) and $\mathrm{WBO} \sim \big((1234), 0\big)$ is a $90°$ rotation of the Rubik's cube. Also, $\mathrm{OGY} \sim \big((\,), 1\big)$ is the inverted symmetry (from swapping opposite colors, e.g., Red and Orange), and $\mathrm{RGW} \sim \big((13)(24), 1\big)$ essentially mirrors the cube across a horizontal cross-section of the cube.









3.1 Types of Symmetries

With $\mathbb{S} \cong \mathbb{Z}_2 \times S_4$, it follows that there are $|\mathbb{S}| = |\mathbb{Z}_2| \cdot |S_4| = 2 \cdot 4! = 48$ different symmetries to consider. It would be daunting to have to consider each one-by-one, especially when we get to analyzing which permutations "display" each symmetry, so we will want to classify them into "similar" categories.

For example, consider how $\mathrm{WBO} \sim \big((1234), 0\big)$ and $\mathrm{RWG} \sim \big((1243), 0\big)$ both rotate the Rubik's cube $90°$. In standard position, the rotation axis of WBO goes through the up and down sides of the Rubik's cube (which are Blue and Green, respectively), and if we look at the top of the Rubik's cube, this rotation is counterclockwise. Meanwhile, if we arrange the cube such that the Blue side is on the front side and the Orange side is on the up side, the rotation axis of RWG goes through the up and down sides of the Rubik's cube (which are Orange and Red, respectively), and if we look at the top of the Rubik's cube, this rotation is still counterclockwise. These symmetries move cubelets with the same movements, the exception being that they may use each movement on a different cubelet.

A good place to start is to consider the $4! = 24$ elements of S_4, which are as follows:

$$(\ )$$

$$(1234) \quad (1243) \quad (1324) \quad (1342) \quad (1423) \quad (1432)$$

$$(12)(34) \quad (13)(24) \quad (14)(23)$$

$$(123) \quad (132) \quad (124) \quad (142) \quad (134) \quad (143) \quad (234) \quad (243)$$

$$(12) \quad (13) \quad (14) \quad (23) \quad (24) \quad (34)$$

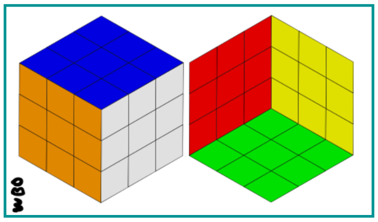
We can see that there are five types of elements in S_4, one for each row above. These categories can be firmly established by considering the cycle type of each (the definition of which is exactly as you expect if you look at how each row is organized), but it may be more intuitive to consider how each permutation in a row perform similar "movements" of each value, although they may differ on *which* values are moved in which way.

The idea I am trying to convey is that, for example, (1234) and (1324) both permute the four numbers in one cycle, while the other values of S_4 do not cycle all four numbers in one cycle. This allows us to differentiate these two elements from others, such as (12) or (123) which may leave one number in place, or like $(13)(24)$ which cycles the numbers across two disjoint cycles.

Using these five classes of elements in S_4, we can add the other differentiation across the first coordinate of $\mathbb{Z}_2 \times S_4$ to obtain ten classes of symmetries in $\mathbb{S}$. We can pick one from each to act as a "representative" for each class:

$\mathrm{RBW} \sim \big((\ ), 0\big)$	$\mathrm{WBO} \sim \big((1234), 0\big)$	$\mathrm{OBY} \sim \big((13)(24), 0\big)$	$\mathrm{BWR} \sim \big((234), 0\big)$	$\mathrm{BRY} \sim \big((12), 0\big)$
$\mathrm{OGY} \sim \big((\ ), 1\big)$	$\mathrm{YGR} \sim \big((1234), 1\big)$	$\mathrm{RGW} \sim \big((13)(24), 1\big)$	$\mathrm{GYO} \sim \big((234), 1\big)$	$\mathrm{GOW} \sim \big((12), 1\big)$

Each class has a unique way in which they affect the cubelets of the Rubik's cube. To help demonstrate this, I introduce the notion of a *rotation set*, which is basically a glorified element of S_{20} that shows how the symmetry permutes the cubelets when it is applied to the Rubik's cube in standard position. While each symmetry moves nearly all of the 27 cubelets on the cube (namely, the face-center cubelets are also moved), we only are concerned about how they move the 20 cubelets on the edges and vertices of the cube.



For an example of a rotation set, consider the WBO symmetry, which is displayed to the right. We will first focus on the four vertices on the top. Vertex 1—which is red, blue, and white—is moved to the position of Vertex 2. And Vertex 2 moves to the position of Vertex 3. This causes Vertex 3 to move to the position of Vertex 4, and Vertex 4 is moved to the position of Vertex 1. We can then collect these together as the rotation set $\{1,2,3,4\}$. Since a similar scenario occurs for the bottom vertices, we also have the rotation set $\{5,6,7,8\}$. The edges are similarly placed into rotation sets of four elements each.

The rotation sets can have varying numbers of elements. Some rotation sets may have up to six elements, while others may consist of only one cubelet.

The following subsections provides a brief summary of each of the ten classes of symmetries in $\mathbb{S}$. Each class will list all of the symmetries that belong to that class with a general description for the class. Additionally, an example from each class will be provided to help visualize each class.

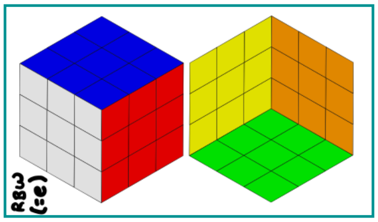
The remainder of Chapter 3 is listing these classes. You can read through each type or reference back to them in later chapters as needed. Honestly, the format is repetitive, which could make for a boring read; I would recommend you to briefly look through them, then reference back as needed later.

3.1.1 Type $\mathrm{RBW} \sim \big((\ ),0\big)$

$$\mathrm{RBW} \sim \big((\,),0\big)$$

This category has only one element: the identity symmetry of the cube. This is the symmetry produced by not rotating the cube at all. This symmetry has a trivial color map:

$$\mathrm{R} \mapsto \mathrm{R} \quad \mathrm{B} \mapsto \mathrm{B} \quad \mathrm{W} \mapsto \mathrm{W} \quad \mathrm{O} \mapsto \mathrm{O} \quad \mathrm{G} \mapsto \mathrm{G} \quad \mathrm{Y} \mapsto \mathrm{Y}$$



This symmetry does not generate any other symmetries. Rather, every symmetry will eventually produce this symmetry when iterated multiple times.

The cubelets are not moved around at all by this symmetry, so every cubelet is in its own rotation set. Thus, the rotation sets are:

$$\{1\}\ \{2\}\ \{3\}\ \{4\}\ \{5\}\ \{6\}\ \{7\}\ \{8\}$$

$$\{9\}\ \{10\}\ \{11\}\ \{12\}\ \{13\}\ \{14\}\ \{15\}\ \{16\}\ \{17\}\ \{18\}\ \{19\}\ \{20\}$$

3.1.2 Type $\mathrm{WBO} \sim \big((1234),0\big)$

$$\mathrm{WBO} \sim \big((1234),0\big) \qquad \mathrm{BOW} \sim \big((1324),0\big) \qquad \mathrm{RWG} \sim \big((1243),0\big)$$

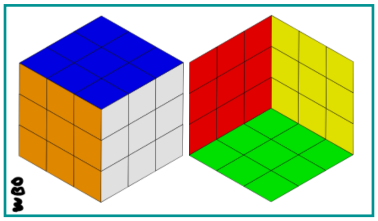
$$\mathrm{YBR} \sim \big((1432),0\big) \qquad \mathrm{GRW} \sim \big((1423),0\big) \qquad \mathrm{RYB} \sim \big((1342),0\big)$$

These take our cube and rotate it 90° as if it were on a flat surface. These can be either clockwise or counterclockwise rotations: $\mathrm{WBO} \sim \big((1234),0\big)$ is a counterclockwise rotation, and $\mathrm{YBO} \sim \big((1432),0\big)$ is a clockwise rotation. (Of course, “clockwise” and “counterclockwise” are relative to the way you orient the cube when looking at the rotation.) These rotations need not be horizontal: for example, $\mathrm{RWG} \sim \big((1243),0\big)$ is also in this category. Each has their own color map; for example, the color map of WBO will be:

$$\mathrm{W} \mapsto \mathrm{R} \quad \mathrm{B} \mapsto \mathrm{B} \quad \mathrm{O} \mapsto \mathrm{W} \quad \mathrm{Y} \mapsto \mathrm{O} \quad \mathrm{G} \mapsto \mathrm{G} \quad \mathrm{R} \mapsto \mathrm{Y}$$

These elements come in pairs: for example, if we apply WBO on a Rubik's cube three times, it will be the same as performing YBR. Additionally, each will generate an element of the type $\mathrm{OBY} \sim \big((13)(24),0\big)$ after being performed twice, and each will generate the identity element $\mathrm{RGB} \sim \big((\,),0\big)$ after being performed four times.

Every vertex is in a rotation set of four elements, so there are two disjoint vertex rotation sets. Similarly, the edges are in rotation sets of four elements, so there are three disjoint edge rotation sets. For example, the rotation sets of WBO are as follows:



$$\{1,2,3,4\} \quad \{5,6,7,8\}$$

$$\{9,10,11,12\} \quad \{13,14,15,16\} \quad \{17,18,19,20\}$$

3.1.3 Type $\mathrm{OBY} \sim \big((13)(24),0\big)$

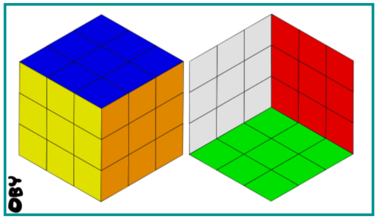
$$\mathrm{OBY} \sim \big((13)(24),0\big) \qquad \mathrm{OGW} \sim \big((12)(34),0\big) \qquad \mathrm{RGY} \sim \big((14)(23),0\big)$$

These take our cube and rotate it 180° as if it were on a flat surface. However, these rotations need not be horizontal: for example, $\mathrm{OGW} \sim \big((12)(34),0\big)$ is a vertical rotation. Each has their own color map; for example, the color map of OBY will be:

$$\mathrm{O} \mapsto \mathrm{R} \quad \mathrm{B} \mapsto \mathrm{B} \quad \mathrm{Y} \mapsto \mathrm{W} \quad \mathrm{R} \mapsto \mathrm{O} \quad \mathrm{G} \mapsto \mathrm{G} \quad \mathrm{W} \mapsto \mathrm{Y}$$

These elements will generate the identity when performed on a Rubik's cube two times.

Every vertex is in a rotation set of two elements, so there are four disjoint rotation sets. Similarly, the edges are in rotation sets of two elements, so there are six disjoint rotation sets. For example, the rotation sets of OBY are as follows:



$$\{1,3\} \quad \{2,4\} \quad \{5,7\} \quad \{6,8\}$$

$$\{9,11\} \quad \{10,12\} \quad \{13,15\} \quad \{14,16\} \quad \{17,19\} \quad \{18,20\}$$

3.1.4 Type $\mathrm{BWR} \sim \big((234),0\big)$

$$\mathrm{BWR} \sim \big((234),0\big) \qquad \mathrm{YRG} \sim \big((134),0\big) \qquad \mathrm{GYR} \sim \big((124),0\big) \qquad \mathrm{YOB} \sim \big((123),0\big)$$

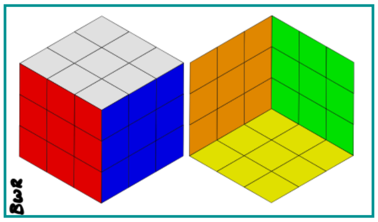
$$\mathrm{WRB} \sim \big((243),0\big) \qquad \mathrm{BYO} \sim \big((143),0\big) \qquad \mathrm{WOG} \sim \big((142),0\big) \qquad \mathrm{GWO} \sim \big((132),0\big)$$

These take our cube and rotate it 120° about a vertex. These can be either clockwise or counterclockwise rotations: $\mathrm{BWR} \sim \big((234),0\big)$ is a clockwise rotation, and $\mathrm{WRB} \sim \big((243),0\big)$ is a counterclockwise rotation. Each has their own color map; for example, the color map of BWR will be:

$$\mathrm{B} \mapsto \mathrm{R} \quad \mathrm{W} \mapsto \mathrm{B} \quad \mathrm{R} \mapsto \mathrm{W} \quad \mathrm{G} \mapsto \mathrm{O} \quad \mathrm{Y} \mapsto \mathrm{G} \quad \mathrm{O} \mapsto \mathrm{Y}$$

These elements come in pairs: for example, if we apply BWR on the cube twice, the result will be WRB. Additionally, each will generate the identity element $\mathrm{RGB} \sim \big((\,), 0\big)$ when applied to the cube three times.

The rotation's axis goes through two vertices; these will be in rotation sets of size 1. The remaining vertices are in rotation sets of 3 elements. Meanwhile, all of the edges are in rotation sets of size 3. Hence, there are two 1-element rotation sets of vertices, and two 3-element rotation sets of vertices. Also, there are four rotation sets of edges. For example, the rotation sets of BWR are as follows:



$$\{1\} \quad \{2,5,4\} \quad \{3,6,8\} \quad \{7\}$$

$$\{9,16,12\} \quad \{10,17,15\} \quad \{11,13,20\} \quad \{14,18,19\}$$

3.1.5 Type $\mathrm{BRY} \sim \big((12), 0\big)$

$$\mathrm{BRY} \sim \big((12), 0\big) \qquad \mathrm{WGR} \sim \big((13), 0\big) \qquad \mathrm{OWB} \sim \big((14), 0\big)$$

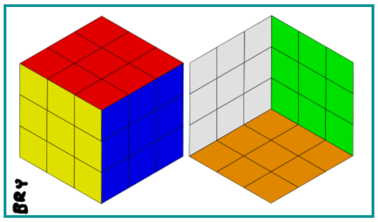
$$\mathrm{OYG} \sim \big((23), 0\big) \qquad \mathrm{YGO} \sim \big((24), 0\big) \qquad \mathrm{GOY} \sim \big((34), 0\big)$$

These take our cube and rotates it 180° about an edge. These are probably the strangest rotations that can be done on the cube. Each has their own color map; for example, the color map of BRY will be:

$$\mathrm{B} \mapsto \mathrm{R} \quad \mathrm{R} \mapsto \mathrm{B} \quad \mathrm{Y} \mapsto \mathrm{W} \quad \mathrm{G} \mapsto \mathrm{O} \quad \mathrm{O} \mapsto \mathrm{G} \quad \mathrm{W} \mapsto \mathrm{Y}$$

If we apply any of these rotations on a cube twice, we will end up with the identity element $\mathrm{RGB} \sim \big((\,), 0\big)$.

The rotation's axis goes through two edges; these will be in rotation sets of size 1. The remaining edges are in rotation sets of 2 elements. Meanwhile, all of the vertices are in rotation sets of size 2. Hence, there are two 1-element rotation sets of edges, and five 2-element rotation sets of edges. Also, there are four rotation sets of vertices. For example, the rotation sets of BRY will be:



$$\{1,2\} \quad \{3,5\} \quad \{4,6\} \quad \{7,8\}$$

$$\{9\} \quad \{10,16\} \quad \{11,17\} \quad \{12,13\} \quad \{14,20\} \quad \{15,18\} \quad \{19\}$$

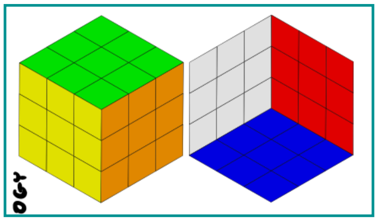
3.1.6 Type $\mathrm{OGY} \sim \big((\ \ ), 1\big)$

$$\mathrm{OGY} \sim \big((\,), 1\big)$$

This symmetry comes from inverting every diagonal of the cube: imagine mapping each vertex to the vertex furthest from it, or imagine scaling the cube by "a factor of -1". This will have the inversion color map that maps each color to its "opposite" color:

$$\mathrm{O} \mapsto \mathrm{R} \quad \mathrm{G} \mapsto \mathrm{B} \quad \mathrm{Y} \mapsto \mathrm{W} \quad \mathrm{R} \mapsto \mathrm{O} \quad \mathrm{B} \mapsto \mathrm{G} \quad \mathrm{W} \mapsto \mathrm{Y}$$

This symmetry will produce the identity symmetry when iterated with itself.



Each vertex and edge is swapped with its opposing vertex or edge, so it follows that each cubelet will be in a rotation set of two elements. Thus, there are four vertex rotation sets, and six edge rotation sets. The rotation sets of this symmetry will be:

$$\{1,7\} \quad \{2,8\} \quad \{3,5\} \quad \{4,6\}$$

$$\{9,19\} \quad \{10,20\} \quad \{11,17\} \quad \{12,18\} \quad \{13,15\} \quad \{14,16\}$$

3.1.7 Type $\mathrm{YGR} \sim \big((1234),1\big)$

$$\mathrm{YGR} \sim \big((1234),1\big) \qquad \mathrm{GRY} \sim \big((1324),1\big) \qquad \mathrm{OYB} \sim \big((1243),1\big)$$

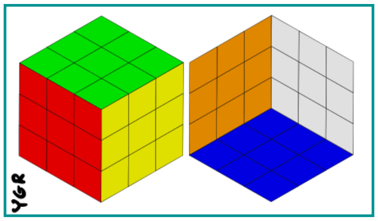
$$\mathrm{WGO} \sim \big((1432),1\big) \qquad \mathrm{BOY} \sim \big((1423),1\big) \qquad \mathrm{OWG} \sim \big((1342),0\big)$$

These can be thought of as rotating the cube 90° on a flat surface, then mirroring the cube horizontally. The direction rotated will be opposite the direction of its corresponding version of type $\mathrm{WBO} \sim \big((1234),0\big)$: in particular, YGR involves a clockwise rotation, but WBO is a counterclockwise rotation. This is because $\mathrm{YGR} = \mathrm{OGY} \circ \mathrm{WBO}$; similar facts hold for the other elements. Of course, the "flat surface" does not need to be horizontal, as is the case with most of the other elements in this category. Each has their own color map; for example, the color map of YGR will be:

$$\mathrm{Y} \mapsto \mathrm{R} \quad \mathrm{G} \mapsto \mathrm{B} \quad \mathrm{R} \mapsto \mathrm{W} \quad \mathrm{W} \mapsto \mathrm{O} \quad \mathrm{B} \mapsto \mathrm{G} \quad \mathrm{O} \mapsto \mathrm{Y}$$

These elements come in pairs: for example, if we apply YGR on a Rubik's cube three times, it will be the same as performing WGO. Additionally, each will generate an element of the type $\mathrm{OBY} \sim \big((13)(24),0\big)$ after being performed twice, and each will generate the identity element $\mathrm{RGB} \sim \big((\,),0\big)$ after being performed four times.

Every vertex is in a rotation set of four elements, so there are two disjoint vertex rotation sets. Similarly, the edges are in rotation sets of four elements, so there are three disjoint edge rotation sets. For example, the rotation sets of YGR will be as follows:



$$\{1,8,3,6\} \quad \{2,5,4,7\}$$

$$\{9,20,11,18\} \quad \{10,17,12,19\} \quad \{13,16,15,14\}$$

3.1.8 Type $\mathrm{RGW} \sim \big((13)(24),1\big)$

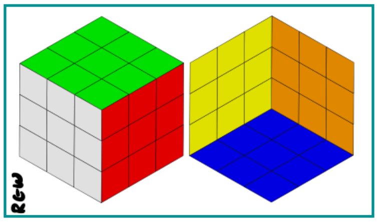
$$\mathrm{RGW} \sim \big((13)(24),1\big) \qquad \mathrm{RBY} \sim \big((12)(34),1\big) \qquad \mathrm{OBW} \sim \big((14)(23),1\big)$$

These take our cube and reflect it across a horizontal axis. Alternatively, it can be considered as rotating the cube by 180° on a flat surface, then inverting the cube: This latter perspective is informed by the fact that $\mathrm{RGW} = \mathrm{OGY} \circ \mathrm{OBY}$ and the fact that $\mathrm{OBY} \sim \big((13)(24),0\big)$ is a rotation of the cube by 180° on a flat surface. Each has their own color map; for example, the color map of RGW will be:

$$\mathrm{R} \mapsto \mathrm{R} \quad \mathrm{G} \mapsto \mathrm{B} \quad \mathrm{W} \mapsto \mathrm{W} \quad \mathrm{O} \mapsto \mathrm{O} \quad \mathrm{B} \mapsto \mathrm{G} \quad \mathrm{Y} \mapsto \mathrm{Y}$$

These elements will generate the identity when performed on a Rubik's cube two times.

Every vertex is in a rotation set of two elements, so there are four rotation sets. Meanwhile, the edges along the axis of the reflection remain in place while the other edges are in rotation groups of size 2. Hence, there are four 1-element rotation sets and four 2-element rotation sets. Therefore, the rotation set of RGW looks like:



$$\{1,5\} \quad \{2,6\} \quad \{3,7\} \quad \{4,8\}$$

$$\{9,17\} \quad \{10,18\} \quad \{11,19\} \quad \{12,20\} \quad \{13\} \quad \{14\} \quad \{15\} \quad \{16\}$$

3.1.9 Type $\mathrm{GYO} \sim \big((234),1\big)$

$$\mathrm{GYO} \sim \big((234),1\big) \qquad \mathrm{WOB} \sim \big((134),1\big) \qquad \mathrm{BWO} \sim \big((124),1\big) \qquad \mathrm{WRG} \sim \big((123),1\big)$$

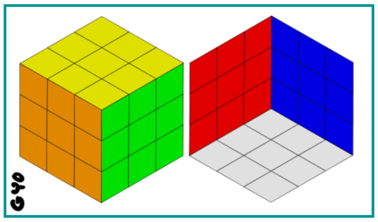
$$\mathrm{YOG} \sim \big((243),1\big) \qquad \mathrm{GWR} \sim \big((143),1\big) \qquad \mathrm{YRB} \sim \big((142),1\big) \qquad \mathrm{BYR} \sim \big((132),1\big)$$

These invert the Rubik's cube and then rotate it 120° about a vertex. Ultimately, this will "cycle" the colors in a zig-zag around the cube. The rotation can be either clockwise or counterclockwise rotation: $\mathrm{GYO} \sim \big((234),1\big)$ is made with a clockwise rotation, and $\mathrm{YOG} \sim \big((243),1\big)$ is made with a counterclockwise rotation. Each has their own color map; for example, the color map of GYO will be:

$$\mathrm{G} \mapsto \mathrm{R} \quad \mathrm{Y} \mapsto \mathrm{B} \quad \mathrm{O} \mapsto \mathrm{W} \quad \mathrm{B} \mapsto \mathrm{O} \quad \mathrm{W} \mapsto \mathrm{G} \quad \mathrm{R} \mapsto \mathrm{Y}$$

These elements come in pairs: for example, if we apply GYO on the cube five times, the result will be YOG; alternatively, we can see that $\mathrm{YOG} \circ \mathrm{GYO}$ is the identity symmetry. Additionally, each will generate two symmetries of type $\mathrm{BWR} \sim \big((234),0\big)$, the inversion symmetry $\mathrm{OGY} \sim \big((\,),1\big)$, and the identity element $\mathrm{RGB} \sim \big((\,),0\big)$. For GYO, the two elements of type $\mathrm{BWR} \sim \big((234),0\big)$ are $\mathrm{GYO}^2 = \mathrm{WRB} \sim \big((243),0\big)$ and $\mathrm{GYO}^4 = \mathrm{BWR} \sim \big((234),0\big)$.

When cycling the colors of the cube, two vertices remain in place while the remaining six vertices form one rotation set. The three edges near either of the one-element rotation set vertices form a rotation set, and the remaining six edges form another rotation set. For example, the rotation sets for GYO will be:



$$\{1\} \quad \{2,3,4,8,5,6\} \quad \{7\}$$

$$\{9,12,16\} \quad \{10,11,15,20,17,13\} \quad \{14,19,18\}$$

3.1.10 Type $\mathrm{GOW} \sim \big((12),1\big)$

$$\mathrm{GOW} \sim \big((12),1\big) \qquad \mathrm{YBO} \sim \big((13),1\big) \qquad \mathrm{RYG} \sim \big((14),1\big)$$

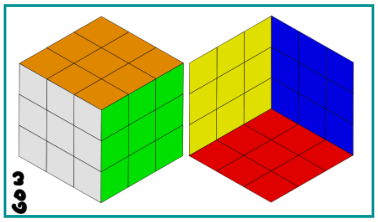
$$\mathrm{RWB} \sim \big((23),1\big) \qquad \mathrm{WBR} \sim \big((24),1\big) \qquad \mathrm{BRW} \sim \big((34),1\big)$$

These reflect the cube across a plane that divides the cube into two triangular prisms. In other words, they fix two opposite colors, then swap the remaining colors in adjacent pairs. Each has their own color map; for example, the color map of GOW will be:

$$\mathrm{G} \mapsto \mathrm{R} \quad \mathrm{O} \mapsto \mathrm{B} \quad \mathrm{W} \mapsto \mathrm{W} \quad \mathrm{B} \mapsto \mathrm{O} \quad \mathrm{R} \mapsto \mathrm{G} \quad \mathrm{Y} \mapsto \mathrm{Y}$$

If we apply any of these rotations on a cube twice, we will end up with the identity element $\mathrm{RGB} \sim \big((\,),0\big)$.

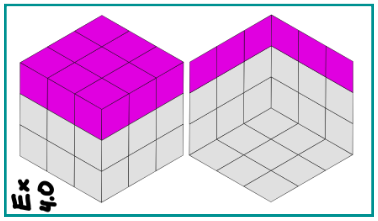
The plane of reflection goes through two edges and four vertices; these will all be in rotation sets of size 1. The remaining vertices and edges are in rotation sets of 2 elements.
For example, the rotation sets of GOW will be:



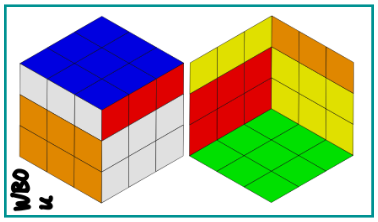
$$\{1,8\} \quad \{2,7\} \quad \{3\} \quad \{4\} \quad \{5\} \quad \{6\}$$

$$\{9,19\} \quad \{10,14\} \quad \{11\} \quad \{12,15\} \quad \{13,18\} \quad \{16,20\} \quad \{17\}$$

4 Get Excited!



Consider the hypothetical cube to the right. Intuitively, we know that it can be rotated 90° and maintain its pattern: more specifically, it *displays* the 90° rotation symmetry WBO. If we let $\sigma = \text{WBO}$ and let g represent the hypothetical cube, it follows that $\sigma(g) = g$. Notably, we are using σ as a function on g. While clearly $g \notin \mathbb{G}$, the idea is to find a way to transfer this idea to $g \in \mathbb{G}$.



We'll start this idea with the $\sigma(g)$ side. Our template will be for $\sigma = \text{WBO}$ and $g = u$. When we consider $\sigma(u)$, we first apply the permutation u onto the Rubik's cube, then rotate the Rubik's cube according to WBO. This results in the diagram to the right:

We can see that u and $\sigma(u)$ have the same pattern, but they have different colors. This is in opposition to our opening example where $\sigma(g) = g$ were the same: there was no color difference. To rectify this, we can invent a "color map" that repaints the colors of the cube with different colors. Letting this be τ_σ, we then have $\sigma(g) = \tau_\sigma(u)$.

We need to make this notion of a "color map" more rigorous. The following parts are designed to specify how this concept works, a better way of writing these concepts, and to ultimately design a plan for concluding this project.

4.1 Rewriting $\sigma(g)$

When we perform a symmetry $\sigma \in \mathbb{S}$ on our cube, we are essentially permuting the cubelets around the cube. This would suggest some way of considering σ as an element of $\mathbb{G}$, but this is not quite right: for example, OGY causes every corner cubelet to invert, which deforms them to the point where we cannot physically rotate our cube in a way to make it work. One case of this is the Red-Blue-White corner. The normal Rubik's cube has these colors in counterclockwise order on the cubelet. After applying OGY, though, this becomes a clockwise order; there isn't a way to apply some element of $\mathbb{G}$ to get this to occur normally.

This is why we need to introduce a broader set than $\mathbb{G}$ in order to house all $\sigma \in \mathbb{S}$. We can define $\mathbb{G}!$ to be this set. We want this set to allow for permuting *any* cubelets in any logical way (i.e. we cannot move vertex 1 to the position of edge 9). This means that permutations that are not possible in $\mathbb{G}$, such as $\{1, 2\}$, will become possible in $\mathbb{G}!$. Hence, the permutations of the cubelets can be represented by any element of $S_\mathcal{A} S_\mathcal{B}$.

We also want vertices and edges to be invertible, such as the example provided with the Red-Blue-White corner. For vertices, this means that each vertex position will be associated with an element of S_3 instead of an element of $\mathbb{Z}_3$; that way, we can essentially permute the colors of any cubelet position in any way we want. We will define 1 to represent the side with an "X," then 2 and 3 to be the remaining sides in counterclockwise order. The coordinates of S_3^8 list each vertex position in order; thus if this looks like $(\delta_1, \delta_2, \cdots, \delta_8)$ for some $\delta_i \in S_3$, then δ_1 represents vertex position 1, and δ_2 represents vertex position 2, and so forth.

Meanwhile, edge positions would technically be associated with S_2 for a similar reason; however, we can stick with $\mathbb{Z}_2$ since $S_2 \cong \mathbb{Z}_2$. We shall let $0 \in \mathbb{Z}_2$ represent that the edge in the edge position has not been flipped, and $1 \in \mathbb{Z}_2$ represent that the edge in the edge position has been flipped. We also list these edge positions in order in $\mathbb{Z}_2^{12}$: if this looks like $(\delta_9, \delta_{10}, \cdots, \delta_{20})$ for some $\delta_i \in \mathbb{Z}_2$, then δ_9 represents the state of edge position 9, and δ_{10} represents the state of edge position 10, and so forth.

We then have an isomorphism:

$$\Phi: \mathbb{G}! \to S_{\mathcal{A}} S_{\mathcal{B}} \ltimes_\rho (S_3^8 \times \mathbb{Z}_2^{12})$$

Where ρ is the action of permuting the coordinates of $S_3^8 \times \mathbb{Z}_2^{12}$ by $S_{\mathcal{A}} S_{\mathcal{B}}$ in the intended way:

$$(\forall \sigma \in S_{\mathcal{A}} S_{\mathcal{B}}) \left(\forall (\delta_i)_{i=1}^{20} \in (S_3^{20} \times \mathbb{Z}_2^{12})\right) \qquad \rho(\sigma)\left((\delta_i)_{i=1}^{20}\right) = \left(\delta_{\sigma(i)}\right)_{i=1}^{20}$$

This will operate in a similar way to how ϕ operated in Chapter 2. We can consider $\mathbb{G}$ as a subgroup of $\mathbb{G}!$ by the injective map $(\Phi^{-1} \circ \pi \circ \tau \circ \phi): \mathbb{G} \to \mathbb{G}!$, where $\pi: (A_{20} \cap S_{\mathcal{A}} S_{\mathcal{B}}) \ltimes_\rho (S_3^7 \times \{(\,)\} \times \mathbb{Z}_2^{11} \times \{0\}) \to S_{\mathcal{A}} S_{\mathcal{B}} \ltimes_\rho (S_3^8 \times \mathbb{Z}_2^{12})$ is the homomorphism created by projection and $\tau \coloneqq (A_{20} \cap S_{\mathcal{A}} S_{\mathcal{B}}) \ltimes_\rho (S_3^7 \times \mathbb{Z}_2^{11}) \to (A_{20} \cap S_{\mathcal{A}} S_{\mathcal{B}}) \ltimes_\rho (S_3^7 \times \{(\,)\} \times \mathbb{Z}_2^{11} \times \{0\})$ is a clear isomorphism. From now on, we will regard $\mathbb{G} \subseteq \mathbb{G}!$.

We shall define $\pi_{\mathbb{H}}: S_{\mathcal{A}} S_{\mathcal{B}} \ltimes_\rho (S_3^8 \times \mathbb{Z}_2^{12}) \to S_{\mathcal{A}} S_{\mathcal{B}}$ and $\pi_{\mathbb{K}}: S_{\mathcal{A}} S_{\mathcal{B}} \ltimes_\rho (S_3^8 \times \mathbb{Z}_2^{12}) \to (S_3^8 \times \mathbb{Z}_2^{12})$ to be the projection functions. We then define:

$$\mathbb{H}! = \left\{ h \in S_{\mathcal{A}} S_{\mathcal{B}} \ltimes_\rho (S_3^8 \times \mathbb{Z}_2^{12}) \;\middle|\; \pi_{\mathbb{K}}(h) = \left(\left((\,), \cdots, (\,)\right); (0, \cdots, 0)\right) \right\}$$

$$\mathbb{K}! \coloneqq \left\{ h \in S_{\mathcal{A}} S_{\mathcal{B}} \ltimes_\rho (S_3^8 \times \mathbb{Z}_2^{12}) \;\middle|\; \pi_{\mathbb{H}}(h) = (\,) \right\}$$

to be the "kernels" of $\pi_{\mathbb{K}}$ and $\pi_{\mathbb{H}}$ respectively.

4.1.1 Are Projections Homomorphisms?

The prior discussion begs the question, are $\pi_{\mathbb{H}}$ and $\pi_{\mathbb{K}}$ homomorphisms? While $\pi_{\mathbb{H}}$ is a group homomorphism, $\pi_{\mathbb{K}}$ is not a group homomorphism.

Let $\gamma, \delta \in S_{\mathcal{A}} S_{\mathcal{B}} \ltimes_\rho (S_3^8 \times \mathbb{Z}_2^{12})$ be arbitrary. Then there exist some $a, h \in S_{\mathcal{A}} S_{\mathcal{B}}$ and some $b, k \in (S_2^8 \times \mathbb{Z}_2^{12})$ such that $\gamma = (a, b)$ and $\delta = (h, k)$. We can note that $\gamma\delta = (a, b) \cdot (h, k) = (ah, \rho(h)(b) \cdot k)$. Then $\pi_{\mathbb{H}}(\gamma\delta) = ah$, which happens to be equal to $\pi_{\mathbb{H}}(\gamma) \cdot \pi_{\mathbb{H}}(\delta) = a \cdot h$. Therefore $\pi_{\mathbb{H}}$ preserves the group operation of $\mathbb{G}!$, so it is a homomorphism. ◩

Meanwhile we can let γ and δ be defined so that $a = (\;)$, $h = (9, 10)$, $b = \left(\left((\,), \cdots, (\,)\right); (1, 0, \cdots, 0)\right)$, and $k = \left(\left((\,), \cdots, (\,)\right); (0, \cdots, 0)\right)$. Then it follows that:

$$\pi_{\mathbb{H}}(\gamma\delta) = \rho(h)(b) \cdot k \qquad \pi_{\mathbb{H}}(\gamma) = b \qquad \pi_{\mathbb{H}}(\delta) = k$$

But $\rho(h)(b) = \left(\left((\,), \cdots, (\,)\right); (0, 1, 0, \cdots, 0)\right)$ is not equal to b, so we can see that $\pi_{\mathbb{H}}(\gamma\delta) \neq \pi_{\mathbb{H}}(\gamma) \cdot \pi_{\mathbb{H}}(\delta)$. Then $\pi_{\mathbb{K}}$ cannot be a homomorphism. ■

4.1.2 Definition of Homomorphism $\Phi_{\mathbb{H}}$ and Lack of Homomorphism $\Phi_{\mathbb{K}}$

We then define $\Phi_{\mathbb{H}}: \mathbb{G}! \to S_{\mathcal{A}} S_{\mathcal{B}}$ via $\Phi_{\mathbb{H}} \coloneqq \pi_{\mathbb{H}} \circ \Phi$; we can also define $\Phi_{\mathbb{K}} \coloneqq \pi_{\mathbb{K}} \circ \Phi$. The prior theorem showed that $\Phi_{\mathbb{H}}$ will be a homomorphism, and it suggests that $\Phi_{\mathbb{K}}$ is not a homomorphism.

We can actually show that it is not: with Φ an isomorphism, we can find some $\Phi^{-1}(\gamma)$ and $\Phi^{-1}(\delta)$ that will lead to the conclusion that $\Phi_{\mathbb{K}}(\gamma\delta) \neq \Phi_{\mathbb{K}}(\gamma) \cdot \Phi_{\mathbb{H}}(\delta)$.

We also receive the alternative definitions $\mathbb{H}! \coloneqq \left\{ h \in \mathbb{G}! \;\middle|\; \Phi_{\mathbb{K}}(h) = \Big(\big((),\cdots,()\big);(0,\cdots,0)\Big) \right\}$ and $\mathbb{K}! \coloneqq \ker \Phi_{\mathbb{H}}$, where only the second is an actual kernel.

4.1.3 Defining $\psi: \mathbb{S} \to \mathbb{G}!$ an Injection

With these changes, we can then define the injection $\psi: \mathbb{S} \to \mathbb{G}!$ in the obvious way. This will allow us to consider each $\sigma \in \mathbb{S}$ as an element of $\mathbb{G}!$. Instead of writing $\psi(\sigma)$, we will write σ unless this causes clarity to suffer. This allows us to write $\sigma(g)$ as σg, which is a product of elements in $S_{\mathcal{A}}S_{\mathcal{B}} \ltimes_\rho (S_3^8 \times \mathbb{Z}_2^{12})$.

Is this well-defined? Essentially, the elements of $\mathbb{G}!$ are any way to move the cubelets of a Rubik's cube around, and each $\sigma \in \mathbb{S}$ can be considered as a way to move the cubelets around: we just rotate them in a way that makes it look like the Rubik's cube is rotating. For example, the WBO symmetry can be considered as an element of $\mathbb{G}!$ that moves the Red-Blue-White vertex to the position of the Red-Yellow-Blue vertex, among other things. This will then allow us to write:

$$\mathrm{WBO} = \sigma \sim \begin{pmatrix} \{1,2,3,4\}\{5,6,7,8\}\{9,10,11,12\}\{13,14,15,16\}\{17,18,19,20\} \\ \big((),\ (),\ (),\ (),\ (123),\ (132),\ (123),\ (132)\big) \\ (0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0) \end{pmatrix}$$

Note that the first coordinate is the rotation set for WBO mentioned in the prior paragraph; while the rotation sets were used to introduce these symmetries before, they become important in this chapter.

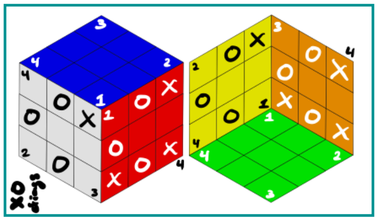
4.1.4 Finding The Element of $S_{\mathcal{A}}S_{\mathcal{B}} \ltimes_\rho (S_3^8 \times \mathbb{Z}_2^{12})$

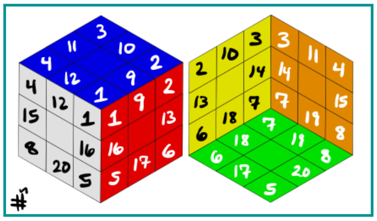
Determining the value of $\Phi(\sigma)$ for any $\sigma \in \mathbb{S}$ is a lengthy process. Starting with a blank Rubik's cube, we have to manually check the following steps:

- Determining $S_{\mathcal{A}}S_{\mathcal{B}}$
    - When we look at σ, we need to look at how applying σ to a cube moves the cubelets around.
    - By tracking where each cubelet moves to, we can create this coordinate by converting this data into symmetry group notation.
    - We know that vertex cubelets cannot be moved to edge cubelet positions and vice versa. As long as each cubelet is numbered properly, we are then guaranteed to have an element of $S_{\mathcal{A}}S_{\mathcal{B}}$
- Determining S_3^8
    - For each of the eight vertex positions, we need to see how the "X" marker of the cubelet relates to the "X" marker of the position. If these are in the same place, then $1 \to 1$. If the cubelet's "X" is counterclockwise of the position's "X," we have $1 \to 2$. Otherwise $1 \to 3$ since the "X" of the cubelet is clockwise from the position's "X."
    - We then repeat this for sides 2 and 3 for the vertex and position.
    - We then take these data $1 \to A$, $2 \to B$, and $3 \to C$ and write it as an element of S_3.
    - This is repeated for the other vertex positions, and we list them in order starting with vertex position 1 and ending with vertex position 8.

- Determining $\mathbb{Z}_2^{12}$
  - This step is nearly identical to the prior step. Instead of tracking where each "X" goes, we track whether the "O" of the cubelet in each position matches with the "O" of the position. If so, this cubelet position has a 0. Otherwise, it will be a 1.
  - We then list the coordinates in order, starting with edge position 9 and ending with edge position 20.

During this process, it may be useful to have a copy of the diagonals and XO notation of the Rubik's Cube on-hand, along with the numbers of each cubelet. The following is a combination of both that can be used for the checking process.





4.2 Rewriting $\tau_\sigma(g)$

We know that τ_σ is a color map that recolors the Rubik's cube so that $\sigma(g) = \tau_\sigma(g)$ for some $g \in \mathbb{G}$. We can note that the center cubelets of the Rubik's cube are unaffected by g, yet σ does affect these center cubelets. In order for this equality to be true, we need for τ_g to recolor these center cubelets in the same way. But this will fully determine the color map of τ_g since each color has one center cubelet, and the color map must be consistent across the whole cube.

The thing is that we can apply the color map at any time: since g is defined by moving cubelets around to specific positions, we could apply the color map *before* applying g onto the cube. Since g is not based on the colors of each cubelet but rather their positions, the recolored cubelets will be moved to the correct position regardless. This suggests that we can rewrite $\tau_\sigma(g)$ as $g \cdot \tau_\sigma(e)$, where $\tau_\sigma(e)$ is just applying the color map to the Rubik's cube prior to permuting the cubelets with g.

With this in mind, consider $\sigma(e)$. We can see that σ will permute the sides of the cube in some manner. For example, WBO moves the Red side to the position of the Yellow side. This will actually define a color map, which we can base this color map on how the center cubelets are moved around. This then implies that $\sigma(e)$ should be the same as $\tau_\sigma(e)$, since τ_σ is known to be the color map that is defined by these center cubelets. As a result:

$$\sigma g = \sigma(g) = \tau_\sigma(g) = g \cdot \tau_\sigma(e) = g \cdot \sigma(e) = g\sigma e = g\sigma$$

In summary, we have that $\sigma(g) = \tau_\sigma(g)$ is equivalent to $\sigma g = g\sigma$, in the notation of the prior subsection. We can also rewrite this as $\sigma g \sigma^{-1} = g$.

4.3 Major Consequence

For any $\sigma \in \mathbb{S}$ and any $g \in \mathbb{G}$ that "displays" σ, the prior section argued that the desired property is for $\sigma g \sigma^{-1} = g$ as elements of $\mathbb{G}$!. We will let this be the requirement that we are looking for in order to say that some $g \in \mathbb{G}$ will "display" some symmetry $\sigma \in \mathbb{S}$.

4.4 The Symmetries for a Permutation of $\mathbb{G}$ is a Group

Let $g \in \mathbb{G}$ be fixed and let $\mathbb{S}_g$ be the subset of $\mathbb{S}$ consisting of all the symmetries that g displays. It follows that $\mathbb{S}_g$ is a subgroup.

We can prove this by using the One Step Subgroup Test. Let $\sigma \in \mathbb{S}_g$ and $\tau \in S_g$ be two symmetries that g displays. We then have that $\sigma g \sigma^{-1} = g$ and $\tau g \tau^{-1} = g$. On the latter, we can multiply both sides by τ^{-1} on the left and τ on the right to show that $g = \tau^{-1} g \tau$. We can then combine these together to show:

$$(\sigma\tau^{-1})g(\sigma\tau^{-1})^{-1} = \sigma(\tau^{-1}g\tau)\sigma^{-1} = \sigma g \sigma^{-1} = g$$

As a result, g will also display $\sigma\tau^{-1}$; in other words, $\sigma\tau^{-1} \in \mathbb{S}_g$. Therefore, by the One-Step Subgroup Test we know that $\mathbb{S}_g$ is a subgroup. ■

4.4.1 Plan of Action

Because the set of displayed symmetries for any $g \in \mathbb{G}$ is a subgroup of $\mathbb{S}$, we can categorize the elements of $g \in \mathbb{G}$ based on which ones have the same subgroup $\mathbb{S}_g \subseteq \mathbb{S}$. So if we can find each possible subgroup of $\mathbb{S}$, we can find all the possible elements of $\mathbb{G}$ for each.

But how should we go about finding these subgroups, and how do we find the elements of $\mathbb{G}$ that are compatible with each subgroup?

Each question will require a chapter to answer it. For the former, we need to find all the subgroups of $\mathbb{S}$: the methodology of which is explained in Chapter 5. The latter will rely on this result to find all the elements of $\mathbb{G}$ that match each symmetry group. Our plan is listed in this final sub-section:

4.4.2 Plan In List Form

So overall, our plan now looks like this:

1. We find all possible subgroups $M \subseteq \mathbb{S}$.
    a. In particular, we will find some $\sigma_i \in \mathbb{S}$ such that $M = \langle \sigma_1, \cdots, \sigma_n \rangle$
    b. To help streamline the proof, we want to choose representatives that "overlap" with other subgroups, allowing for us to perform less calculations.
2. For each M, we will use the $\langle \sigma_i \rangle$ presentation to find its commutators in $\mathbb{G}$.
    a. We will show that we can simplify this process by breaking each $g \in \mathbb{G}$ into its $\mathbb{H}$ and $\mathbb{K}$ parts, essentially allowing us to "mix and match" parts together.
    b. We may need to consider only a subset of all the possible patterns since otherwise there may be too many to consider.
3. Once these steps are done, we need to write each into a presentable form.
    a. I plan to make colored diagrams, such as those shown previously when discussing the symmetry classes in Chapter 3, for example.

Time to get things on the road!

5 Super Special Symmetry Subgroups of $\mathbb{S}$

In this chapter, we will be looking at the subgroups of $\mathbb{S}$. We will start with S_4 since $\mathbb{S} \cong S_4 \times \mathbb{Z}_2$, and we will extend this to subgroups of $S_4 \times \mathbb{Z}_2$. These then correspond to the subgroups of $\mathbb{S}$ in a clear way.

5.1 Subgroups of S_4

It is difficult to rigorously go through and determine with proof all the different subgroups of S_4: without advanced tools, we would have to manually find all of them. Rather, we will appeal to prior results[2] for the subgroups of S_4:

$$S_4 \quad A_4 \quad \langle(1234),(13)\rangle \quad \langle(1243),(14)\rangle \quad \langle(1324),(12)\rangle$$

$$\langle(123),(12)\rangle \quad \langle(124),(12)\rangle \quad \langle(134),(13)\rangle \quad \langle(234),(23)\rangle$$

$$\langle(1234)\rangle \quad \langle(1243)\rangle \quad \langle(1324)\rangle \quad \langle(13),(24)\rangle \quad \langle(14),(23)\rangle \quad \langle(12),(34)\rangle$$

$$\langle(12)(34),(13)(24)\rangle \quad \langle(123)\rangle \quad \langle(124)\rangle \quad \langle(134)\rangle \quad \langle(234)\rangle$$

$$\langle(13)(24)\rangle \quad \langle(14)(23)\rangle \quad \langle(12)(34)\rangle$$

$$\langle(12)\rangle \quad \langle(13)\rangle \quad \langle(23)\rangle \quad \langle(14)\rangle \quad \langle(24)\rangle \quad \langle(34)\rangle \quad \langle(\,)\rangle$$

5.1.1 Subgroup Representatives

We will focus on a few of these subgroups as representatives and extrapolate the others based off of these representatives. We can start with the following eleven subgroup representatives:

S_4	A_4
$\langle(1234),(13)\rangle$	$\langle(234),(23)\rangle$
$\langle(1234)\rangle$	$\langle(12),(34)\rangle$
$\langle(12)(34),(13)(24)\rangle$	$\langle(234)\rangle$
$\langle(13)(24)\rangle$	$\langle(12)\rangle$
$\langle(\,)\rangle$	

We can substitute for equivalent representations for some of the groups above:

$$S_4 \quad \longrightarrow \quad \langle(1234),(12)\rangle$$

$$A_4 \quad \longrightarrow \quad \langle(234),(13)(24)\rangle$$

$$\langle(1234),(13)\rangle \quad \longrightarrow \quad \langle(1234),(24)\rangle$$

$$\langle(234),(23)\rangle \quad \longrightarrow \quad \langle(234),(24)\rangle$$

$$\langle(13),(24)\rangle \quad \longrightarrow \quad \langle(13),(13)(24)\rangle$$

This allows us to rewrite the eleven representative subgroups as follows:

$\langle(1234),(12)\rangle$	$\langle(234),(13)(24)\rangle$
$\langle(1234),(24)\rangle$	$\langle(234),(24)\rangle$

[2] Here are some links:
http://faculty.smcm.edu/sgoldstine/Math321f09/S4subgroups.pdf
http://www.math.hawaii.edu/~williamdemeo/groups/S4Subgroups.pdf

$\langle (1234) \rangle$	$\langle (12), (12)(34) \rangle$
$\langle (12)(34), (13)(24) \rangle$	$\langle (234) \rangle$
$\langle (13)(24) \rangle$	$\langle (12) \rangle$
$\langle (\,) \rangle$	

The reasoning behind this is that it allows us to represent all of the different subgroups with just a handful of elements in S_4:

$$(\,) \quad (12) \quad (24) \quad (234) \quad (12)(34) \quad (13)(24) \quad (1234)$$

Additionally, each subgroup is generated by at most two elements. We will use these seven elements to create representatives for the subgroups of $S_4 \times \mathbb{Z}_2$, and we will be able to expand each representative based on how these seven representatives can be used to expand the representative subgroups of S_4, as provided in the following section.

5.1.2 List of Subgroups Relating to S_4 Representatives

Since we changed the way in which we wrote some of the subgroup representatives, we should update the other subgroups to match.

- Because $\langle (1234), (13) \rangle \rightarrow \langle (1234), (24) \rangle$, we then update the following subgroups as well:
  $$\langle (1243), (14) \rangle \rightarrow \langle (1243), (23) \rangle \qquad \langle (1324), (12) \rangle \rightarrow \langle (1324), (34) \rangle$$

- Because $\langle (234), (23) \rangle \rightarrow \langle (234), (24) \rangle$, we then update the following subgroups as well:
  $$\langle (123), (12) \rangle \rightarrow \langle (123), (23) \rangle \qquad \langle (124), (12) \rangle \rightarrow \langle (124), (24) \rangle$$
  $$\langle (134), (13) \rangle \rightarrow \langle (134), (34) \rangle$$

- Because $\langle (12), (34) \rangle \rightarrow \langle (12), (12)(34) \rangle$, we then update the following subgroups as well:
  $$\langle (13), (24) \rangle \rightarrow \langle (13), (13)(24) \rangle \qquad \langle (14), (23) \rangle \rightarrow \langle (14), (14)(23) \rangle$$

Then the following list contains all subgroups, categorized by their representatives:

$\langle (1234), (12) \rangle$	$\langle (234), (13)(24) \rangle$
$\langle (1234), (24) \rangle \quad \langle (1243), (23) \rangle$ $\langle (1324), (34) \rangle$	$\langle (123), (23) \rangle \quad \langle (124), (24) \rangle$ $\langle (134), (34) \rangle \quad \langle (234), (24) \rangle$
$\langle (1234) \rangle \quad \langle (1243) \rangle \quad \langle (1324) \rangle$	$\langle (13), (13)(24) \rangle \quad \langle (14), (14)(23) \rangle$ $\langle (12), (12)(34) \rangle$
$\langle (12)(34), (13)(24) \rangle$	$\langle (123) \rangle \quad \langle (124) \rangle \quad \langle (134) \rangle \quad \langle (234) \rangle$
$\langle (13)(24) \rangle \quad \langle (14)(23) \rangle \quad \langle (12)(34) \rangle$	$\langle (12) \rangle \; \langle (13) \rangle \; \langle (23) \rangle \; \langle (14) \rangle \; \langle (24) \rangle \; \langle (34) \rangle$
$\langle (\,) \rangle$	

5.2 Generators of a Subgroup Derived with π

Suppose G is a group, p is a prime, and $\pi: G \times \mathbb{Z}_p \rightarrow G$ is the projection homomorphism. Let H be a subgroup of $G \times \mathbb{Z}_p$ and let $K = \pi(H)$ be its image: this is known to be a subgroup of G because π is a homomorphism.

If $\{\pi(a_i)\}_{i \in I}$ are generators of K for some index set I (as in $K = \langle\, \pi(a_i) \mid i \in I \,\rangle$) and $(e_G, 1) \notin H$, then $H = \langle\, a_i \mid i \in I \,\rangle$ will be generated by $\{a_i\}_{i \in I}$.

Suppose that the conditions of G, p, and π are as above, and define H to be any subgroup of $G \times \mathbb{Z}_p$ not containing $(e_G, 1)$. Noting that $z^{p-1} = 1$ for all $z \in \mathbb{Z}_p^{\times}$, it follows that the sum $z + \cdots + z$ consisting of z^{p-2} copies of z (where z^{p-2} is interpreted as an integer in $\mathbb{Z}$), it follows that $(e_G, z)^{z^{p-2}} = (e_G, 1)$. Because $(e_G, 1) \notin H$, it follows that $(e_G, z) \notin H$ for any $z \in \mathbb{Z}_p \setminus \{0\}$.

Let $K \coloneqq \pi(H)$ and consider the restricted map $\pi|_H^K \colon H \to K$. Now suppose that $\pi|_H^K(a_1) = \pi|_H^K(a_2)$ for any $a_1 = (g_1, z_1)$ and $a_2 = (g_2, z_2)$ in H. This then implies that $\pi(a_1) = g_1$ must equal $\pi(a_2) = g_2$; we can then conclude that $a_1 a_2^{-1} = (g_1 g_2^{-1}, z_1 z_2^{-1})$ will equal (e_G, z) for $z \coloneqq z_1 z_2^{-1}$ in $\mathbb{Z}_p$. But we previously proved that this is impossible for $z \in \mathbb{Z}_p \setminus \{0\}$; we can then conclude that $z = z_1 z_2^{-1}$ must be $0 \in \mathbb{Z}_p$, so $a_1 a_2^{-1} = \left(e_g, 0\right)$. Since $(e_G, 0)$ is the identity of $G \times \mathbb{Z}_p$, it follows that $a_1 = a_2$ after multiplying both sides by a_2 on the right. Because a_1 and a_2 were any two elements of H such that $\pi|_H^K(a_1) = \pi|_H^K(a_2)$, this then implies that $\pi|_H^K$ will be one-to-one.

Let I be an index set and let $\{a_i\}_{i \in I}$ be a set of elements in $G \times \mathbb{Z}_p$ such that $\{\pi(a_i)\}_{i \in I}$ generates the group $K \coloneqq \pi(H)$—namely that $K = \langle\, \pi(a_i) \mid i \in I \,\rangle$. If $b \in H$, it then follows that $\pi(b) \in K$, so there exists some indexes $1 \le n_j \le N$ such that $\pi(b) = \prod_{j=1}^{N}\left(\pi\left(a_{n_j}\right)\right)$, where $N \in \mathbb{N}$ is some upper bound. (An element of a group cannot be equal to a product of an infinite number of elements.) Using the fact that π is a homomorphism, it follows that $\pi(b) = \pi\left(\prod_{j \in J} a_{n_j}\right)$. Because both b and $\prod_{j \in J} a_{n_j}$ are elements in H, we can restrict π to $\pi|_H^K$ to find that $\pi|_H^K(b) = \pi|_H^K\left(\prod_{j \in J} a_{n_j}\right)$. In the prior paragraph, we showed that $\pi|_H^K$ is one-to-one, so we can conclude that $b = \prod_{j \in J} a_{n_j}$. This implies that $b \in \langle\, a_i \mid i \in I \,\rangle$ for all $b \in H$. Since $a_i \in H$ for all $i \in I$, we can then conclude that $H = \langle\, a_i \mid i \in I \,\rangle$. ■

5.3 Subgroup Representatives of $S_4 \times \mathbb{Z}_2$

The subgroups representatives of $S_4 \times \mathbb{Z}_2$ are as follows, where each $\phi, \psi \in \mathbb{Z}_2$ are independent of each other and produces a new subgroup representative for each value:

$\langle ((1234),0); ((12),0); ((\,),1) \rangle$	$\langle ((234),0); ((13)(24),0); ((\,),1) \rangle$
$\langle ((1234),0); ((24),0); ((\,),1) \rangle$	$\langle ((234),0); ((24),0); ((\,),1) \rangle$
$\langle ((1234),0); ((\,),1) \rangle$	$\langle ((12),0); ((12)(34),0); ((\,),1) \rangle$
$\langle ((12)(34),0); ((13)(24),0); ((\,),1) \rangle$	$\langle ((234),0); ((\,),1) \rangle$
$\langle ((13)(24),0); ((\,),1) \rangle$	$\langle ((12),0); ((\,),1) \rangle$
$\langle ((\,),1) \rangle$	
$\langle ((1234),\phi); ((12),0) \rangle$	$\langle ((234),0); ((13)(24),0) \rangle$
$\langle ((1234),\phi); ((24),\psi) \rangle$	$\langle ((234),0); ((24),\phi) \rangle$
$\langle ((1234),\phi) \rangle$	$\langle ((12),\phi); ((12)(34),\psi) \rangle$
$\langle ((12)(34),\phi); ((13)(24),\psi) \rangle$	$\langle ((234),0) \rangle$
$\langle ((13)(24),\phi) \rangle$	$\langle ((12),\phi) \rangle$
$\langle ((\,),0) \rangle$	

We will start with the projection homomorphism $\pi \colon S_4 \times \mathbb{Z}_2 \to S_4$. We can observe that $\ker \pi = \langle ((\,),1) \rangle$. We will let H be an arbitrary subgroup of $S_4 \times \mathbb{Z}_2$ and define $K \coloneqq \pi(H)$.

Suppose $\big((\,),1\big) \in H$. We can then conclude that $\ker \pi = \big\langle\big((\,),1\big)\big\rangle$ will be in H, so by the Lattice Isomorphism Theorem, we then have that H will have the same index as K: namely, $[S_4 \times \mathbb{Z}_2 : H] = [S_4 : K]$. This then implies that $\frac{|S_4 \times \mathbb{Z}_2|}{|H|} = \frac{|S_4|}{|K|}$, which will then show that $2|K| = |H|$. Recalling that π is a homomorphism that maps H onto K and that $|\ker \pi| = 2$, we can conclude that π will be a two-to-one map. This will then require for $H = \{\,(k,0),(k,1) \mid k \in K\,\}$. In other words, if $K = \langle k_1, \cdots, k_n\rangle$, it would follow that $H = \big\langle (k_1,0), \cdots, (k_n,0), \big((\,),1\big)\big\rangle$. If we use the modified subgroup representatives of S_4, we then get the first set of subgroup representatives provided above.

Otherwise $\big((\,),1\big) \notin H$. We can then apply Section 5.2: for k_i such that $K = \langle k_1, \cdots, k_n\rangle$, we define the values $h_i \coloneqq (k_i, z_i)$ where $z_i \in \mathbb{Z}_2$ are unknown. We then observe that $H = \langle h_1, \cdots, h_n\rangle$. For this, we will use the modified forms of the subgroup representatives of S_4:

$\langle(1234),(12)\rangle$	$\langle(234),(13)(24)\rangle$
$\langle(1234),(24)\rangle$	$\langle(234),(24)\rangle$
$\langle(1234)\rangle$	$\langle(12),(12)(34)\rangle$
$\langle(12)(34),(13)(24)\rangle$	$\langle(234)\rangle$
$\langle(13)(24)\rangle$	$\langle(12)\rangle$
$\langle(\,)\rangle$	

We can manually check the cases for the subgroups K generated by one element: with the exceptions of $\langle(234)\rangle$ and $\langle(\,)\rangle$, we can set $z_1 = 1$ without issue. This addresses the following subgroups:

$$\langle(1234)\rangle \quad \langle(234)\rangle \quad \langle(13)(24)\rangle \quad \langle(12)\rangle \quad \langle(\,)\rangle$$

If $k_i \in K$ has an odd order (i.e. $|k_i|$ is an odd integer), then its corresponding z_i must be 0 since $(k_i, z_i)^{|k_i|} = (e_G, |k_i|z_i)$ is in H, and we cannot have $(e_G, 1) \in H$. This can be used twice to show that the only option for $\langle(234),(143)\rangle$ is for $z_1 = z_2 = 0$. (Note that $(234) \cdot (13)(24) = (143)$.) This is precisely the subgroup $\langle(234),(13)(24)\rangle$, so it follows that the corresponding H must be $\big\langle\big((234),0\big);\big((23)(24),0\big)\big\rangle$.

Meanwhile, suppose that $K = \langle k_1, k_2\rangle$ where $k_1 \in A_4$ and $k_2 \notin A_4$. Set $z_1 = 0$ and $z_2 = 1$; if $(h,z) \in H$, then it follows that $h = \prod_{i=1}^{t} k_{n_i}$ for some $n_i \in \{1,2\}$. Then $z = \sum_{i=1}^{t} z_{n_i}$ will be 0 iff there is an even number of copies of z_2 in the sum, and 1 otherwise. Hence, $h = \prod_{i=1}^{t} k_{n_i}$ will contain an even number of copies of k_2 iff $z = 0$. If we set $h = (\,) \in A_4$, then $\prod_{i=1}^{t} k_{n_i}$ must contain an even number of copies of k_2 since $k_2 \notin A_n$, so by the prior statements, we can conclude $z = 0$. Thus, any time where $K = \langle k_1, k_2\rangle$, we can set $z_1 = 0$ and $z_2 = 1$ and have a valid group $H = \langle(k_1,0);(k_2,1)\rangle$.

These two considerations allow us to consider most of the cases above: for example, let $K = \langle(234),(24)\rangle$. We set $k_1 = (234)$ and $k_2 = (24)$. By the first point, we know that $z_1 = 0$ since $|k_1| = 3$. Meanwhile, $k_2 \notin A_4$, so the second point implies that we can set $z_2 = 1$ without issue; we can also set $z_2 = 0$ since then every element of H has 0 in the second coordinate. We can then use these to address $\langle(1234),(12)\rangle = \langle(1234),(234)\rangle$ and $\langle(234),(24)\rangle$. (Note that $(12)(1234) = (234)$.)

We will now address $K = \langle(1234),(24)\rangle$. We can take $H = \big\langle\big((1234),z_1\big);\big((24),z_2\big)\big\rangle$ for some $z_1, z_2 \in \mathbb{Z}_2$ and let $M = \big\langle\big((1234),z_1\big)\big\rangle$ and $N = \big\langle\big((24),z_2\big)\big\rangle$. Note that $\overline{M} = \langle(1234)\rangle$ has four elements and is a subgroup of K, which has 8 elements. Then $[K:\overline{M}] = 2$, so $\overline{M} \trianglelefteq K$; this shows that $k\overline{M}k^{-1} = \overline{M}$ for all

$k \in K$. Now let $h = (k, z)$ be an arbitrary element of H; we can then see that $hMh^{-1} = (k\bar{M}k^{-1}, z + z_1 - z)$. Using the prior fact, we then have $hMh^{-1} = (\bar{M}, z_1) = M$, so it follows that $M \trianglelefteq H$. Meanwhile, N is a subgroup of H; we can then conclude that MN is a subgroup of H. But also $\big((1234), z_1\big)$ and $\big((24), z_2\big)$ are in MN, so we can conclude that $H \subseteq MN$. Thus, $H = MN$. We can then conclude that $|H| = |MN| = \frac{|M||N|}{|M \cap N|} = \frac{4 \cdot 2}{1} = 8$. But because K has 8 elements as well, this means that H "does not have enough room" to contain $\big((\,), 1\big)$: recall that $\pi(H) = K$, where $\pi: S_4 \times \mathbb{Z}_2 \to S_4$ is the projection homomorphism, and that $\big((\,), 0\big) \in H$. Therefore, $= \big\langle \big((1234), z_1\big); \big((24), z_2\big) \big\rangle$ is a subgroup of $S_4 \times \mathbb{Z}_2$ regardless of $z_1, z_2 \in \mathbb{Z}_2$.

The remaining groups to address are $\langle (12), (12)(34) \rangle$ and $\langle (12)(34), (13)(24) \rangle$. These can also be addressed manually: note that both are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. (We can alternatively use the methods of the prior paragraph if we wanted to make this more rigorous.) We find that z_1 and z_2 can be either 0 or 1 independently. This concludes all of the possible groups of K, so we then have addressed all possible groups H in $S_4 \times \mathbb{Z}_2$ that do not include $\big((\,), 1\big)$.

Since we have addressed all groups H containing $\big((\,), 1\big)$ and those not including $\big((\,), 1\big)$, we have found all possible subgroups of $S_4 \times \mathbb{Z}_2$. ■

5.3.1 Subgroups of $S_4 \times \mathbb{Z}_2$

Each of the representatives presented in Section 5.3 is based on a subgroup representative of S_4. Recall the following subgroups of S_4, as grouped by their subgroup representatives defined in Section 5.1.1:

$\langle (1234), (12) \rangle$	$\langle (234), (13)(24) \rangle$
$\langle (1234), (24) \rangle$ $\langle (1243), (23) \rangle$ $\langle (1324), (34) \rangle$	$\langle (123), (23) \rangle$ $\langle (124), (24) \rangle$ $\langle (134), (34) \rangle$ $\langle (234), (24) \rangle$
$\langle (1234) \rangle$ $\langle (1243) \rangle$ $\langle (1324) \rangle$	$\langle (13), (13)(24) \rangle$ $\langle (14), (14)(23) \rangle$ $\langle (12), (12)(34) \rangle$
$\langle (12)(34), (13)(24) \rangle$	$\langle (123) \rangle$ $\langle (124) \rangle$ $\langle (134) \rangle$ $\langle (234) \rangle$
$\langle (13)(24) \rangle$ $\langle (14)(23) \rangle$ $\langle (12)(34) \rangle$	$\langle (12) \rangle$ $\langle (13) \rangle$ $\langle (23) \rangle$ $\langle (14) \rangle$ $\langle (24) \rangle$ $\langle (34) \rangle$
$\langle (\,) \rangle$	

If we replace the appropriate values into the result of Section 5.3, we have a full list of all the subgroups of $S_4 \times \mathbb{Z}_2$, where each $\phi, \psi \in \mathbb{Z}_2$ are independent of each other and produces a new subgroup for each value:

$\big\langle \big((1234), 0\big); \big((12), 0\big); \big((\,), 1\big) \big\rangle$	$\big\langle \big((234), 0\big); \big((13)(24), 0\big); \big((\,), 1\big) \big\rangle$
$\big\langle \big((1234), 0\big); \big((24), 0\big); \big((\,), 1\big) \big\rangle$ $\big\langle \big((1243), 0\big); \big((23), 0\big); \big((\,), 1\big) \big\rangle$ $\big\langle \big((1324), 0\big); \big((34), 0\big); \big((\,), 1\big) \big\rangle$	$\big\langle \big((123), 0\big); \big((23), 0\big); \big((\,), 1\big) \big\rangle$ $\big\langle \big((124), 0\big); \big((24), 0\big); \big((\,), 1\big) \big\rangle$ $\big\langle \big((134), 0\big); \big((34), 0\big); \big((\,), 1\big) \big\rangle$ $\big\langle \big((234), 0\big); \big((24), 0\big); \big((\,), 1\big) \big\rangle$
$\big\langle \big((1234), 0\big); \big((\,), 1\big) \big\rangle$ $\big\langle \big((1243), 0\big); \big((\,), 1\big) \big\rangle$ $\big\langle \big((1324), 0\big); \big((\,), 1\big) \big\rangle$	$\big\langle \big((13), 0\big); \big((13)(24), 0\big); \big((\,), 1\big) \big\rangle$ $\big\langle \big((14), 0\big); \big((14)(23), 0\big); \big((\,), 1\big) \big\rangle$ $\big\langle \big((12), 0\big); \big((12)(34), 0\big); \big((\,), 1\big) \big\rangle$
$\big\langle \big((12)(34), 0\big); \big((13)(24), 0\big); \big((\,), 1\big) \big\rangle$	$\big\langle \big((123), 0\big); \big((\,), 1\big) \big\rangle$ $\big\langle \big((124), 0\big); \big((\,), 1\big) \big\rangle$ $\big\langle \big((134), 0\big); \big((\,), 1\big) \big\rangle$ $\big\langle \big((234), 0\big); \big((\,), 1\big) \big\rangle$

$\langle((13)(24),0);((\,),1)\rangle$ $\langle((14)(23),0);((\,),1)\rangle$ $\langle((12)(34),0);((\,),1)\rangle$	$\langle((12),0);((\,),1)\rangle$ $\langle((13),0);((\,),1)\rangle$ $\langle((14),0);((\,),1)\rangle$ $\langle((23),0);((\,),1)\rangle$ $\langle((24),0);((\,),1)\rangle$ $\langle((34),0);((\,),1)\rangle$
$\langle((\,),1)\rangle$	
$\langle((1234),\phi);((12),0)\rangle$	$\langle((234),0);((13)(24),0)\rangle$
$\langle((1234),\phi);((24),\psi)\rangle$ $\langle((1243),\phi);((23),\psi)\rangle$ $\langle((1324),\phi);((34),\psi)\rangle$	$\langle((123),0);((23),\phi)\rangle$ $\langle((124),0);((24),\phi)\rangle$ $\langle((134),0);((34),\phi)\rangle$ $\langle((234),0);((24),\phi)\rangle$
$\langle((1234),\phi)\rangle$ $\langle((1243),\phi)\rangle$ $\langle((1324),\phi)\rangle$	$\langle((13),\phi);((13)(24),\psi)\rangle$ $\langle((14),\phi);((14)(23),\psi)\rangle$ $\langle((12),\phi);((12)(34),\psi)\rangle$
$\langle((12)(34),\phi);((13)(24),\psi)\rangle$	$\langle((123),0)\rangle$ $\langle((124),0)\rangle$ $\langle((134),0)\rangle$ $\langle((234),0)\rangle$
$\langle((13)(24),\phi)\rangle$ $\langle((14)(23),\phi)\rangle$ $\langle((12)(34),\phi)\rangle$	$\langle((12),\phi)\rangle$ $\langle((13),\phi)\rangle$ $\langle((14),\phi)\rangle$ $\langle((23),\phi)\rangle$ $\langle((24),\phi)\rangle$ $\langle((34),\phi)\rangle$
$\langle((\,),0)\rangle$	

5.3.2 Number of Subgroups of $S_4 \times \mathbb{Z}_2$

There are 98 subgroups of $S_4 \times \mathbb{Z}_2$.

We can come to this conclusion by counting the subgroups listed in Section 5.3.1. ■

5.3.3 Verification of Results

These subgroups are also listed online on Wikipedia[3], but they do not have listed generators that are usable However, we can agree with it that there will be 98 subgroups, as shown in the prior section. Their subgroup representatives are nearly identical to those specified above with some exceptions. The following charts label the subgroup representatives of Section 5.3 (not the subgroups of Section 5.3.1) with the representatives given at (source here). The "#2" at the end of a label indicates it is the second group listed on the page; similar definitions hold for "#1" and "#3".

O_h	T_h
D_{4h}	D_{2d}
C_{4h}	$D_{2h\#2}$
$D_{2h\#1}$	S_6
$C_{2h\#1}$	$C_{2h\#2}$
S_2	
$\phi = 0{:}\, T_d$ $\quad$ $\phi = 1{:}\, O$	T

[3] The precise link is: https://en.wikiversity.org/wiki/Full_octahedral_group

$\phi=\psi=0: D_{2d\#2}$ $\phi=1,\psi=0: C_{4v}$ $\phi=0,\psi=1: D_{2d\#1}$ $\phi=\psi=1: D_4$	$\phi=0: C_{3v}$ $\phi=1: D_3$
$\phi=0: S_4$ $\phi=1: C_4$	$\phi=\psi=0: C_{2v\#3}$ $\phi=0,\psi=1: D_{2\#1}$ $\phi=1,\psi\in\{0,1\}: C_{2v\#1}$
$\phi=\psi=0: D_{2\#2}$ otherwise: $C_{2v\#2}$	C_3
$\phi=0: C_{2\#1}$ $\phi=1: C_{3\#1}$	$\phi=0: C_{3\#2}$ $\phi=1: C_{2\#2}$
C_1	

5.4 Theorem 5.3: Subgroup Representatives of $\mathbb{S}$

We know that $\mathbb{S}\cong S_4\times\mathbb{Z}_2$. Then due to the result of Section 5.3.2, there are 98 subgroups of $\mathbb{S}$. Further, we can use Section 5.3 to find the subgroup representatives. We can observe the following comparisons:

RBW ∼ $(( ),0)$ BRY ∼ $((12),0)$ YGO ∼ $((24),0)$ BWR ∼ $((234),0)$

OGW ∼ $((12)(34),0)$ OBY ∼ $((13)(24),0)$ WBO ∼ $((1234),0)$

OGY ∼ $(( ),1)$ GOW ∼ $((12),1)$ WBR ∼ $((24),1)$ ~~GYO ∼ ((234), 1)~~

RBY ∼ $((12)(34),1)$ RGW ∼ $((13)(24),1)$ YGR ∼ $((1234),1)$

These then provide the following subgroup representatives of $\mathbb{S}$:

⟨WBO; BRY; OGY⟩	⟨BWR; OBY; OGY⟩
⟨WBO; YGO; OGY⟩	⟨BWR; YGO; OGY⟩
⟨WBO; OGY⟩	⟨BRY; OGW; OGY⟩
⟨OGW; OBY; OGY⟩	⟨BWR; OGY⟩
⟨OBY; OGY⟩	⟨BRY; OGY⟩
⟨OGY⟩	
⟨WBO; BRY⟩ ⟨YGR; BRY⟩	⟨BWR; OBY⟩
⟨WBO; YGO⟩ ⟨YGR; YGO⟩ ⟨WBO; WBR⟩ ⟨YGR; WBR⟩	⟨BWR; YGO⟩ ⟨BWR; WBR⟩
⟨WBO⟩ ⟨YGR⟩	⟨BRY; OGW⟩ ⟨GOW; OGW⟩ ⟨BRY; RBY⟩ ⟨GOW; RBY⟩
⟨OGW; OBY⟩ ⟨RBY; OBY⟩ ⟨OGW; RGW⟩ ⟨RBY; RGW⟩	⟨BWR⟩
⟨OBY⟩ ⟨RGW⟩	⟨BRY⟩ ⟨GOW⟩
⟨RBW⟩	

Continued on Next Page →

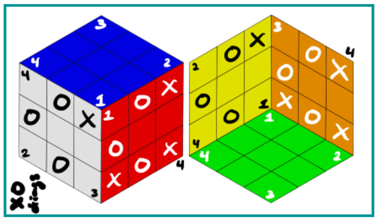
5.5 Subgroups of 𝕊

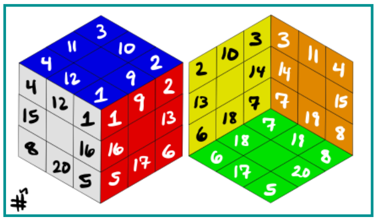
Recall that Section 3.1 provided a complete list of all the elements of 𝕊, categorized by "type." We can use their names to create a list of all the subgroups of 𝕊 using the results provided in Section 5.3.1. We then have the following list of all the subgroups of 𝕊:

⟨WBO; BRY; OGY⟩	⟨BWR; OBY; OGY⟩
⟨WBO; YGO; OGY⟩ ⟨RWG; OYG; OGY⟩ ⟨BOW; GOY; OGY⟩	⟨YOB; OYG; OGY⟩ ⟨GYR; YGO; OGY⟩ ⟨YRG; GOY; OGY⟩ ⟨BWR; YGO; OGY⟩
⟨WBO; OGY⟩ ⟨RWG; OGY⟩ ⟨BOW; OGY⟩	⟨WGR; OBY; OGY⟩ ⟨OWB; RGY; OGY⟩ ⟨BRY; OGW; OGY⟩
⟨OGW; OBY; OGY⟩	⟨YOB; OGY⟩ ⟨GYR; OGY⟩ ⟨YRG; OGY⟩ ⟨BWR; OGY⟩
⟨OBY; OGY⟩ ⟨RGY; OGY⟩ ⟨OGW; OGY⟩	⟨BRY; OGY⟩ ⟨WGR; OGY⟩ ⟨OWB; OGY⟩ ⟨OYG; OGY⟩ ⟨YGO; OGY⟩ ⟨GOY; OGY⟩
⟨OGY⟩	
⟨WBO; BRY⟩ ⟨YGR; BRY⟩	⟨BWR; OBY⟩
⟨WBO; YGO⟩ ⟨YGR; YGO⟩ ⟨WBO; WBR⟩ ⟨YGR; WBR⟩ ⟨RWG; OYG⟩ ⟨OYB; OYG⟩ ⟨RWG; RWB⟩ ⟨OYB; RWB⟩ ⟨BOW; GOY⟩ ⟨GRY; GOY⟩ ⟨BOW; BRW⟩ ⟨GRY; BRW⟩	⟨YOB; OYG⟩ ⟨YOB; RWB⟩ ⟨GYR; YGO⟩ ⟨GYR; WBR⟩ ⟨YRG; GOY⟩ ⟨YRG; BRW⟩ ⟨BWR; YGO⟩ ⟨BWR; WBR⟩
⟨WBO⟩ ⟨YGR⟩ ⟨RWG⟩ ⟨OYB⟩ ⟨BOW⟩ ⟨GRY⟩	⟨WGR; OBY⟩ ⟨YBO; OBY⟩ ⟨WGR; RGW⟩ ⟨YBO; RGW⟩ ⟨OWB; RGY⟩ ⟨RYG; RGY⟩ ⟨OWB; OBW⟩ ⟨RYG; OBW⟩ ⟨BRY; OGW⟩ ⟨GOW; OGW⟩ ⟨BRY; RBY⟩ ⟨GOW; RBY⟩
⟨OGW; OBY⟩ ⟨RBY; OBY⟩ ⟨OGW; RGW⟩ ⟨RBY; RGW⟩	⟨YOB⟩ ⟨GYR⟩ ⟨YRG⟩ ⟨BWR⟩
⟨OBY⟩ ⟨RGW⟩ ⟨RGY⟩ ⟨OBW⟩ ⟨OGW⟩ ⟨RBY⟩	⟨BRY⟩ ⟨GOW⟩ ⟨WGR⟩ ⟨YBO⟩ ⟨OWB⟩ ⟨RYG⟩ ⟨OYG⟩ ⟨RWB⟩ ⟨YGO⟩ ⟨WBR⟩ ⟨GOY⟩ ⟨BRW⟩
⟨RBW⟩	

6 Call Your Representative

Despite section 5.5 providing an excellent list of all the subgroups of $\mathbb{S}$, it will be sufficient to find the values of $\langle\sigma\rangle_{\mathbb{H}}$ and $\langle\sigma\rangle_{\mathbb{K}}$ of each representative σ listed in 5.4 since these can be "reoriented" to match the subgroups needed. Fortunately, there are only 13 unique representatives used in 5.4, so we only need to find $\psi(\sigma) \in \mathbb{G}!$ for these values. As a reminder, here are the diagrams used in section 4.1.4:





Additionally, recall that, in S_3, the 1 corresponds to the "X" side and the 2 corresponds to the side $120°$ counterclockwise from the "X" side. Then (123) is a $120°$ counterclockwise rotation of the vertex.

6.1 Embedding $\mathbb{S}$ into $\mathbb{G}!$

Our first course of action is finding how each $\mathbb{S}$ is represented in $\mathbb{G}!$. We can do this by considering how each $\mathbb{S}$ maps into $S_{\mathcal{A}}S_{\mathcal{B}} \ltimes_\rho (S_3^8 \times \mathbb{Z}_2^{12})$ via $\Phi \circ \psi$ (the notation presented in Section 4.1); since this group is isomorphic to $\mathbb{G}!$, this essentially tells us how the elements of $\mathbb{S}$ are presented in $\mathbb{G}!$. I decided to list these presentations in Appendix A in order to support a larger paper width.

6.2 Commutator Notation

For any subgroup $M \subseteq \mathbb{S}$, we will let $\mathrm{C}(M)$ represent the set of commutators in $\mathbb{G}!$. For the subsets $\mathbb{G}$ and $\mathbb{H}$, we introduce a subscript to give $\mathrm{C}_{\mathbb{G}}(M)$ and $\mathrm{C}_{\mathbb{H}}(M)$, respectively.

6.2.1 Finding Commutators

We have that $\mathrm{C}_{\mathbb{G}}(M) = \mathbb{G} \cap \mathrm{C}(M)$ and $\mathrm{C}_{\mathbb{H}}(M) = \mathbb{H} \cap \mathrm{C}(M)$.

Essentially, these are the definitions of $\mathrm{C}_{\mathbb{G}}(M)$ and $\mathrm{C}_{\mathbb{H}}(M)$, so there is nothing to prove. ■

6.2.2 Intersection of Commutators

For any subgroups $M, N \subseteq \mathbb{S}$ we have that $\mathrm{C}(\langle M, N\rangle) = \mathrm{C}(M) \cap \mathrm{C}(N)$. A similar statement applies to $\mathrm{C}_{\mathbb{G}}(\langle M, N\rangle)$ and to $\mathrm{C}_{\mathbb{H}}(\langle M, N\rangle)$.

Suppose that $g \in \mathrm{C}(\langle M, N\rangle)$. Then it follows that $gm = mg$ for all $m \in M = \langle M, N\rangle$, so then $g \in \mathrm{C}(M)$. Likewise, $gn = ng$ for all $n \in N \subseteq \langle M, N\rangle$, so we conclude that $g \in \mathrm{C}(N)$. Therefore $g \in \mathrm{C}(M) \cap \mathrm{C}(N)$. With $g \in \mathrm{C}(\langle M, N\rangle)$ being generic, we then have that $\mathrm{C}(\langle M, N\rangle) \subseteq \mathrm{C}(M) \cap \mathrm{C}(N)$.

Now suppose that $g \in \mathrm{C}(M) \cap \mathrm{C}(N)$. Then it follows that $gh = hg$ for all $h \in M \cup N$. Let $k \in \langle M, N\rangle$. It then follows that $k = \prod_{i=1}^{t} h_i$ for some $h_i \in M \cup N$. Then we have that:

$$gk = gh_1 \cdot h_2h_3 \cdots h_t = h_1g \cdot h_2h_3 \cdots h_t = h_1 \cdot h_2g \cdot h_3 \cdots h_t = \quad \cdots \quad = h_1h_2 \cdots h_{t-1} \cdot gh_t = kg$$

This shows that g commutes with k, but since $k \in \langle M, N\rangle$ was arbitrary, it follows that $g \in \mathrm{C}(\langle M, N\rangle)$. But then this means that—because $g \in \mathrm{C}(M) \cap \mathrm{C}(N)$ was arbitrary as well—that $\mathrm{C}(M) \cap \mathrm{C}(N) \subseteq \mathrm{C}(\langle M, N\rangle)$. Therefore, the two sets are equal to one another. ■

6.2.3 Generated by a Single Element

The commutator of a single element, $\mathrm{C}(\sigma)$, is equal to the commutator of the subgroup it generates, $\mathrm{C}(\langle\sigma\rangle)$.

We have one direction that is clear: if $g \in \mathrm{C}(\langle\sigma\rangle)$, then in particular we have $g\sigma = \sigma g$. Therefore $g \in \mathrm{C}(g)$, so we have $\mathrm{C}(\langle\sigma\rangle) \subseteq \mathrm{C}(\sigma)$.

The opposite direction follows a proof similar to before: Suppose that $g \in \mathrm{C}(\sigma)$. Then $g\sigma = \sigma g$, which implies that $\sigma^{-1}g = g\sigma^{-1}$. Then consider an arbitrary $k \in \langle\sigma\rangle$. There are two cases: $k = \sigma^{n}$ or $k = \sigma^{-n}$ for some $n \in \mathbb{N}_0$. In either case, we can use the facts that $g\sigma = \sigma g$ or that $g\sigma^{-1} = \sigma^{-1}g$ to conclude that $g\sigma^{\pm n} = \sigma^{\pm 1}g\sigma^{\pm(n-1)} = \cdots = \sigma^{\pm n}g$, so g commutes with k. Since g commutes with all possible k, we have that $g \in \mathrm{C}(\langle\sigma\rangle)$. Then $\mathrm{C}(\sigma) \subseteq \mathrm{C}(\langle\sigma\rangle)$, so $\mathrm{C}(\sigma) = \mathrm{C}(\langle\sigma\rangle)$ as described previously. ■

6.2.4 Finding $\mathrm{C}_{\mathbb{H}}(M)$ for Arbitrary M

Suppose $M = \langle\sigma_i\rangle_{i=1}^{t}$ is a subgroup of $\mathbb{S}$ for some $\sigma_i \in \mathbb{S}$. Then $\mathrm{C}(M) = \bigcap_{i=1}^{t}\mathrm{C}(\sigma_i)$.

With $M = \langle\sigma_i\rangle_{i=1}^{t}$, we can use Theorem 6.2.2 to write $\mathrm{C}(M) = \bigcap_{i=1}^{t}\mathrm{C}(\langle\sigma_i\rangle)$. Then Theorem 6.2.3 allows us to write $\bigcap_{i=1}^{t}\mathrm{C}(\langle\sigma_i\rangle) = \bigcap_{i=1}^{t}\mathrm{C}(\sigma_i)$. Therefore $\mathrm{C}(M) = \bigcap_{i=1}^{t}\mathrm{C}(\sigma_i)$ as desired. ■

6.2.5 Defining $M_{\mathbb{G}}$ and Determining its Values

For every subgroup $M \subseteq \mathbb{S}$, define $M_{\mathbb{G}} \coloneqq \mathrm{C}_{\mathbb{G}}(M)$ and $M_{\mathbb{H}} \coloneqq \mathrm{C}_{\mathbb{H}}(M)$. Then $M_{\mathbb{G}} \subseteq M_{\mathbb{H}}\mathbb{K}$.

Suppose that $g \in M_{\mathbb{G}}$. We then have that $g\sigma = \sigma g$ for all $\sigma \in M$. Let us write $g = hk$ and $\sigma = ab$ for some $h \in \mathbb{H}$, some $a \in \mathbb{H}!$, some $k \in \mathbb{K}$, and some $b \in \mathbb{K}!$. We then have that:

$$ha \cdot \rho(a)(k)b = g\sigma = \sigma g = ah \cdot \rho(h)(b)k$$

If we apply the homomorphism $\Phi_{\mathbb{H}}$ to both sides (defined in Section 4.1.2), we then have:

$$\Phi_{\mathbb{H}}(ha) \cdot \mathbb{0}_{\mathbb{H}!} = \Phi_{\mathbb{H}!}(ah) \cdot \mathbb{0}_{\mathbb{H}!}$$

since $b, k, \rho(a)(k)$, and $\rho(h)(b)$ are elements of $\mathbb{K}!$, the kernel of $\Phi_{\mathbb{H}!}$. Therefore $\Phi_{\mathbb{H}!}(hah^{-1}a^{-1}) = \mathbb{0}_{\mathbb{H}!}$, which implies that $(hah^{-1}a^{-1}) \in \mathbb{K}!$, but with $h, a \in \mathbb{H}!$, we also know that $(hah^{-1}a^{-1}) \in \mathbb{H}!$. Therefore we know that $(hah^{-1}a^{-1}) \in \mathbb{H}! \cap \mathbb{K}! = \{\mathbb{0}_{\mathbb{G}!}\}$, so that $ha = ah$ in $\mathbb{G}!$. Then it follows that $h \in M_{\mathbb{H}}$.

In summary we have that if $g = hk$ is in $M_{\mathbb{G}}$ for $h \in \mathbb{H}$ And $k \in \mathbb{K}$, then it follows that $h \in M_{\mathbb{H}}$. Then with $k \in \mathbb{K}$, it follows that $g \in M_{\mathbb{H}}\mathbb{K}$. With $g \in M_{\mathbb{G}}$ being arbitrary, it follows that $M_{\mathbb{G}} \subseteq M_{\mathbb{H}}\mathbb{K}$. ■

6.3 Plan for Finishing

As the prior theorem shows, we must have that every $g \in M_{\mathbb{G}}$ will be in $M_{\mathbb{H}}\mathbb{K}!$. This provides us with a strategy for calculating the symmetric permutations, but we first need to calculate $M_{\mathbb{H}}$ for every subgroup representative M of the subgroup classes established in 5.1.1: these representatives can be expanded to the other subgroups later. Using Maple, we can find the elements in each $\langle\sigma\rangle_{\mathbb{H}}$; these can then be intersected to find the subgroup representatives' $M_{\mathbb{H}}$. The results of the Maple calculations are shown in the following table, which shows how many elements are in each $M_{\mathbb{H}}$. Note that the table was split into halves in order to reduce the amount of space used.

If you wish to investigate the Maple calculations, they are attached in Appendix B.

Subgroup Rep. M	# Centralizers in $M_{\mathbb{H}}$	Subgroup Rep.	# Centralizers in $M_{\mathbb{H}}$
$\langle WBO; BRY; OGY \rangle$	4	$\langle WBO; BRY \rangle$	4
		$\langle YGR; BRY \rangle$	4
$\langle BWR; OBY; OGY \rangle$	8	$\langle BWR; OBY \rangle$	24
$\langle WBO; YGO; OGY \rangle$	32	$\langle WBO; YGO \rangle$	128
		$\langle YGR; YGO \rangle$	128
		$\langle WBO; WBR \rangle$	128
		$\langle YGR; WBR \rangle$	1536
$\langle BWR; YGO; OGY \rangle$	8	$\langle BWR; YGO \rangle$	72
		$\langle BWR; WBR \rangle$	24
$\langle WBO; OGY \rangle$	128	$\langle WBO \rangle$	6144
		$\langle YGR \rangle$	6144
$\langle BRY; OGW; OGY \rangle$	256	$\langle BRY; OGW \rangle$	2048
		$\langle GOW; OGW \rangle$	4096
		$\langle BRY; RBY \rangle$	2048
		$\langle GOW; RBY \rangle$	2048
$\langle OGW; OBY; OGY \rangle$	512	$\langle OGW; OBY \rangle$	12288
		$\langle RBY; OBY \rangle$	1024
		$\langle OGW; RGW \rangle$	4096
		$\langle RBY; RGW \rangle$	4096
$\langle BWR; OGY \rangle$	432	$\langle BWR \rangle$	34992
$\langle OBY; OGY \rangle$	737280	$\langle OBY \rangle$	8847360
		$\langle RGW \rangle$	1769472
$\langle BRY; OGY \rangle$	2048	$\langle BRY \rangle$	1474560
		$\langle GOW \rangle$	737280
$\langle OGY \rangle$	8847360	$\langle RBW \rangle$	9656672256000

6.3.1 Limiting the Results

I developed an algorithm for finding the elements of $M_{\mathbb{G}}$ that uses the elements $h \in M_{\mathbb{H}}$ to find all the elements of $k \in \mathbb{K}$ such that $hk \in M_{\mathbb{G}}$. However, I have only been able to partially automate each case of $h \in \mathbb{H}$. Therefore, I only did the cases where $|M_{\mathbb{H}}| \leq 8$. I leave the other cases to the viewer to complete if they choose to do so.

6.4 Commutators $M_{\mathbb{K}}$ in $\mathbb{K}$

For each fixed $h \in M_{\mathbb{H}}$, we now need to find the $k \in \mathbb{K}$ such that $hk \in M_{\mathbb{G}}$. This is the same as finding k such that $hk \in \mathrm{C}(\sigma_i)$ for all the generators σ_i of M.

This is true due to Corollary 6.2.4: We have that $g = hk$ is in $M_{\mathbb{G}}$, iff it is in $\bigcap_{i=1}^{t} \mathrm{C}(\sigma_i)$. ■

6.4.1 Inverse of a $g \in \mathbb{G}!$

Let $g \in \mathbb{G}!$. Then if $g = ab$ for $a \in \mathbb{H}!$ and $b \in \mathbb{K}!$, we have that $g^{-1} = a^{-1}b'$ for $b' \coloneqq \rho(a^{-1})(b^{-1})$.

Writing $g^{-1} = a'b'$ for some $a' \in \mathbb{H}!$ and $b' \in \mathbb{K}!$, we have that $g^{-1}g = \mathbb{O}_{\mathbb{G}!}$ implies that $a'b'ab = \mathbb{O}_{\mathbb{G}!}$. Then it follows that $a'a \cdot \rho(a)(b') \cdot b = \mathbb{O}_{\mathbb{G}!}$. With $a'a \in \mathbb{H}!$ and $\rho(a)(b') \cdot b \in \mathbb{K}!$, we then have that $a'a = \mathbb{O}_{\mathbb{G}!}$ and $\rho(a)(b') \cdot b = \mathbb{O}_{\mathbb{G}!}$. The former implies that $a' = a^{-1}$,and the latter implies that $\rho(a)(b') = b^{-1}$. We can apply the inverse automorphism $\rho(a^{-1})$ to both sides to determine that $b' = \rho(a^{-1})(b^{-1})$ as expected. ■

6.4.2 Criterion for Finding Elements

Let $\sigma \in M$ and $h \in M_{\mathbb{H}}$ be fixed. Write $\sigma = ab$ for $a \in \mathbb{H}!$ and $b \in \mathbb{K}!$. Then $hk \in \mathrm{C}(\sigma_i)$ for $k \in \mathbb{K}$ iff the following property is true in $\mathbb{K}!$:

$$\rho(h)(b^{-1}) \cdot \rho(a)(k) \cdot b = k$$

The following statements are designed to be bijections of one another. Therefore, we are proving both implications simultaneously.

Let $hk \in M_{\mathbb{G}}$. Then it follows that $\sigma^{-1}(hk)\sigma = (hk)$. From the prior section, we know that $\sigma^{-1} = a^{-1} \cdot \rho(a^{-1})(b^{-1})$. We then calculate the left side to be:

$$\begin{aligned} \sigma^{-1}(hk)\sigma &= a^{-1} \cdot \rho(a^{-1})(b^{-1}) \cdot (hk) \cdot ab \\ &= a^{-1}h \cdot \rho(ha^{-1})(b^{-1}) \cdot k \cdot ab \\ &= a^{-1}ha \cdot \rho(aha^{-1})(b^{-1}) \cdot \rho(a)(k) \cdot b \end{aligned}$$

We know that $h \in M_{\mathbb{H}}$, so it follows that $a^{-1}ha = aha^{-1} = h$. Therefore we have:

$$h \cdot \rho(h)(b^{-1}) \cdot \rho(a)(k) \cdot b = \sigma^{-1}(hk)\sigma = hk$$

$$\rho(h)(b^{-1}) \cdot \rho(a)(k) \cdot b = k$$

This concludes the proof. ■

6.5 Conclusion

I then used the prior theorem repeatedly to calculate each possible k using trial-and-error for each coordinate of k. I made these calculations in a spreadsheet which is attached as Appendix C. I then created a set of images that show all the Rubik's cube permutations that I found, and I attached them as Appendix D.

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7 Appendix A: Embedding $\mathbb{S}$ into $\mathbb{G}$!

	$S_4 \times \mathbb{Z}_2$	$S_{\mathcal{A}}S_{\mathcal{B}}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
RBW	(( ),0)	{1}{2}{3}[4}{5}{6}{7}{8} {9}{10}{11}{12}{13}{14}{15} ⋯ {20}	( )	( )	( )	( )	( )	( )	( )	( )	0	0	0	0	0	0	0	0	0	0	0	0
BRY	((12), 0)	{1, 2}{3, 5}{4, 6}{7, 8} {9}{10, 16}{11, 17}{12, 13}{14, 20}{15, 18}{19}	(132)	(123)	(132)	(132)	(123)	(123)	(132)	(123)	1	1	1	0	0	1	0	1	1	0	1	1
YGO	((24), 0)	{1, 7}{2, 6}{3, 5}{4, 8} {9, 18}{10, 17}{11, 20}{12, 19}{13}{14, 16}{15}	( )	(123)	( )	(123)	( )	(132)	( )	(132)	0	0	0	0	1	1	1	1	0	0	0	0
BWR	((234), 0)	{1}{2, 5, 4}{3, 6, 8}{7} {9, 16, 12}{10, 17, 15}{11, 13, 20}{14, 18, 19}	(132)	(123)	( )	(123)	(123)	(123)	(123)	(132)	1	1	1	0	0	1	0	1	1	0	1	1
OGW	((12)(34), 0)	{1, 8}{2, 7}{3, 6]{4, 5} {9, 19}{10, 18}{11, 17}{12, 20}{13, 14}{15, 16}	(123)	( )	(123)	( )	(132)	( )	( )	(132)	0	0	0	0	1	1	1	1	0	0	0	0
OBY	((13)(24), 0)	{1, 3}{2, 4}{5, 7}{6, 8} {9, 11}{10, 12}{13, 15}{14, 16}{17, 19}{18, 20}	( )	( )	( )	( )	( )	( )	( )	( )	0	0	0	0	0	0	0	0	0	0	0	0
WBO	((1234), 0)	{1, 2, 3, 4}{5, 6, 7, 8} {9, 10, 11, 12}{13, 14, 15, 16}{17, 18, 19, 20}	( )	( )	( )	( )	(123)	(132)	(123)	(132)	0	0	0	0	0	0	0	0	0	0	0	0
OGY	(( ), 1)	{1, 7}{2, 8}{3, 5}{4, 6} {9, 19}{10, 20}{11, 17}{12, 18}{13, 15}{14, 16}	(12)	(23)	(12)	(23)	(12)	(23)	(12)	(23)	0	0	0	0	0	0	0	0	0	0	0	0
GOW	((12), 1)	{1, 8}{2, 7}{3}{4}{5}{6} {9, 19}{10, 14}{11}{12, 15}{13, 18}{16, 20}{17}	(13)	(13)	(23)	(13)	(13)	(12)	(13)	(13)	1	1	1	0	0	1	0	1	1	0	1	1
WBR	((24), 1)	{1}{2, 4}{3}{5}{6, 8}{7} {9, 12}{10, 11}{13, 15}{14}{16}{17, 20}{18, 19}	(12)	(12)	(12)	(12)	(12)	(13)	(12)	(13)	0	0	0	0	1	1	1	1	0	0	0	0
RBY	((12)(34), 1)	{1, 2}{3, 4}{5, 6}{7, 8} {9}{10, 12}{11}{13, 16}{14, 15}{17}{18, 20}{19}	(12)	(12)	(12)	(12)	(23)	(23)	(23)	(23)	0	0	0	0	1	1	1	1	0	0	0	0
RGW	((13)(24), 1)	{1, 5}{2, 6}{3, 7}{4, 8} {9, 17}{10, 18}{11, 19}{12, 20}{13}{14}{15}{16}	(12)	(23)	(12)	(23)	(12)	(23)	(12)	(23)	0	0	0	0	0	0	0	0	0	0	0	0
YGR	((1234), 1)	{1, 8, 3, 6}{2, 5, 4, 7} {9, 20, 11, 18}{10, 17, 12, 19}{13, 16, 15, 14}	(23)	(12)	(23)	(12)	(12)	(23)	(12)	(23)	0	0	0	0	0	0	0	0	0	0	0	0