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The axiom of choice for countable sets.

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THE AXIOM OF CHOICE FOR COUNTABLE SETS

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CHAPTER I

INTRODUCTION AND METAMATHEMATICAL BACKGROUND

In this thesis axioms for set theory will be presented which include the well-known axiom of choice. These axioms, together with their associated primitives, defined terms, and theorems will be referred to as the BGN (for Bernays, Gödel, and von Neumann) theory of sets, or just BGN. BGN, without the axiom of choice and theorems requiring it, will be denoted by BGN'. It will be shown that if a proposition attributing a property to countable sets can be proved in BGN, then it can be proved in BGN', by showing that the axiom of choice (Zermelo's form), when stated for countable sets, is a theorem of BGN'.

We remark at the outset that we are not claiming to prove what is known as the "countable axiom of choice". The distinction between the "countable axiom of choice" and the "axiom of choice for countable sets" is made in Chapter IV.

In order to show that Zermelo's form of the axiom of choice, when stated for countable sets, is a theorem of BGN', we must set forth carefully the axioms and principal definitions of BGN. But first we will need to develop some metamathematical ideas concerning axiomatic theories. An informal approach, following R. L. Wilder [9], is well-suited to our needs.

By an axiomatic theory we will mean a totality consisting of the following.

1. A formal object language with primitive (undefined) terms.
2. A fundamental set of axioms expressed in the object language.
3. Theorems deducible from the axioms by logic.

For convenience, the primitives and fundamental axioms are called an axiom system and given a name, e.g., X. Object language statements which make "meaningful" assertions about the primitives are called X-statements. In proving theorems, that part of logic known as "predicate calculus with identity theory" will be assumed.

Definition 1.1 If X is an axiom system, an *interpretation* I of X is an assignment of meanings to the undefined terms of X in such a way that the axioms of X are all true for these meanings. The concept arising from the interpretation I is called a *model* for X, and denoted M(X,I).

Definition 1.2 An axiom system X is called *satisfiable* if there is an interpretation I leading to a model M(X,I) for X.

Definition 1.3 If X is an axiom system and T is an X-statement which can be proved by the rules of predicate calculus with identity theory, using any or all of the axioms of X as hypotheses, then T is called a theorem of X. We say "X implies T".

Definition 1.4 An axiom system X is called consistent if there are no two contradictory X-statements implied by X.

Without delving into the fine points of logic, we will assume the following two logical principles, following Wilder [9, p. 27].

Principle 1 Let X be an axiom system and I an interpretation of X. If X implies an X-statement T, then T(I), the statement resulting from T by replacing the undefined terms of X by their meanings in I, is true of M(X,I).

In other words, every theorem of a satisfiable axiom system, when interpreted, becomes a true statement about the model arising from the interpretation.

Principle 2 Let X be an axiom system and I an interpretation of X. Then there are no two theorems P(I) and Q(I) about M(X,I) which are contradictory.

Using these two principles, we have a simple but important metatheorem.

Metatheorem 1 Let X be an axiom system. If X is satisfiable, then X is consistent.

Proof Let I be an interpretation of X, and suppose that P and Q are contradictory X-statements implied by X. Then P(I) and Q(I) are true of M(X,I) by Principle 1. But by Principle 2, this cannot be. So P and Q cannot be contradictory, and hence X is consistent.

Definition 1.5 Let X be an axiom system and A be an axiom of X. Let X' be X with A excluded, and let X" be X' with the negation of A included. Then A is said to be *independent in X* or *independent of X'* if both X and X" are satisfiable.

We will apply the metamathematical ideas just discussed many times in what follows. The primary axiom system we will deal with is of course the axiom system of BGN.

CHAPTER II

THE BGN AXIOMATIC THEORY OF SETS

The nine axioms for set theory which we will use have their origins in the works of P. Bernays, K. Gödel, and J. von Neumann, and for this reason we will refer to them as the BGN axiom system. Lest this chapter become in itself a monograph on axiomatic set theory, it must be assumed that the reader is familiar with the constructions of set theory which are not defined here. The axioms are, except for some minor modifications, those given in E. J. Lemmon's monograph [7], and the reader is referred there for a brief, concise treatment of axiomatic set theory.

In the formal axiomatic BGN theory of sets there is one primitive concept, that of a "class". The object language has symbols x, y, u, v, etc., which stand for variables ranging over the domain of classes. Usually, class variables will be lower case, but some specific classes will be denoted by upper case, for example, the set N of natural numbers. The other symbols of the object language are

$\forall$	for each (universal quantification)
$\rightarrow$	implies (implication)
$=$	equals (equality)

$\sim$	not (negation)
$\in$	is a member of (membership in a class)

All except the last are recognized to be standard symbols from logic. Note that the logical symbols

$\exists$	there exists (existential quantification)
$\exists!$	there exists exactly one (unique existential quantification)
$\leftrightarrow$	if and only if (biconditional)
&	and (conjunction)
or	or (disjunction)

can all be defined in terms of $\forall$, $\rightarrow$, and $\sim$; the appropriate definitions are understood to be a part of the theory by the fact that predicate calculus with identity theory has been assumed.

Parentheses will also appear in the object language, but only for the sake of legibility. The convention, following Lemmon, will be that parentheses are used sparingly; only around statements of the form $A \rightarrow B$, where A and B are object language predicates, will parentheses be mandatory.

Having listed the symbols of the object language, we must specify how the symbols may be combined in order to form admissible strings of symbols. An admissible string will be called a well-formed formula, or wff. The most basic type of wff is defined first.

Definition 2.1 A primitive dyadic predicate, or p.d.p., is a string of the form $x = y$ or of the form $x \in y$.

The rules of wff formation can now be stated.

1. Every p.d.p. is a wff.
2. If A and B are wffs, $\sim A$ and $(A \rightarrow B)$ are wffs.
3. If A is a wff and x is a class, then $\forall x\ A$ is a wff.

The rules of wff formation are concise. Note, for example, that if A, B, and C are wffs, then

$$\exists x\ ((A \text{ or } B)\ \&\ C)$$

is a wff, for it is equivalent to

$$\sim \forall x\ (\sim(\sim A \rightarrow B) \text{ or } \sim C)).$$

Now that the rules for manipulation of the object language symbols have been stated, we turn our attention to the problem of defining derived (non-primitive) concepts in the object language. We give below a policy on definitions which follows that of Lemmon.

Formal definitions will be of two kinds: abbreviations for classes and abbreviations for wffs. As for class definitions, if we can prove a wff in the object language of the form

$$\exists ! x\ A(y_1, \ldots, y_n)$$

where A is a wff in which the y_i are free class variables, then we are entitled to introduce a term $\tau(y_1, \ldots, y_n)$ to stand for x. For example, assuming that an ordered pair (x,y) of classes has already been defined, and assuming that the wff

$$\exists ! x\ \forall t\ (t \in x \leftrightarrow \exists u \exists v\ u \in y_1\ \&\ v \in y_2\ \&\ (u,v) = t)$$

is provable, it would define the cartesian product $y_1 \times y_2$.

Formally, we would write

<u>Definition</u> $y_1 \times y_2$ for x provided $\exists ! x \forall t \, (t \in x \leftrightarrow \exists u \exists v$ $u \in y_1 \; \& \; v \in y_2 \; \& \; (u,v) = t)$.

As for wff definitions, if there is a particular wff $A(y_1,\ldots,y_n)$ which we expect to use so much that it would be cumbersome to write it out at each usage, we introduce a term to stand for it. For example, given the wff

$\exists x \exists y \; t = (x,y)$

we might want to abbreviate this by "opair t". Formally, we would write

<u>Definition</u> Opair t for $\exists x \exists y \; t = (x,y)$.

The two types of definitions are then just metalanguage statements assigning names, purely as a matter of convenience, to class variables (or constants) and wffs.

We are now ready to give the BGN axioms, together with definitions of important terms used in the axioms and in that which follows. First of all, we define what it means for a class to be a set.

<u>Definition 2.2</u> Set x for $\exists w \; x \in w$.

If a class is not a set, we call it a proper class.

<u>Definition 2.3</u> Prop x for $\sim$set x.

The first axiom provides a wff condition sufficient for equality of classes. That this wff condition is necessary for equality of classes follows from identity theory; it need not be postulated.

1. Axiom of Equality

$\forall x\ \forall y\ (\forall z\ (z \in x \leftrightarrow z \in y) \rightarrow x = y)$.

The second axiom allows us to form a unique class from every wff. If we were to postulate that a unique set could be formed from every wff, the wff $\sim(x \in x)$ would quickly lead to Russell's antinomy. There is a good discussion of this in the introductory chapter of Lemmon.

2. Axiom Scheme of Classification

For every wff A (in which y does not occur)

$\exists! y\ \forall x\ (x \in y \leftrightarrow \text{set } x \ \&\ A)$.

The second axiom is actually not a single axiom, but rather an axiom "scheme": an axiom is obtained for each wff A by application of the quantification rule of universal instantiation. This axiom, unlike all others in the BGN system, is not a pure object language statement, but is a mixture of object language and metalanguage. The following fundamental definition is a definition "scheme".

Definition 2.4 For every wff A (in which y does not occur), $\{x: A\}$ for y, provided $\exists! y\ \forall x\ (x \in y \leftrightarrow \text{set } x \ \&\ A)$.

The symbolism $\{x: A\}$ is called a class abstract, and A is called the defining wff. Use of class abstracts will effect a simplification of every subsequent class definition.

Definition 2.5 $\{x,y\}$ for $\{z: \text{set } x \ \& \ \text{set } y \rightarrow z = x \text{ or } z = y\}$.

Definition 2.6 $\{x\}$ for $\{x,x\}$.

The third axiom allows us to form a set from any pair of given sets.

3. Axiom of Pairs

$\forall x \, \forall y \, (\text{set } x \ \& \ \text{set } y \rightarrow \text{set } \{x,y\})$.

We define the sum of a class, and give the fourth axiom, which postulates that the sum of every set is a set.

Definition 2.7 $\bigcup x$ for $\{y: \exists z \ z \in x \ \& \ y \in z\}$.

4. Axiom of Sums

$\forall x \, (\text{set } x \rightarrow \text{set } \bigcup x)$.

We define the inclusion relation for classes and the power class of any class, so that the fifth axiom, the axiom of powers, may be stated.

Definition 2.8 $y \subseteq x$ for $\forall z \, (z \in y \rightarrow z \in x)$.

Definition 2.9 $\mathcal{P} x$ for $\{y: \text{set } y \ \& \ y \subseteq x\}$.

5. Axiom of Powers

$\forall x \, (\text{set } x \rightarrow \text{set } \mathcal{P} x)$.

Definition 2.10 (x,y) for $\{\{x\},\{x,y\}\}$.

Definition 2.11 Opair t for $\exists x \exists y\ t = (x,y)$.

Definition 2.12 Rel x for $\forall y\ (y \in x \rightarrow \text{opair } y)$.

Definition 2.13 Uni x for $\forall u \forall v \forall w\ ((u,v) \in x \ \&\ (u,w) \in x \rightarrow v = w)$.

Definition 2.14 Func f for rel f & uni f.

Definition 2.15 f(x) for y provided func f & $(x,y) \in f$.

Definition 2.16 Inj f for func f & $\forall u \forall v \forall w\ ((u,v) \in f \ \&\ (w,v) \in f \rightarrow u = w)$.

Definition 2.17 $\mathfrak{D}$ f for $\{u: \exists v\ (u,v) \in f\}$ provided func f.

Definition 2.18 $\mathfrak{R}$ f for $\{v: \exists u\ (u,v) \in f\}$ provided func f.

Definition 2.19 $f{:}x \rightarrow y$ for func f & $\mathfrak{D} f = x$ & $\mathfrak{R} f \subseteq y$.

Definition 2.20 f[w] for $\{v: \exists u\ u \in w \ \&\ (u,v) \in f\}$ provided $f{:}x \rightarrow y$.

Note that in the definition just given, $f[x] = \mathfrak{R} f$, and that we have not required that $w \subseteq x$.

Definition 2.21 Bij f for $\exists x \exists y\ (f{:}x \rightarrow y$ & inj f & $\mathfrak{R} f = y)$.

Definition 2.22 $f|u$ for $\{t: \exists x \exists y\ (t = (x,y) \ \&\ t \in f \ \&\ x \in u)\}$ provided func f & $u \subseteq \mathfrak{D} f$.

6. Axiom of Images

$\forall x \forall f\ (\text{set } x \ \&\ \text{func } f \rightarrow \text{set } f[x])$.

Let set x & $y \subseteq x$, then if 1_x is the identity function on x, $1_x|y : y \to x$, so $(1_x|y)[x] = y$ is a set by the axiom of images. So every subclass of a set is a set. We call this result the "theorem of subsets".

The seventh axiom, the axiom of regularity, rules out classes being members of themselves, and the provability of cyclic strings such as $x \in y$ & $y \in z$ & $z \in x$. Before stating this axiom, we will need to give some more definitions.

Note that the wffs

$\exists! t\ \forall x\ (x \in t \leftrightarrow \text{set } x\ \&\ x \neq x)$

$\exists! t\ \forall x\ (x \in t \leftrightarrow \text{set } x\ \&\ x = x)$

contain no free variables. This simply means that the terms in BGN that they define are constants, i.e., particular classes. Using Definition 2.4, we can then make the following definitions.

<u>Definition 2.23</u> $\emptyset$ for $\{x: x \neq x\}$.

<u>Definition 2.24</u> U for $\{x: x = x\}$.

The class $\emptyset$ is called the "empty class"; we defined it now because it will appear in the axiom of regularity. The definition of U, the "class of all sets", is a slight, but enlightening digression. BGN, as an axiomatic theory of classes, is distinguished from most other set theories in that it admits non-sets, or proper classes, such as U. In most applications, however, proper classes do not appear. It could be said that they are an unwanted side effect that appeared when set theory was cured of Russell's antinomy by making a distinction between classes and sets.

Definition 2.25 $x \cap y$ for $\{z: z \in x \ \& \ z \in y\}$.

7. Axiom of Regularity

$\forall x \ (x \neq \emptyset \rightarrow \exists y \ y \in x \ \& \ x \cap y = \emptyset)$.

The eighth axiom, the axiom of infinity, postulates the existence of a set which has a subset behaving like the natural numbers, as we will show in Chapter III.

Definition 2.26 $x \cup y$ for $\{z: z \in x \text{ or } z \in y\}$.

8. Axiom of Infinity

$\exists x \text{ set } x \ \& \ \emptyset \in x \ \& \ \forall y \ (y \in x \rightarrow y \cup \{y\} \in x)$.

In many versions of set theory, set $\emptyset$ is postulated. However, it follows in BGN as a theorem, since $\exists x \ \emptyset \in x$ by the axiom of infinity. We now define the difference of two classes, in order to be able to state the ninth and final axiom of BGN, the axiom of choice.

Definition 2.27 $x - y$ for $\{z: z \in x \ \& \ z \notin y\}$.

9. Axiom of Choice

$\forall x \ \exists f \ (f: \mathcal{P} x - \{\emptyset\} \rightarrow x \ \& \ \forall u \ \forall v \ ((u,v) \in f \rightarrow v \in u))$.

Our fundamental assumption, now that the axioms of BGN have been stated, is that the axioms of BGN' are satisfiable, and therefore consistent, by Metatheorem 1. To complete the exposition of BGN, we give a final definition which is used in Chapter III.

Definition 2.28 $\bigcap x$ for $\{z: \forall y \, (y \in x \rightarrow z \in y)\}$.

There are other equally good formulations of set theory. Two which are important here are Gödel-Bernays (GB) set theory and Zermelo-Fraenkel (ZF) set theory. ZF is important to us because P. J. Cohen [3] established the independence of the axiom of choice (AC) from ZF' (ZF excluding AC), and we will use some of his results in Chapter VI. The historical importance of GB is that Gödel [5] showed that AC is consistent with GB' (GB excluding AC); its importance here is that Cohen [3, p. 78] shows every theorem of GB which speaks only about sets, or set-theorem, to be a theorem of ZF. Furthermore, Cohen's demonstration does not assume AC. Hence ZF' implies the set-theorems of GB'. Finally, as it is easy to see from examination of Cohen [3, Chapter II, Section 6] that GB' implies BGN', we have the following.

Metatheorem 2 ZF' implies the set-theorems of BGN'.

This result will be used in Chapter VI.

CHAPTER III

COUNTABILITY IN BGN'

In order to define countability in BGN', we will need to have a model in BGN' for the natural numbers. We therefore construct a couple (N^*, s^*) in BGN' which satisfies a certain set of axioms for the natural numbers. In addition, we will show that there is a partial order relation on N^* that is a well-ordering. This last fact will be an essential part of the proof that the AC for countable sets is a theorem of BGN'.

It has become conventional to associate axioms for the natural numbers with the name of G. Peano, who published a consistent and independent axiom system for the natural numbers in 1894. We therefore call the axioms we are about to present the Peano system (PS), although they closely resemble axioms for the natural numbers given by R. Dedekind [4] in 1888.

The undefined terms of PS are a class N and a function s, called the successor function. There are three axioms, which we now state.

P1. $s: N \rightarrow N$ & inj s.
P2. $\exists u \; u \in N - \mathfrak{R}\, s$.

P3. $\forall x\ ((u \in N - \mathcal{R}s\ \&\ u \in x\ \&\ x \subseteq N\ \&$
$\forall t\ (t \in x \rightarrow s(t) \in x)) \rightarrow x = N).$

Axiom P3 is usually called the induction axiom.

<u>Metatheorem 3</u> PS is satisfiable relative to BGN', i.e., if BGN' is satisfiable, then there is a model (N^*, s^*) in BGN' for which P1, P2, and P3 are all true.

<u>Proof</u> By the axiom of infinity, $\exists x$ set $x\ \&\ \emptyset \in x\ \&\ \forall y\ (y \in x \rightarrow y \cup \{y\} \in x)$, so define N^* as follows

$$N^* = \bigcap\{x: \emptyset \in x\ \&\ \forall t\ (t \in x \rightarrow t \cup \{t\} \in x)\}.$$

N^* is a set by the theorem of subsets, since N^* is a subclass of every set x satisfying the axiom of infinity. Now define

$$s^* = \{x: \exists a\ a \in N^*\ \&\ x = (a, a \cup \{a\})\}.$$

Clearly $s^*: N^* \rightarrow N^*$. To show that s^* is an injection, let $(a, a \cup \{a\})$ and $(b, b \cup \{b\})$ be in s^*, with $a \cup \{a\} = b \cup \{b\}$. Then as $a \in a \cup \{a\}$, $a \in b \cup \{b\}$, and as $b \in b \cup \{b\}$, $b \in a \cup \{a\}$. So $a = b$, or $(a \in b\ \&\ b \in a)$. But if $a \in b\ \&\ b \in a$, $\{a,b\}$ would violate the axiom of regularity: since $(b \in a \rightarrow \{b\} \subseteq a \cap \{a,b\})$ and $(a \in b \rightarrow \{a\} \subseteq b \cap \{a,b\})$, it would be false that $\exists t\ t \in \{a,b\}$ and $t \cap \{a,b\} = \emptyset$. So $a = b$, s^* is an injection, and P1 holds. To show that P2 holds, note that $a \in a \cup \{a\}$ for each $a \in N^*$, therefore each $a \cup \{a\}$ is not empty. So $\emptyset \in N^*$, and yet $\emptyset \notin \mathcal{R}s$. To show that P3 holds, let x be an arbitrary class, let $\emptyset \in x$, let $x \subseteq N^*$, and suppose $\forall t\ (t \in x \rightarrow s^*(t) \in x)$. Then by the way N^* and s^* were defined, $N^* \subseteq x$, so $x = N^*$ as required.

The significance of Metatheorem 3 is that we now have in BGN' a couple (N^*, s^*) which has the properties of the natural numbers, i.e., for the model (N^*, s^*), the axioms of PS become theorems provable in BGN'. Note that N^* and s^* form a particular model for PS, while N and s denote the uninterpreted primitives of PS.

We will summarize the development of PS leading up to the well-ordering of (N,s); the results then hold for the model (N^*, s^*). A good reference for an intuitive (versus axiomatic) treatment is L. Cohen and G. Ehrlich [2]. It is important to explain beforehand exactly what is going on from a set-theoretic standpoint. Cohen and Ehrlich's treatment of the natural numbers assumes, by their own admission, intuitive set theory. Axiomatically speaking, they assume all of BGN' except the axiom of infinity. N and s always remain primitive. In order to circumvent the need for the axiom of infinity, they tacitly assume satisfiability of PS. So to make their treatment fully axiomatic, all that is needed is to throw in the axiom of infinity, then PS becomes satisfiable via the model (N^*, s^*) in BGN'.

Hence, in the theorems about to be presented that lead up to a well-ordering, PS and all of BGN' except the axiom of infinity will be assumed (remember that BGN' excludes the axiom of choice). Throw in the axiom of infinity at the end, then by Metatheorem 3, all the theorems we have proved are true about (N^*, s^*). We are therefore justified in dropping the asterisk notation; henceforth (N,s) means the primitives of PS and BGN' without the axiom of infinity, or the model

(N^*, s^*) in BGN', as required in the context. We now develop N up to a well-ordering.

Theorem 3.1 $\exists! u \; \{u\} = N - \mathcal{R}\, s$.

Proof By P2, $\exists u \; \{u\} \subseteq N - \mathcal{R}\, s$. It suffices then to show that $N - \mathcal{R}\, s \subseteq \{u\}$. So let $x = \mathcal{R}\, s \cup \{u\}$, then clearly x satisfies the hypotheses of P3, so that $x = N$. Thus if t is any element of N not in $\mathcal{R}\, s$, then t must be in $\{u\}$.

We denote the unique non-successor by "1" (in the model (N^*, s^*) for PS, as was shown in the proof of Metatheorem 3, the non-successor is $\emptyset$). P3 now becomes the following.

P3'. $\forall x \; ((x \subseteq N \;\&\; 1 \in x \;\&\; \forall t \; (t \in x \rightarrow s(t) \in x)) \rightarrow x = N)$.

Theorem 3.2 (Recursion) $\forall y \; \forall g \; \forall e \; ((e \in y \;\&\; g{:}y \rightarrow y) \rightarrow \exists! f \; (f{:}N \rightarrow y \;\&\; f(1) = e \;\&\; \forall n \; (f(s(n)) = g(f(n)))))$.

Proof Let $f = \bigcap w$, where $w = \{t: t \subseteq N \times y \;\&\; (1,e) \in t \;\&\; \forall n \; \forall b \; ((n,b) \in t \rightarrow (s(n), g(b)) \in t)\}$. As $N \times y \in w$, $w \neq \emptyset$. Furthermore, $f \in w$, and $f \subseteq t$ for each $t \in w$. Now let $M = \{x: \exists! b \; b \in y \;\&\; (x,b) \in f\}$. We will show using P3' that $M = N$. First of all, $1 \in M$, as $(1,e) \in f$, and if $(1,b) \in f$ for some $b \neq e$, then as $(1,b) \neq (1,e)$, $(1,e) \in f - \{(1,b)\}$. And if $(n,c) \in f - \{(1,b)\}$, $(s(n), g(c)) \in f - \{(1,b)\}$, as $f \in w$. So by the way w was defined, $f - \{(1,b)\} \in w$. But then $f \subseteq f - \{(1,b)\}$, which is a contradiction. Now if $n \in M$, then $\exists! b \; b \in y \;\&\; (n,b) \in f$. As $f \in w$, $(s(n), g(b)) \in f$. We want to show $s(n) \in M$. Suppose not, then $(s(n), c) \in f$ for some $c \neq g(b)$. Then $(1,e)$ is a member of $f - \{(s(n), c)\}$, and if

$(m,d) \in f - \{(s(n),c)\}$, then $(s(m),g(d)) \in f$, and $(s(m), g(d)) \neq (s(n),c)$, else $m = n$ and $g(d) \neq g(b)$, so that $(n,d) \in f$ and $d \neq b$, contradicting that $n \in M$. Thus $f - \{(s(n),c)\} \in w$. As this requires $f \subseteq f - \{(s(n),c)\}$, we have a contradiction, and hence $s(n)$ must be in M after all. So $n \in M \rightarrow s(n) \in M$, and by P3', $M = N$. Hence $f{:}N \rightarrow y$ and $f(1) = e$. Now if $n \in N$, $\exists! b\ (n,b) \in f$, so $(s(n),g(b)) \in f$, and therefore $f(s(n)) = g(f(n))$. All that remains to show is that f is unique. Suppose that h satisfies $h{:}N \rightarrow y$, $h(1) = e$, and $\forall n\ (n \in N \rightarrow h(s(n)) = g(h(n)))$. Let $M = \{x{:}\ x \in N\ \&\ f(x) = h(x)\}$. Then $1 \in M$, and if $n \in M$, $f(n) = h(n)$, so $g(f(n)) = g(h(n))$, therefore $h(s(n)) = f(s(n))$, and so $s(n) \in M$. Again, by P3', $M = N$, so that $h = f$.

<u>Theorem 3.3</u> (Addition) $\exists! f\ (f{:}N \times N \rightarrow N\ \&\ \forall m\ \forall n\ ((m,n) \in N \times N \rightarrow f(m,1) = s(m)\ \&\ f(m,s(n)) = s(f(m,n))))$.

<u>Proof</u> Let m be any element of N, then by Theorem 3.2, with $y = N$, $e = s(m)$, and $g = s$, $\exists! f_m\ (f_m{:}N \rightarrow N\ \&\ f_m(1) = s(m)\ \&\ \forall n\ (n \in N \rightarrow f_m(s(n)) = s(f_m(n))))$. So let $f = \{((m,n),f_m(n)){:}\ (m,n) \in N \times N\}$. Clearly $f{:}N \times N \rightarrow N$. Also, $f(m,1) = f_m(1) = s(m)$, and given $n \in N$, $f(m,s(n)) = f_m(s(n)) = s(f_m(n)) = s(f(m,n))$. To show that f is unique, let h have the properties of f, let m be any element of N, and let $N_m = \{x{:}\ x \in N\ \&\ f(m,x) = h(m,x)\}$. Then $1 \in N_m$, for $f(m,1) = s(m) = h(m,1)$, and if $p \in N_m$, then $p \in N$ and $f(m,p) = h(m,p)$, so $f(m,s(p)) = s(f(m,p)) = s(h(m,p)) = h(m,s(p))$. Thus by P3', $N_m = N$. So for all $m \in N$ and for all $n \in N$,

$f(m,n) = h(m,n)$, so that $h = f$.

The function f is just the familiar binary operation of addition of natural numbers. Henceforth we will write $m + n$ for the value of f at any $(m,n) \in N \times N$. For the function "+" so defined, we then have $\forall m \forall n\ ((m,n) \in N \times N \rightarrow m + 1 = s(m)\ \&\ m + s(n) = s(m + n))$.

Associativity and commutativity of "+" easily follow from P3'.

<u>Definition 3.1</u> R for $\{x: \exists m \exists n \exists p\ (m \in N\ \&\ n \in N\ \&\ p \in N\ \&\ x = (m,n)\ \&\ (m = n \text{ or } m + p = n))\}$.

The relation R is just the familiar "less than or equal to" relation on the natural numbers. Given $(m,n) \in N \times N$, we will usually write mRn for $(m,n) \in R$.

<u>Theorem 3.4</u> $\forall m\ \forall n\ \forall p\ ((m \in N \rightarrow mRm)\ \&\ (mRn\ \&\ nRm \rightarrow m = n)\ \&\ (mRn\ \&\ nRp \rightarrow mRp))$.

<u>Proof</u> (1) Given $m \in N$, let $n = m$ and let $x = (m,n)$, then $x \in R$ by the way R was defined, so mRm.

(2) If mRn & nRm, then $m = n$ or $m + p = n$ for some p, and $n = m$ or $n + p' = m$ for some p'. So $m = n$ or $(\exists p\ (m + p = n)\ \&\ \exists p'\ (n + p' = m))$. But the second disjunct cannot be true. For if it were, then $m + (p + p') = m$. But it may be shown, using P3', that for all m and n in N, $m + n \neq n$.

(3) If mRn and nRp, then $m = n$ or $\exists q\ (m + q = n)$ and $n = p$ or $\exists q'\ (n + q' = p)$. So suppose $m = n$, then $m = p$ or $\exists q'\ (m + q' = p)$, so mRp. Or suppose $\exists q\ (m + q =$

n), then either $m + q = p$, in which case mRp, or $\exists q'$ ($n + q' = p$), in which case $m + (q + q') = p$, so that mRp. As in all cases we have mRp, the proof of (3) is complete.

Theorem 3.4 will be recognized as the assertion that R is a partial order relation on N. We now define what it means for an element of a subset of N to be a first element.

Definition 3.2 First x for a provided $x \subseteq N$ & $a \in x$ & $\forall y$ ($y \in x \rightarrow aRy$).

It is easy to see, by Theorem 3.4, that each non-empty subset of N can have at most one first element. We will show that each non-empty subset of N has at least one first element, hence that R is a well-ordering of N. But first we will need two lemmas. The first asserts that N is a "chain" with respect to R, and the second asserts the principle of "strong induction".

Lemma 1 $\forall m \forall n$ ($m \in N$ & $n \in N \rightarrow$ mRn or nRm).

Proof Let m be an arbitrary set in N, and let $M_m = \{t$: mRt or tRm$\}$. We show that $1 \in M_m$ by showing that $\forall m$ ($m \in N \rightarrow$ 1Rm). Let $Q = \{x$:1Rx$\}$, so that $\forall m$ ($m \in Q \rightarrow$ 1Rm); we must then show that $Q = N$. Now clearly $1 \in Q$, and if arbitrary $n \in Q$, so that 1Rn, then as $1 + n = s(n)$, we have 1Rs(n), so that $s(n) \in Q$. Hence by P3', $Q = N$. Now if $n \in M_m$, then $s(n) \in M_m$, for let n be any set in M_m, then mRn or nRm, so that there are three cases to consider.

Case 1. $m = n$. Then $s(m) = s(n)$, so as $s(m) = m + 1$ by Theorem 3.3, we have $mRs(n)$.

Case 2. $\exists p\ (m + p = n)$. Here $m + s(p) = s(m + p) = s(n)$, and so $mRs(n)$.

Case 3. $\exists p\ (n + p = m)$. If $p = 1$, $s(n) = n + 1 = m$, so $s(n)Rm$. If $p \neq 1$, then $p = s(q)$ for some $q \in N$ by Theorem 3.1. So $s(n) + q = q + s(n) = s(q + n) = s(n + q) = n + s(q) = m$, by Theorem 3.3. So again, $s(n)Rm$.

As in every case $s(n) \in M_m$, $M_m = N$ now follows by P3'. Therefore given arbitrary $n \in N$, then $n \in M_m$, so mRn or nRm.

<u>Lemma 2</u> $\forall x\ ((x \subseteq N\ \&\ 1 \in x\ \&\ \forall t\ (\forall v\ (vRt \rightarrow v \in x) \rightarrow s(t) \in x)) \rightarrow x = N)$.

<u>Proof</u> Let the hypotheses hold for arbitrary x, and let $I_x = \{t\colon t \in x\ \&\ \forall v\ (vRt \rightarrow v \in x)\}$. $1 \in I_x$, for $1 \in x$ and given any v with $vR1$, we have $v = 1$, so that $v \in x$. Now suppose $n \in I_x$ for an arbitrary n, then $\forall v\ (vRn \rightarrow v \in x)$, so that by hypothesis, $s(n) \in x$. We show that the assumption that $n \in I_x$ implies that $s(n) \in I_x$. Given arbitrary v, if $vRs(n)$, then $v = s(n)$ or $\exists p\ (p \in N\ \&\ v + p = s(n))$. There are three cases to consider.

Case 1. $v = s(n)$. Then as $s(n) \in x$, $v \in x$.

Case 2. $v + 1 = s(n)$. Then by Theorem 3.3, $s(v) = v + 1 = s(n)$, so $v = n$ by P1, and hence vRn, so $v \in x$.

Case 3. $\exists q\ (v + s(q) = s(n))$. Then $s(v + q) = v + s(q) = s(n)$, so that $v + q = n$ by P1. Hence vRn, so that $v \in x$.

So $s(n) \in x$ and $\forall v\ (vRs(n) \rightarrow v \in x)$, i.e., $s(n) \in I_x$.

Hence $I_x = N$ by P3', and as $I_x \subseteq x \subseteq N$, it follows that $x = N$.

Theorem 3.5 (Well-ordering) $\forall x\ (x \subseteq N \rightarrow (x = \emptyset \text{ or } \exists! a\ a = \text{first } x))$.

Proof It will suffice to prove that $\forall x\ (x \subseteq N \rightarrow x = \emptyset$ or $\exists a\ a = \text{first } x)$. Let x be any subset of N, and suppose $\forall a\ a \neq \text{first } x$. Define $y = N - x \subseteq N$. Then $1 \in y$, else 1 would be in x, and hence the first element of x. Now for arbitrary $t \in N$, suppose $\forall v\ (vRt \rightarrow v \in y)$, and consider $s(t)$. If we can show that $s(t) \in y$, or equivalently, that $s(t) \notin x$, then $y = N$ by Lemma 2, so that $x = \emptyset$, and the proof will be complete.

Given any $q \in x$, qRt is false, as this would imply $q \in y$. So by Lemma 1, we must have tRq. But as $t = q$ is also ruled out (as $t \in y$), it follows from the definition of R that $\exists p\ (t + p = q)$. Now if $p = 1$, $s(t) = t + 1 = q$, so s(t)Rq, and if $p \neq 1$, $\exists r\ s(r) = p$ and $s(t) + r = r + s(t) = s(r + t) = s(t + r) = t + s(r) = q$, so that s(t)Rq. Now if $s(t) \in x$, then as we have just shown that $\forall q\ (q \in x \rightarrow s(t)Rq)$, s(t) would be the first element of x. As by hypothesis x has no first element, we must have $s(t) \notin x$.

We are now ready to define what it means for a class in BGN' to be countable. This definition is implied trivially by the usual "finite or countably infinite" definition, and implies that definition, given the Recursion

Theorem and the Schröder-Bernstein Theorem. The latter implication is proved in Appendix I.

Definition 3.3 Ctbl x for $\exists f$ $f:x \to N$ & inj f.

Notice that by the theorem of subsets, f[x] is a set. Hence, by the axiom of images applied to the inverse function f^{-1}, $f^{-1}f[x] = x$ is a set. So every countable class x in BGN' is a set.

CHAPTER IV

THREE EQUIVALENT STATEMENTS OF THE AXIOM OF CHOICE

In this chapter three statements of the axiom of choice are given and shown to be equivalent.

AC_1. $\forall x \exists f\ (f : \mathcal{P}x - \{\emptyset\} \rightarrow x\ \&\ \forall y\ (y \in \mathcal{P}x - \{\emptyset\} \rightarrow f(y) \in y))$.

AC_2. $\forall x\ (x \neq \emptyset\ \&\ \emptyset \notin x \rightarrow \exists g\ g : x \rightarrow \bigcup x\ \&\ \forall y\ (y \in x \rightarrow g(y) \in y))$.

AC_3. $\forall x\ (x \neq \emptyset\ \&\ \emptyset \notin x\ \&\ \forall y \forall z\ (y \in x\ \&\ z \in x\ \&\ y \neq z \rightarrow y \cap z = \emptyset) \rightarrow \exists u \forall v\ (v \in x \rightarrow \exists! w\ \{w\} = u \cap v))$.

AC_1, already stated as the ninth axiom of BGN, was first published by E. Zermelo in 1904, in a paper in which he proved that every set can be well-ordered [10]. AC_2, although useful in its own right, is given here only to make the proof that AC_1 implies AC_3 easier to follow. AC_3 is the most commonly known form of the axiom of choice, and is usually what people have in mind when speaking about the axiom of choice. When it does not matter which version of the axiom of choice is referenced, the abbreviation AC will be used, as has been done earlier. We now make the following important distinctions.

<u>Definition 4.1</u> The "countable AC" is AC_3 with the additional hypothesis "ctbl x".

<u>Definition 4.2</u> The "AC for countable sets", denoted AC_1^*, is AC_1 with the hypothesis "ctbl x".

Therefore, in the BGN object language, AC_1^* is the following.

AC_1^*. $\forall x\ (\text{ctbl } x \rightarrow \exists f\ f{:}\mathcal{P}x - \{\emptyset\} \rightarrow x\ \&\ \forall y\ (y \in \mathcal{P}x - \{\emptyset\} \rightarrow f(y) \in y))$.

<u>Theorem 4</u> $AC_1 \rightarrow AC_2\ \&\ AC_2 \rightarrow AC_3\ \&\ AC_3 \rightarrow AC_1$.

<u>Proof</u> $AC_1 \rightarrow AC_2$: Let AC_1 hold and let x be an arbitrary class such that $x \neq \emptyset$ and $\emptyset \notin x$. Then by AC_1, $\exists f\ f{:}\mathcal{P}\cup x - \{\emptyset\} \rightarrow x\ \&\ \forall y\ (y \in \mathcal{P}\cup x - \{\emptyset\} \rightarrow f(y) \in y)$. Since $x \subseteq \mathcal{P}\cup x - \{\emptyset\}$ define $g = f|x$.

$AC_2 \rightarrow AC_3$: Let AC_2 hold and let x be an arbitrary class such that $x \neq \emptyset$, $\emptyset \notin x$, and $\forall y\ \forall z\ (y \in x\ \&\ z \in x\ \&\ y \neq z \rightarrow y \cap z = \emptyset)$. Then by AC_2, $\exists g\ g{:}x \rightarrow \cup x\ \&\ \forall y\ (y \in x \rightarrow g(y) \in y)$. Let $u = g[x]$, then given any $v \in x$ and any $w \in u \cap v$, $w = g(s)$ for some $s \in x$. If $s \neq v$, then $v \cap s \neq \emptyset$ which contradicts the hypothesis that members of x are pairwise disjoint. So $w = g(v)$, and as g is a function, w is unique for each v. Hence $\exists! w\ \{w\} = u \cap v$.

$AC_3 \rightarrow AC_1$: Let AC_3 hold and let x be an arbitrary class. If $x = \emptyset$, AC_1 is trivially true, so let $x \neq \emptyset$. Consider A, defined as follows.

$$A = \{y' : \exists y\ y \in \mathcal{P}x - \{\emptyset\}\ \&\ y' = y \times \{y\}\}.$$

$A \neq \emptyset$, $\emptyset \notin A$, and given y' and z' in A with $y' \neq z'$, we have that $y \neq z$ (as $y = z \rightarrow y \times \{y\} = z \times \{z\}$), so $y' \cap z' = y \times \{y\} \cap z \times \{z\} = (y \cap z) \times (\{y\} \cap \{z\}) = \emptyset$. We can therefore apply AC_3 to the class A: $\exists u \forall v (v \in A \rightarrow \exists! w \{w\} = u \cap v)$. So for each $y \in \mathcal{P}x - \{\emptyset\}$, there is an associated $y' = y \times \{y\}$ in A, and $u \cap y' = \{w\}$ for some unique ordered pair w. As $w \in y'$, $w = (t,y)$ for some $t \in y$; let $f(y) = t$. As w is unique for each y, t is unique for each y. Therefore f is a well-defined function.

It is AC_1^* that we will prove to be a theorem of BGN'. By examination of the proof that AC_1 implies AC_2, we see that AC_1^* implies the following.

AC_2^*. $\forall x$ (ctbl $\bigcup x$ & $x \neq \emptyset$ & $\emptyset \notin x \rightarrow \exists g\ g{:}x \rightarrow \bigcup x$ & $\forall y (y \in x \rightarrow g(y) \in y)$).

By examination of the proof that AC_2 implies AC_3, we see that AC_2^* implies the following.

AC_3^*. $\forall x$ (ctbl $\bigcup x$ & $x \neq \emptyset$ & $\emptyset \notin x$ & $\forall y \forall z (y \in x$ & $z \in x$ & $y \neq z \rightarrow y \cap z = \emptyset) \rightarrow \exists u \forall v (v \in x \rightarrow \exists! w \{w\} = u \cap v)$).

Hence AC_1^* implies AC_3^*, so when in the next chapter we show that BGN' implies AC_1^*, we will have that BGN' implies AC_3^*. We point out again that AC_3^* is not the countable AC.

CHAPTER V

THE DEMONSTRABILITY OF AC_1^* IN BGN'

We now prove the main result, that the axiom of choice for countable sets, as stated in Definition 4.2, is a theorem of BGN'.

<u>Metatheorem 5</u> BGN' implies AC_1^*.

<u>Proof</u> Let x be an arbitrary class, and suppose ctbl x. Then by Definition 3.3, $\exists g\ g:x \longrightarrow N$ & inj g. If $x = \emptyset$, then the theorem holds trivially, so let $x \neq \emptyset$. Then $\mathcal{P}x - \{\emptyset\} \neq \emptyset$, as $\mathcal{P}x - \{\emptyset\} = \emptyset$ implies $x = \emptyset$. Therefore let y be an arbitrary set in $\mathcal{P}x - \{\emptyset\}$, then as $y \neq \emptyset$ and $y \subseteq x$, $g[y] \neq \emptyset$ and $g[y] \subseteq N$. As the set N is well-ordered by R, g[y] has a unique first element a, i.e., $\exists! a\ a \in g[y]$ & $\forall t\ (t \in g[y] \rightarrow aRt)$. Denote this unique element by n_y. Then as g is an injection, $g^{-1}(n_y)$, where g^{-1} is the inverse function $g^{-1}:g[x] \longrightarrow x$, is a unique element of y for each y in the set $\mathcal{P}x - \{\emptyset\}$. Now we can define

$$f = \{t: \exists y\ y \in \mathcal{P}x - \{\emptyset\}\ \&\ t = (y, g^{-1}(n_y))\}.$$

Clearly, f is the required function.

It is important to point out how we have avoided the use of AC, specifically AC_3. True, we have an infinite (and uncountably so, if x is countably infinite) collection $\{g[y]: y \in \mathcal{P}x - \{\emptyset\}\}$ of non-empty sets, and we are choosing from each g[y] an element n_y. AC_3 would be needed to assert that n_y is unique if we had no criterion for the choice other than that each g[y] is non-empty. However, by well-ordering, we can assert the uniqueness of n_y, by taking it as the unique first element of g[y]. In other words, we have a concrete way to specify uniqueness. Thus use of AC has been avoided, as we have seen that the well-ordering of N by R derives from the properties of N via the axiom of infinity and other axioms of BGN'.

Finally, note that the defining property of f is a wff, so that f is a well-defined class by the axiom scheme of classification.

CHAPTER VI

CONCLUSIONS AND REMARKS

Since by Metatheorem 5, AC_1^* is a set-theorem of BGN', it follows by Metatheorem 2 that it can be proved in ZF'. We make two observations related to P. J. Cohen's proof [3] of the independence of AC from ZF'.

First of all, Cohen's proof puts AC in a position with respect to ZF' which is analogous to that of the "parallel postulate" with respect to the other axioms of plane Euclidean geometry: we can assume it or its negation; either way we get a relatively consistent theory. But if we assume its negation, i.e., if we work in a set theory based upon ZF' and the negation of AC, then by Metatheorem 5 and the equivalence results of Chapter IV, AC_1 still holds for countable sets x, and AC_2 and AC_3 still hold for sets x with countable sum $\cup x$.

Secondly, examination of AC_3^*, i.e., AC_3 for sets x with countable sum $\cup x$, possibly leads one to the following argument: since the sum of a countably infinite set of countably infinite sets is countably infinite (we call this theorem "CS"), and this can be shown in BGN or ZF without using AC, we have that BGN' implies the countable AC. The argument is invalid. P. Suppes [8, p. 158] asserts that all known proofs of CS assume AC. All elementary analysis texts we have seen give roughly the same proof of CS, and none acknowledge that AC has been used. Since the proofs are usually given in such a way that the dependence upon AC is not clear, we give in Appendix II a proof of CS in which the use of AC is explicitly shown. What is presently important is that it is in fact impossible to prove CS in ZF or BGN without using AC.

For suppose CS were a theorem of ZF', then it would be a true statement about every model for ZF'. But Cohen [3, p. 146] has found a model for ZF' in which the continuum is the sum of a countably infinite set of countably infinite sets. In other words, CS is false in this model. Hence it is impossible to prove CS in ZF'. Now suppose CS were a theorem of BGN', then as CS is a set-theorem, ZF' would imply CS by Metatheorem 2. Hence it is impossible to prove CS in BGN'.

APPENDIX I

Let N_k = {t: tRk} for each k in N, where R is the relation on N given in Definition 3.1. We define "finite x" to mean "$\exists k\ \exists f\ f:N_k \rightarrow x$ & bij f" and "countably infinite x" to mean "$\exists f\ f:N \rightarrow x$ & bij f". Then we can give, for sets in BGN', the usual definition of countability, which we denote by "ctbl'".

<u>Definition</u> Ctbl' x for finite x or countably infinite x.

We now show that countability as defined in Definition 3.3 implies the notion of countability just given.

<u>Theorem A.1</u> $\forall x$ (ctbl x $\rightarrow$ ctbl' x).

<u>Proof</u> Let x be arbitrary, and let ctbl x, so that $\exists f\ f:x \rightarrow N$ & inj f.

Case 1. $\exists t\ (t \in N\ \&\ \forall y\ (y \in f[x] \rightarrow yRt))$. Then $f[x] \subseteq N_t$, so by the easily shown fact that every subset of a finite set is finite, f[x] is finite, with say, $h:N_k \rightarrow f[x]$ for some k in N and bijection h. Thus $f^{-1}h:N_k \rightarrow x$ and bij $f^{-1}h$, and hence finite x.

Case 2. $\forall t\ (t \notin N$ or $\exists y\ (y \in f[x]\ \&\ \sim yRt))$. Then let $g:f[x] \rightarrow f[x]$ be defined as follows: for each $t \in f[x]$, $g(t)$ = first $\{y: y \in f[x]\ \&\ \sim yRt\}$. Then by Theorem 3.2, $\exists! h\ h:N \rightarrow f[x]$, $h(1)$ = first f[x], and for each n in N,

$h(n + 1) = g(h(n)) =$ first $\{y: y \in f[x]\ \&\ \sim yRh(n)\}$. It can be shown using P3' that h is an injection. So $f^{-1}h:N \rightarrow x$ is an injection. Therefore, by the Schröder-Bernstein Theorem, $\exists f'\ f':N \rightarrow x$ and bij f'. So countably infinite x.

J. L. Kelley [5, pp. 28-9, 276] gives a proof of the Schröder-Bernstein Theorem which does not use AC. The underlying set theory of Kelley, as given in his Appendix, is quite similar to BGN, which is not surprising, since both evolved from Gödel-Bernays set theory.

APPENDIX II

<u>Theorem A.2</u> Let $F = \{A_n : n \in N\}$ be a countably infinite collection of pairwise disjoint, countably infinite sets. Then $\cup F$ is countably infinite.

<u>Proof</u> Let $x \in \cup F$, then $\exists n\ x \in A_n$, and n is unique because the A_k are pairwise disjoint. Now x is the m^{th} element of A_n, for some m, and m is obviously unique. This follows because as A_n is countably infinite, $\exists f_n\ f_n : N \to A_n$ and f_n is a bijection. So f_n "counts" A_n and allows us to tag x as the m^{th} element of A_n. Let $f(x) = (m,n)$, then we have shown how to define an injection $f : \cup F \to N \times N$, so that $\cup F$ must be countably infinite.

But let $T_n = \{g : g : N \to A_n \ \&\ \text{bij } g\}$. This is certainly an infinite set. We have asserted that from the set $T = \{T_n : n \in N\}$, which is a non-empty, countably infinite collection of infinite, pairwise disjoint sets, we can form a set consisting of exactly one element f_n from each set T_n in the collection, without specifying any kind of procedure to make a unique choice of f_n. We can only appeal to the "countable axiom of choice": if T is a countably infinite collection of pairwise disjoint infinite sets, then there is a set consisting of exactly one element from each set in the collection.

Theorem A.2 is not quite the theorem CS which was stated in Chapter VI, but theorem CS follows immediately from Theorem A.2 and the fact that to any countably infinite collection of sets there corresponds a countably infinite collection of pairwise disjoint sets with the same sum. A proof of this last fact may be found in Apostol [1, p. 35].

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