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Eigenvalues of the mean sigma and weighted mean integral operators related to a Gronwall inequality.

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EIGENVALUES OF THE MEAN SIGMA AND WEIGHTED MEAN INTEGRAL OPERATORS RELATED TO A GRONWALL INEQUALITY

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CHAPTER ONE: INTRODUCTION

In their paper [1], Schmaedeke and Sell have established a Gronwall inequality which holds for the Stieltjes mean sigma and Dushnik integrals. In Chapter 2, we shall investigate an eigenvalue problem related to the Gronwall inequality for the Stieltjes mean sigma integral. In Chapter 3, we shall extend the results in [1] to the weighted refinement Stieltjes integral introduced by F. M. Wright and J. D. Baker [2].

We list for reference the main theorem of [1] and a lemma used to establish it.

<u>Theorem 1.1</u>: Let f and g be functions of bounded variation on $[0,T]$, and let $\varepsilon > 0$. Further let f and g be right continuous, $f \geq 0$, and g nondecreasing. If

$$f(t) \leq \varepsilon + \int_0^t f(s)\, dg(s), \quad 0 \leq t \leq T$$

then there exist constants T' and K, depending on g only, such that $0 < T' \leq T$, $0 \leq K$ and $f(t) \leq K\varepsilon$ for $0 \leq t < T'$. Further, T' is maximal in the sense that either $T' = T$ or $\Delta g(T') \geq 2$.

<u>Notation</u>: In this paper we use the notation introduced in [1], $\Delta g(t) = g(t) - g(t-)$. In addition, we shall make use of I^+ and $\mathbf{R}$ to denote the positive integers and the reals, respectively.

As a consequence of the bounded variation of f and g, we also have that $\lim_{t \to b-} g(t)$ and $\lim_{t \to b-} f(t)$ both exist for $b \in [0,T]$.

Lemma 1.2: If $f(t) \leqslant K\varepsilon$ for $0 \leqslant t \leqslant t_1$ then there are a $t_2 > t_1$ and a K' such that $f(t) \leqslant K'\varepsilon$ for $0 \leqslant t < t_2$.

We shall introduce some notation at this point which will facilitate our future discourse. Assume f and g are functions from an interval $[0,T]$ into $\mathbb{R}$

Definition 1.3: $f \in QC_{TR}$ means that f is quasicontinuous on $[0,T]$, and continuous from the right.

Definition 1.4: $g \in BV_{TR}$ means that $g \in QC_{TR}$ and that g is of bounded variation on $[0,T]$.

Definition 1.5: That the function f is nondecreasing will be denoted by $f\uparrow$.

It has been shown [3] that for $\int_0^t f(s)\, dg(s)$ to exist in the mean sigma sense, it is sufficient that f be quasicontinuous and g be of bounded variation. We may further note that $f \in QC_{TR}$ implies that there exists $B > 0$ such that $|f(t)| \leqslant B$ for $t \in [0,T]$. Thus both Theorem 1.1 and Lemma 1.2 hold if $f \in QC_{TR}$ and $g \in BV_{TR}$. (See details of proofs in [1].)

In Chapter 2, we shall explore the eigenvalues of the following operator:

Definition 1.6: Let $g \in BV_{TR}$ with $g\uparrow$ on $[0,T]$. Then the operator U is defined at each $f \in QC_{TR}$ with $f \geqslant 0$ on $[0,T]$ by

$$Uf(t) = \int_0^t f(s)\, dg(s), \quad 0 \leqslant t \leqslant T,$$

where the integral is the Stieltjes mean sigma integral.

CHAPTER TWO: EIGENVALUES OF THE OPERATOR U

Our major investigation in this chapter is motivated by a statement in [1] that the integral equation

$$f(t) = \int_0^t f(s)\, dg(s)$$

may have nontrivial positive solutions. Schmaedeke and Sell have indicated that a proof of the following theorem will appear in a future paper. Let U denote the operator of Definition 1.6.

Theorem 2.1: $\lambda > 0$ is an eigenvalue of U iff. there is a T', $0 < T' \leqslant T$, such that $\Delta g(T') = 2\lambda$.

We shall be concerned with proving this theorem with some added conditions to insure that the eigenfunction $f \geqslant 0$.

The failure of Gronwall's inequality when $\Delta g(T') \geqslant 2$ is related to Theorem 2.1 in the following manner:

We note that for $\lambda > 0$, if there exists $f \in QC_{TR}$, $f \neq 0$ such that

(i) $\lambda f(t) = \int_0^t f(s)\, dg(s)$, $0 \leqslant t \leqslant T$,

one may state the equivalent equality

(ii) $f(t) = \int_0^t f(s)\, d\, {}^1\!/\!_\lambda g(s)$, $0 \leqslant t \leqslant T$.

Theorem 2.1 says that there exists $T' > 0$ such that $\Delta[{}^1\!/\!_\lambda g](T') = 2$. Since $\Delta[{}^1\!/\!_\lambda g](T') = 2$, Theorem 1.1 gives us the result that the Gronwall inequality fails for ${}^1\!/\!_\lambda\, g$ at T'.

Since (i) and (ii) are equivalent, for the remainder of this chapter we shall be concerned with the eigenvalue problem for U with $\lambda = 1$.

PROPERTIES OF f: In this section let $f \in QC_{TR}$ and $g \in BV_{TR}$ with $g\uparrow$ be such that $f(t) = \int_0^t f(s)\, dg(s)$, i.e. f is an eigenfunction for U corresponding to $\lambda = 1$. Suppose also that $f \geqslant 0$.

Lemma 2.2: Let $\varepsilon = 0$. Then there is a T^* with $0 < T^* \leqslant T$ such that $ft = 0$ for $0 \leqslant t < T^*$.

Proof: By Theorem 1.1, since

$$f(t) = \int_0^t f(s)\, dg(s) \quad \text{for} \quad t \in [0,T],$$

there exist a T^* and $K \geqslant 0$ such that $T^* > 0$ and $f(t) \leqslant K\varepsilon = K \cdot 0$ for for $0 \leqslant t < T^*$. Thus $f(t) = 0$ for $0 \leqslant t < T^*$.

Lemma 2.3: If $f(t) = 0$ for $0 \leqslant t \leqslant t_1$ then there exists a t_2 with $T \geqslant t_2 > t_1$ such that $f(t) = 0$ for $0 \leqslant t < t_2$.

Proof: Let $K = 1$. For $\varepsilon = 0$, $f(t) \leqslant K\varepsilon$ for $0 \leqslant t \leqslant t_1$. By Lemma 1.2, there exists $t_2 > t_1$ and a $K' > 0$ such that $f(t) \leqslant K'\varepsilon$ for $0 \leqslant t < t_2$. Thus $f(t) = 0$ for $0 \leqslant t < t_2$.

This lemma has an immediate consequence.

Corollary 2.4: If f is an eigenfunction of U such that $f(t) > 0$ for $T \geqslant t > T^* > 0$ then $f(T^*) > 0$ also.

Proof: Suppose not, i.e. suppose $f(T^*) = 0$ and $f(t) > 0$ for $t > T^*$. By Lemma 2.3, there exists $t_2 > T^*$ such that $f(t) = 0$ for $0 \leqslant t < t_2$. This contradicts $f(t) > 0$ for $t > T^*$.

We now are in a position to prove the necessity of Theorem 2.1.

PROOF OF NECESSITY OF THEOREM 2.1: Suppose $\lambda = 1$ is an eigenvalue of U and $f \in QC_{TR}$ such that f is not identically 0 and $f = Uf$, i.e.

$$f(t) = \int_0^t f(s)\, dg(s), \text{ for } 0 \leqslant t \leqslant T.$$

Since $f(0) = 0$, by Lemma 2.3, there exists $t^* > 0$ such that $f(t) = 0$ for $0 \leqslant t < t^*$. Let $Z = \{ t \in [0,T]$ such that $f(s) = 0$ for $0 \leqslant s \leqslant t \}$. Z is not empty since $t = \frac{1}{2}t^*$ is in Z. Since Z is a bounded set, $Z \subset [0,T]$, there exists $T' \in [0,T]$ such that $T' =$ l.u.b. Z. We must have that $fT' \neq 0$ since $fT' = 0$ implies that there exists $t'' > T'$ such that $ft = 0$ for $0 \leq t < t''$ by Lemma 2.3 and T' would not be a l.u.b. for Z. We then have $f(t) = 0$ for $t < T'$ and $fT' \neq 0$.

Thus $f(T') = \int_0^{T'} f(s)\, dg(s) = \frac{1}{2}[fT' + f(T'-)]\, \Delta g(T')$ or $f(T') = \frac{1}{2}f(T')\Delta g(T')$. Since $f(T') \neq 0$, $\Delta g(T') = 2$.

<u>STEP FUNCTIONS</u>: Before we try to prove the sufficiency part of Theorem 2.1, let us look at a simple example which will illustrate that the condition in Theorem 2.1 is not enough to guarantee that the eigenfunction for λ be nonnegative.

Let the step function g be defined on $[0,3]$ as follows:

$gt = 0, \quad 0 \leqslant t < 1$

$gt = 2, \quad 1 \leqslant t < 2$

$gt = 5, \quad 2 \leqslant t \leqslant 3$

Since $\Delta g(1) = 2$, by Theorem 2.1, $\lambda = 1$ is an eigenvalue of U. We note that the following function f is an eigenfunction of U for $\lambda = 1$:

$ft = 0, \quad 0 \leqslant t < 1$

$ft = 1, \quad 1 \leqslant t < 2$

$ft = -5, \quad 2 \leqslant t \leqslant 3.$

As shown here, we have no immediate guarantee that $f \geqslant 0$. To remain in the context of [1], we require that $f \geqslant 0$ on $[0,T]$. We shall see later that a sufficient condition for this is that $\Delta g(t) < \Delta g(T')$ for all $T \geqslant t > T'$. We therefore restate the sufficiency part of Theorem 2.1 as a separate theorem:

<u>Theorem 2.5</u>: Let $g \in BV_{TR}$ and $g\uparrow$, $\lambda = 1$. If we let the operator U be defined by $U: f \longrightarrow \int_0^t f(s)\, dg(s)$, then $\lambda = 1$ is an eigenvalue of U having a nonnegative eigenfunction f if there exists a T', $T \geqslant T' > 0$ such that $\Delta g(T') = 2$ and

(iii) $\Delta g(t) < 2$ for $T' < t \leqslant T$.

In order to prove this theorem, we first investigate the nature of operators and eigenfunctions with g restricted to be a step function.

Let g be a step function on $[0,T]$ such that $g \in BV_{TR}$ and $g\uparrow$. Note that since g is a step function, g is of bounded variation. $g \in BV_{TR}$ and $g\uparrow$ implies that if $\{t_0, t_1, \ldots, t_m\}$ is the set of points at which g is discontinuous, then $gt = gt_j$ for $t_j \leqslant t < t_{j+1}$ and that $gt_j < gt_{j+1}$ for $j = 0,1,\ldots,m-1$. Further assume that $\Delta g(t_0) = 2$ and $\Delta g(t_j) \neq 2$ for $t_j > t_0$.

<u>Lemma 2.6</u>: Let g be a step function on $[t_0, T]$ such that $g \in BV_{TR}$ and $g\uparrow$ on $[0,T]$ and $\Delta g(t_0) = 2$. Then there exists an $f \in QC_{TR}$, $f \neq 0$, such that $f(t) = \int_0^t f(s)\, dg(s)$.

<u>Proof</u>: We shall construct f. Let $ft = 0$ for $0 \leqslant t < t_0$ where t_0 is such that $\Delta gt \neq 2$ for $t_0 < t \leqslant T$. In considering how to define ft_0 we note that we must have

$$\int_0^{t_0} fs\, dgs = \tfrac{1}{2}\left[ft_0 + 0\right] \Delta g(t_0) = f(t_0).$$

Since we wish to have $f \geq 0$, we let $ft_0 = K_0 > 0$. Since $gt = gt_0$, $\Delta g(t) = 0$ for $t_0 \leq t < t_1$. Thus we must also define $ft = K_0$ for $t_0 \leq t < t_1$ since for such t's

$$f(t) = \int_0^t f(s)\, dg(s) = \int_0^{t_0} f(s)\, dg(s) + \int_{t_0}^t f(s)\, dg(s)$$

$$\text{or} \quad f(t) = \int_0^{t_0} f(s)\, dg(s) + 0.$$

At t_1 we have

$$ft_1 = \int_0^{t_1} fs\ dgs = \tfrac{1}{2}\, ft_0 \Delta g(t_0) + \tfrac{1}{2}\left[ft_0 + ft_1\right]\Delta g(t_1)$$

Solving for ft_1 we get

$$ft_1 = K_0 \cdot \frac{1 + \frac{1}{2}\Delta gt_1}{1 - \frac{1}{2}\Delta gt_1}$$

which we will call K_1. By the same reasoning as above, we must have $ft = K_1$ for $t_1 \leq t < t_2$. Continuing in this manner we have for t_j

$$\text{(iv)} \quad K_j = \frac{K_0 + \frac{1}{2}(K_0 + K_1)\Delta gt_1 + \ldots + \frac{1}{2}(K_{j-2} + {}_{j-1})\Delta gt_{j-1} + \frac{1}{2}K_{j-1}\Delta gt_j}{1 - \frac{1}{2}\Delta g(t_j)}$$

Thus our construction yields a step function f such that $ft = 0$ for $0 \leq t < t_0$ and $ft = K_n$ if $t_n \leq t < t_{n+1}$ for $n = 0, \ldots, m-1$ and $ft = K_m$, $t_m \leq t \leq T$. To see that f is in fact an eigenfunction, let $t \in [0,T]$. Then there exists an $n \in \{0,1,\ldots,m\}$ such that $t_n \leq t < t_{n+1}$. By (iv)

$$\int_0^t fs\ dgs = K_0 + \tfrac{1}{2}(K_0 + K_1)\Delta gt_1 + \ldots + \tfrac{1}{2}(K_{n-1} + K_n)\Delta gt_n$$

$$= K_n(1 - \tfrac{1}{2}\Delta gt_n) + \tfrac{1}{2}K_n\, \Delta gt_n .$$

Therefore, $\int_0^t fs\ dgs = K_n$ which is ft. Thus $ft = \int_0^t fs\ dgs$ for $t \in [0,T]$.

<u>Corollary 2.7</u>: Using the notation of Lemma 2.6.

(a) If $\Delta g(t_n) < 2$ for $t_0 < t_n \leq T$, then $K_n > 0$, i.e. $f \geq 0$ on $[0,T]$.

(b) If $\Delta g(t_n) < 2$ for $t_0 < t_n \leqslant T$, f is nondecreasing, i.e. $K_n \leqslant K_{n+1}$, $n = 0, 1, \ldots, m-1$.

Proof: (a) We have chosen $K_0 > 0$ and if $\Delta g(t_1) < 2$, it follows that $K_1 = K_0 \dfrac{1 + \frac{1}{2}\Delta g t_1}{1 - \frac{1}{2}\Delta g t_1} > 0$ also. By (iv), if $K_0, K_1, \ldots, K_{n-1} > 0$, we get $K_n > 0$ if $\Delta g(t_n) < 2$.

(b) Again by (iv)

$$K_n = \frac{K_{n-1}(1 - \frac{1}{2}\Delta g t_{n-1}) + \frac{1}{2}K_{n-1}\Delta g t_{n-1} + \frac{1}{2}K_{n-1}\Delta g t_n}{1 - \frac{1}{2}\Delta g(t_n)}$$

or (v) $$K_n = K_{n-1}\,\frac{1 + \frac{1}{2}\Delta g t_n}{1 - \frac{1}{2}\Delta g t_n}$$

If $\Delta g t_n < 2$, we have by (v), $K_n > K_{n-1}$. We also get from (v),

(vii) $$K_n = K_0 \prod_{i=1}^{n} \frac{1 + \frac{1}{2}\Delta g t_i}{1 - \frac{1}{2}\Delta g t_i}, \text{ for } n = 1, 2, \ldots, m.$$

We may now state

Lemma 2.8: With the notation of Lemma 2.6, $K_n > K_{n-1} > 0$ for $n = 1, 2, \ldots, m$ iff. $\Delta g(t_n) < 2$.

Proof: By Corollary 2.7, $\Delta g(t_n) < 2$ implies that $K_n \geqslant 0$ and $K_n \geqslant K_{n-1}$ for $n = 1, 2, \ldots, m$. Now suppose $K_{n-1} \leqslant K_n$ for $n = 1, 2, \ldots, m$ and $K_0 > 0$. Then solving (v) for $\Delta g(t_n)$ we obtain

$$\Delta g(t_n) = 2\,\frac{K_n - K_{n-1}}{K_n + K_{n-1}}\,.$$

Since $K_n \geqslant K_{n-1} > 0$, we have $\Delta g(t_n) < 2$.

Thus $\Delta g(t_n) < 2$ if and only if $K_n \geqslant K_{n-1}$ and $K_n > 0$ for $n = 1, 2, \ldots, m$.

A CONVERGENCE LEMMA: We shall now investigate the convergence of a sequence of eigenfunctions.

To begin, we cite two useful theorems which are used in our proof. From reference [2] we have

Theorem 2.9: If g is of bounded variation on $[a,b]$ then there exists a sequence of step functions g_n on $[a,b]$ such that $|gt - g_n t| < 1/n$ if $a \leqslant t \leqslant b$, i.e. $\{g_n\}_{n=1}^{\infty}$ converges to g uniformly on $[a,b]$.

We also have from [2] :

Theorem 2.10: Suppose $f \in QC_{TR}$ and there is a sequence $\{g_n\}_{n=1}^{\infty}$ of functions such that

(1) There exists a $V > 0$ such that $V_0^T(g_n) \leqslant V$ if $n \in I^+$

(2) There exists a g such that $g_n \rightarrow g$ uniformly on $[0,T]$.

Then the function sequence $\left\{\int_0^t fs \, dg_n s\right\}_{n=1}^{\infty}$ converges uniformly to $\int_0^t fs \, dgs$ on $[0,T]$.

With these two theorems, we may state and prove the following Lemma:

Lemma 2.11: Suppose $g \in BV_{TR}$ and there exists a sequence $\{g_n\}_{n=1}^{\infty}$ such that $g_n \in BV_{TR}$, $g_n \rightarrow g$ uniformly on $[0,T]$, and $V_0^T(g_n) \leqslant V_0^T(g)$. Let $\lambda = 1$ be an eigenvalue for each operator U_n where

$$(U_n f)(t) = \int_0^t f(s) \, dg_n(s) \text{ if } 0 \leqslant t \leqslant T .$$

If $\{f_n\}_{n=1}^{\infty}$ is the corresponding sequence of eigenfunctions such that $U_n f_n = \lambda f_n$ and $f_n \in QC_{TR}$ for each $n \in I^+$, and if there exists $f \in QC_{TR}$ such that $f_n \to f$, then

$$ft = \int_0^t f(s)\ dg(s),$$

i.e. f is the eigenfunction corresponding to the operator U defined by g.

Proof: Let $\varepsilon > 0$. Then for $t \in [0,T]$,

$$\left| ft - \int_0^t fs\ dgs \right| \leqslant |ft - f_n t| + \left| f_n t - \int_0^t f_n s\ dg_n s \right| + \left| \int_0^t f_n s\ dg_n s - \int_0^t fs\ dg_n s \right| + \left| \int_0^t fs\ dg_n s - \int_0^t fs\ dgs \right|$$

Let $V = V_0^T(g)$. By Theorem 2.10 there exists $N \in I^+$ such that

$$\left| \int_0^t fs\ dg_n s - \int_0^t fs\ dgs \right| < \varepsilon/3 \quad \text{if } n \geqslant N \text{ and } t \in [0,T].$$

Since $f_n t \to ft$ for $t \in [0,T]$, there exists $N' \in I^+$ such that

$$|f_n t - ft| < \varepsilon/3V \text{ if } n \geqslant N'.$$

Then $|f_n t - ft| < \varepsilon/3$ and

$$\left| \int_0^t f_n s\ dg_n s - \int_0^t fs\ dg_n s \right| < (\varepsilon/3V)\ V_0^T(g_n) < \varepsilon/3 \text{ for } n \geqslant N'.$$

Since $f_n(t) = \int_0^t f_n s\ dg_n s$, the second term is zero.

Thus for $t \in [0,T]$, $\left| ft - \int_0^t fs\ dgs \right| < \varepsilon$ for each $\varepsilon > 0$.

Thus $ft = \int_0^t fs\ dgs.$

Corollary 2.12: With the conditions of Lemma 2.11, if each $f_n \in \{f_j\}_{j=1}^{\infty}$ is such that $f_n \geqslant 0$, then $f \geqslant 0$ also.

Proof: Suppose $f_n \geqslant 0$ for all $n \in I^+$ and $ft < 0$ at some $t \in [0,T]$. $f_n t \to ft$ implies, for $\varepsilon = \frac{1}{4}|ft|$, there exists an $N \in I^+$ such that $|f_n t - ft| < \varepsilon$ for $n \geqslant N$. Then

$$ft - \varepsilon < f_n t < ft + \varepsilon = ft + \tfrac{1}{4}|ft| = 3/4\ ft < 0$$

Thus $f_n t < 0$ for $n \geqslant N$. This is a contradiction.

CONSTRUCTION OF THE FUNCTION SEQUENCES: Suppose we have a function $g \in BV_{TR}$ such that $g\uparrow$ on $[0,T]$. Further, let us assume that there is a $t_0 \in [0,T]$ such that $0 < t_0 \leqslant T$, $\Delta g(t_0) = 2$ and $\Delta g(t) < 2$ if $t_0 < t \leqslant T$. We shall construct a sequence of step functions $\{g_n\}_{n=1}^{\infty}$ which converges to g and whose sequence of eigenfunctions $\{f_n\}_{n=1}^{\infty}$ converges to some $f \in QC_{TR}$. To this end, we state the following two lemmas.

Lemma 2.13: With $g \in BV_{TR}$ and $g\uparrow$ on $[a,b]$, let $\varepsilon > 0$. If $gb - ga \geqslant \varepsilon$, there exists a $t^* \in [a,b]$, such that $gt^* - ga \geqslant \varepsilon$ and $gt - ga < \varepsilon$ for $a \leqslant t < t^*$.

Proof: Suppose $\varepsilon > 0$ and $gb - ga \geqslant \varepsilon$. Let $t_1 = \frac{1}{2}(a + b)$. We consider these two cases:

(1) If b is such that $gt - ga < \varepsilon$ for all $t < b$ then $t^* = b$.

(2) The alternative to Case 1 is that there exists a $t' < b$ such that $gt' - ga \geqslant \varepsilon$.

Let $\Delta = \{t \in [a,b]$ such that $gt - ga \geqslant \varepsilon\}$. Let $t^* = \inf \Delta$. Since g is right continuous, $t^* \in \Delta$. Thus $gt - ga < \varepsilon$ for $a \leqslant t < t^*$ and $gt^* - ga \geqslant \varepsilon$.

Lemma 2.14: Let $g \in BV_{TR}$, $g\uparrow$. Let $[a,b] \subset [0,T]$. Suppose $\Delta g(b) < \gamma$ for $\gamma > 0$. Then there exists a $t^* \in [a,b)$ such that $gb - gt^* < \gamma$.

Proof: $\Delta g(b) < \gamma$ means that $g(b) - g(b-) < \gamma$, or $\lim_{t \to b-} gt > gb - \gamma$

Thus there exists a $\delta > 0$ such that $gt > gb - \gamma$ if $b - \delta < t < b$. Let $t^* = \min\{b - \delta/2, a\}$. Then $gb - gt^* < \gamma$.

<u>Corollary 2.15</u>: With the notation of Lemma 2.14, $gb - gt < \gamma$ for $t^* \leq t < b$.

<u>Proof</u>: Since $g\uparrow$, $gt \geq gt^* > gb - \gamma$. Thus $gb - gt < \gamma$ for $t^* \leq t < b$.

Since $g \in BV_{TR}$ there are only a finite number of $t \in [0,T]$ such that $\Delta g(t) = 2$. Since we may pick t_0 to be the largest of these, condition 2 below may be satisfied. If $t_0 = T$, Theorem 2.5 is satisfied with $ft = 0$ for $0 \leq t < T$ and $fT = 1$. In the remainder of this chapter we shall assume $t_0 < T$.

We shall now construct the step function g_1, assuming that

(1) $g \in BV_{TR}$ and $g\uparrow$,

(2) there exists a $t_0 \in [0,T]$ such that $\Delta g(t_0) = 2$ and $\Delta g(t) < 2$ for $t_0 < t \leq T$.

Let $g_1 t = gt$ for $0 \leq t \leq t_0$. Using Lemma 2.13 with $\varepsilon = 1$, we can find a t_1' such that $gt - gt_0 < 1$ for $t_0 \leq t < t_1'$ and $gt_1' - gt_0 \geq 1$. We then let $g_1 t = gt_0$ for $t_0 \leq t < t_1'$ and $g_1 t_1' = gt_1'$. If it happens that $gt_1' - gt_0 \geq 2$, by Lemma 2.14 there exists a $t^* \in (t_0, t_1')$ such that $gt_1' - gt^* < 2$. In this case we let $t_1 = t^*$ and $t_2 = t_1'$ and define

$$g_1 t = gt_0 \text{ for } t_0 \leq t < t_1 \quad \text{and}$$

$$g_1 t = gt_1 \text{ for } t_1 \leq t < t_2 \quad \text{and}$$

$$g_1 t_2 = gt_2 .$$

We then have $|gt - g_1 t| < 1$ for $0 \leq t \leq t_2$ and $\Delta g_1(t) < 2$ for $t_0 < t \leq t_2$.

We repeat the above construction on $[t_2, T]$ to get t_3, or both t_3 and t_4, which satisfy the conditions that $|gt - g_1 t| < 1$ for $0 \leqslant t \leqslant t_3$ (or t_4) and $\Delta g_1(t) < 2$ for $t_0 < t \leqslant t_3$ (or t_4). We may continue constructing $\{t_i\}_{i=1}^{m_1}$ in this manner, letting $t_{m_1} = T$. The set of t_i's is finite since $g \in BV_{TR}$ and $g\uparrow$.

Thus we have constructed a function, g_1, such that g_1 is a step function on $[t_0, T]$ and $g_1 t = gt$ for $0 \leqslant t \leqslant t_0$. g_1 has the properties that for $n = 1$

(ix) $|gt - g_1 t| < 1$ for $0 \leqslant t \leqslant T$,

(x) $\Delta g_n(t_0) = 2$,

(xi) $\Delta g_n(t) < 2$ if $t_0 < t \leqslant T$, and

(xii) $g_n t = gt_j$ if $t_j \leqslant t < t_{j+1}$.

Thus $g_1 \in BV_{TR}$ and $g_1\uparrow$.

Now suppose that for $n \in I^+$, g_n is a step function satisfying properties (x), (xi), and (xii) above and that $|gt - g_n t| < 1/k$ on $[0,T]$ for some $k \in I^+$ but $|gt - g_n t| \geqslant \frac{1}{k+1}$ somewhere on $[0,T]$. We apply Lemma 2.13 to g_n with $\varepsilon = \frac{1}{k+1}$ on each interval in $\left\{ [t_i, t_{i+1}] \right\}_{i=0}^{m_n - 1}$ stopping when we first find a t^* in some interval $[t_i, t_{i+1}]$ such that $t_i < t^* < t_{i+1}$, $|gt - g_n t| < \frac{1}{k+1}$ if $t_0 \leqslant t < t^*$, and $|gt^* - g_n t^*| \geqslant \frac{1}{k+1}$.

We define $g_{n+1} t = g_n t$ for $0 \leqslant t < t^*$, $g_{n+1} t = gt^*$ for $t^* \leqslant t < t_{i+1}$ and $g_{n+1} t = g_n t$ for $t_{i+1} \leqslant t \leqslant T$. Thus g_{n+1} differs from g_n only on $[t^*, t_{i+1})$. We then have $|gt - g_{n+1} t| < \frac{1}{k+1}$ on $[0, t^*]$ and $|gt - g_{n+1} t| < 1/k$ on $[t^*, T]$.

We continue in this manner with $\varepsilon = \frac{1}{k+1}$ until we have g_{n+p}

at some point in our construction such that $|gt - g_{n+p}t| < \frac{1}{k+1}$ for all $t \in [0,T]$. We then let $\varepsilon = 1/_{k'}$, where $k' \geqslant k+2$ is such that $|gt - g_{n+p}t| < \frac{1}{k'-1}$ on $[0,T]$ but $|gt - g_{n+p}t| \geqslant 1/_{k'}$, for some $t \in [0,T]$. If for some $n \in I^+$, $|gt - g_n t| < 1/_p$ for all $p \geqslant k$, $gt = g_n t$ for all $t \in [0,T]$.

Using the results of Lemma 2.6 and Corollary 2.7, for each g_n we can find a function $f_n \in QC_{TR}$ such that $f_n \geqslant 0$ on $[0,T]$, $f_n t = 0$ on $[0,t_0)$, and $f_n t_0 = K_0$. Note that even though g_n may not be a step function on $[0,t_0)$, since $f_n t = 0$ there, $f_n(t) = \int_0^t f_n(s)\, dg_n(s)$ for $t \in [0,t_0)$. We also have $f_n \uparrow$ by Corollary 2.7.

We shall now explore certain relationships among the f_n's. Let us adopt the following notation which will be used in the remainder of this chapter:

1) For g_n we have $\{t_i\}_{i=0}^{m_n}$ such that $0 < t_0 < t_1 < \dots < t_{m_n} = T$ and $g_n t = g_n t_i = g t_i$ for $t_i \leqslant t < t_{i+1}$ $i = 0,1,\dots,m_n - 1$ and $g_n t_{m_n} = gT$.

2) For f_n corresponding to g_n we have

$$f_n t = K_i \quad \text{if } t_i \leqslant t < t_{i+1}, \quad i = 0,1,\dots,m_n$$

3) If t^* is the point added to the set of points $\{t_i\}_{i=1}^{m_n}$ of discontinuity of g_n to form g_{n+1}, let $f_{n+1}t = K^*$ if $t^* \leqslant t < t_{i+1}$, where $t_i < t^* < t_{i+1}$.

4) Let $f_{n+1}t = K_j^*$ for $t_j \leqslant t < t_{j+1}$ if $i+1 \leqslant j < m_n$

5) $f_{n+1}T = K_{m_n}^*$

<u>Lemma 2.16</u>: Using the above notation, $K_i < K^* < K_{i+1}^* < K_{i+1}$ and $K_j^* > K_j$ if $j = i+1,\dots,m_n$. Thus $f_{n+1}t = f_n t$ if $0 \leqslant t < t^*$,

$f_{n+1}t > f_n t$ if $t^* \leqslant t < t_{i+1}$ and $f_{n+1}t < f_n t$ if $t_{i+1} \leqslant t \leqslant T$.

<u>Proof</u>: Using equation (v) on page 8, we have

$$K^* = K_i \quad \frac{1 + \frac{1}{2}\Delta g_{n+1}t^*}{1 - \frac{1}{2}\Delta g_{n+1}t^*} \quad .$$

Since $0 < \Delta g_{n+1}t^* < \Delta g_n t_{i+1}$ we have

$$K_i < K^* = K_i \frac{1 + \frac{1}{2}\Delta g_{n+1}t^*}{1 - \frac{1}{2}\Delta g_{n+1}t^*} < K_i \frac{1 + \frac{1}{2}\Delta g_n t_{i+1}}{1 - \frac{1}{2}\Delta g_n t_{i+1}} = K_{i+1}$$

$$\text{or} \quad K_i < K^* < K_{i+1} \ .$$

To show $K^*_{i+1} < K_{i+1}$ note $\Delta g_{n+1}t^* + \Delta g_{n+1}t_{i+1} = \Delta g_n t_{i+1}$. Thus consider $q_1, q_2, q > 0$ such that $q_1 + q_2 = q < 1$. Then

$$\frac{1+q_1}{1-q_1} \cdot \frac{1+q_2}{1-q_2} = \frac{1 + q/(1+q_1q_2)}{1 - q/(1+q_1q_2)}$$

Since $q/(1+q_1q_2) < q$, we get

$$\frac{1+q_1}{1-q_1} \cdot \frac{1+q_2}{1-q_2} < \frac{1+q}{1-q} \ .$$

Substituting $\frac{1}{2}\Delta g_{n+1}t^*$, $\frac{1}{2}\Delta g_{n+1}t_{i+1}$, $\frac{1}{2}\Delta g_n t_{i+1}$ for q_1, q_2, q respectively, we have

$$K^*_{i+1} = K_i \quad \frac{1 + \frac{1}{2}\Delta g_{n+1}t^*}{1 - \frac{1}{2}\Delta g_{n+1}t^*} \cdot \frac{1 + \frac{1}{2}\Delta g_{n+1}t_{i+1}}{1 - \frac{1}{2}\Delta g_{n+1}t_{i+1}}$$

$$< K_i \quad \frac{1 + \frac{1}{2}\Delta g_n t_{i+1}}{1 - \frac{1}{2}\Delta g_n t_{i+1}} = K_{i+1}$$

Thus $\qquad K_i < K^* < K^*_{i+1} < K_{i+1} \ .$

For $j = i+2, i+3, \ldots, m_n$, we have

$$K_j^* = K_{i+1}^* \prod_{k=i+2}^{j} \frac{1+\frac{1}{2}\Delta g_{n+1}t_k}{1-\frac{1}{2}\Delta g_{n+1}t_k} = K_{i+1}^* \prod_{k=i+2}^{j} \frac{1+\frac{1}{2}\Delta g_n t_k}{1-\frac{1}{2}\Delta g_n t_k}$$

$$< K_{i+1} \prod_{k=i+2}^{j} \frac{1+\frac{1}{2}\Delta g_n t_k}{1-\frac{1}{2}\Delta g_n t_k} = K_j \quad .$$

Thus $K_j^* < K_j$ for $j = i+1, \ldots, m_n$ and $f_{n+1} = f_n$ on $[0, t^*)$, $f_{n+1} > f_n$ on $[t^*, t_{i+1})$, and $f_{n+1} < f_n$ on $[t_{i+1}, T]$.

Lemma 2.17: Let $\varepsilon > 0$. If $|f_n t_{j+1} - f_n t_j| < \varepsilon$, then $|f_{n+1} t_{j+1} - f_{n+1} t_j| < \varepsilon$.

Proof: Case 1 - $t^* > t_{j+1}$. Then $f_{n+1} = f_n$ on $[0, t_{j+1}] \subset [0, t^*)$ and the result is obvious.

Case 2 - $t_j < t^* < t_{j+1}$. By Lemma 2.16

$f_{n+1} t_{j+1} < f_n t_{j+1}$ and $f_{n+1} t_j = f_n t_j$ thus

$$|f_{n+1} t_{j+1} - f_{n+1} t_j| < |f_n t_{j+1} - f_n t_j| < \varepsilon \quad .$$

Case 3 - $t^* < t_j$. Then we have

$$K_{j+1}^* = K_j^* \frac{1+\frac{1}{2}\Delta g_{n+1} t_{j+1}}{1-\frac{1}{2}\Delta g_{n+1} t_{j+1}} = K_j^* \frac{1+\frac{1}{2}\Delta g_n t_{j+1}}{1-\frac{1}{2}\Delta g_n t_{j+1}}$$

and
$$K_{j+1} = K_j \frac{1+\frac{1}{2}\Delta g_n t_{j+1}}{1-\frac{1}{2}\Delta g_n t_{j+1}} \equiv \gamma K_j$$

Therefore

$$K_{j+1} - K_j = (\gamma - 1) K_j > (\gamma - 1) K_j^* = K_{j+1}^* - K_j^*$$

or
$$\varepsilon > |f_n t_{j+1} - f_n t_j| > |f_{n+1} t_{j+1} - f_{n+1} t_j| \quad .$$

Lemma 2.18: There exists a $B > 0$ such that $f_n t \leqslant B$ for all $n \in I^+$ and for all $t \in [0,T]$.

Proof: By Corollary 2.7, part b, we have that $f_1 t_{m_1} = f_1 T \geqslant f_1 t$ for all $t \in [0,T]$. By Lemma 2.16, since $t^* < T$, $f_{n+1} T < f_n T$. Then for each $n = 2, 3, \ldots f_n t \leqslant f_n T < f_1 T$. Let $B = f_1 T$. Then $f_n t \leqslant B$ for all $n \in I^+$ and for all $t \in [0,T]$.

Lemma 2.19: Let $\varepsilon > 0$. Using the notation preceeding Lemma 2.16, there exists an $N \in I^+$ such that $|K^* - K_i| < \varepsilon$ for each g_{n+1} such that $n \geqslant N$.

Proof: Let $\delta > 0$. Then there exists an $N \in I^+$ such that $|gt - g_n t| < 2\delta$ for $n \geqslant N$. Then $\frac{1}{2}\Delta g_{n+1} t^* = \frac{1}{2}(gt^* - g_n t^*) < \delta$. Therefore

$$|K^* - K_i| = \left| \frac{1 + \frac{1}{2}\Delta g_{n+1} t^*}{1 - \frac{1}{2}\Delta g_{n+1} t^*} - 1 \right| K_i$$

$$< \left| \frac{1 + \delta}{1 - \delta} - 1 \right| K_i$$

$$< \frac{2\delta}{1 - \delta} K_i$$

Since $K_i \leqslant B$, by Lemma 2.18,

$$|K^* - K_i| < \frac{2\delta}{1 - \delta} B .$$

If we let $\delta = \dfrac{\varepsilon}{2B + \varepsilon}$, then there exists an $N \in I^+$ such that $|K^* - K_i| < \varepsilon$ for $n \geqslant N$.

With Lemmas 2.16, 2.17, 2.18, and 2.19, we can now state and prove

Lemma 2.20: Given the sequence $\{f_n\}_{n=1}^{\infty}$ described on page 14 preceeding Lemma 2.16, there exists f such that

$$ft = \lim_{n\to\infty} f_n t \text{ for each } t \in [0,T] .$$

Proof: Case 1 - Suppose $t \in [0,T]$ such that for some n, $t = t_i$, i.e. t is a point of discontinuity for some g_n. Then t is also a discontinuity point for all g_m such that $m \geqslant n$. Then for each $m \geqslant n$, $f_m t \geqslant f_{m+1} t$ by Lemma 2.16. Thus $\{f_m t\}_{m=n}^{\infty}$ is a nonincreasing sequence. Let $ft = \lim_{m\to\infty} f_m t$; the limit exists since $f_m t \geqslant 0$ and $f_m t \downarrow$ for $m \geqslant n$.

Case 2 - Suppose $t \in [t_0, T]$ such that for each g_n there exist t_{i_n}, $t_{i_n + 1}$ such that $t_{i_n} < t < t_{i_n + 1}$. We must have $\Delta g(t) = 0$. Otherwise $g_n t_{i_n} \leqslant g(t-) < gt$ implies that

$$|gt - g_n t| = |gt - g_n t_{i_n}| \geqslant |gt - g(t-)| = \gamma > 0$$

for all $n \in I^+$. But there exists an $N \in I^+$ such that $|gt - g_n t| < \gamma$ for all $n \geqslant N$, so we have a contradiction. This then gives us the fact that $\Delta g_n(t_{i_n + 1}) \longrightarrow 0$ as $n \to \infty$.

By the construction of the g_n's, we have

$$t_{i_1} \leqslant t_{i_2} \leqslant \cdots < t < \cdots \leqslant t_{i_2 + 1} \leqslant t_{i_1 + 1} .$$

By Corollary 2.7, for each n, we get

$$f_n t_{i_1} \leqslant f_n t_{i_2} \leqslant \cdots \leqslant f_n t_{i_n} < f_n t_{i_n + 1} \leqslant \cdots \leqslant f_n t_{i_2 + 1} \leqslant f_n t_{i_1 + 1}$$

Since $\Delta g_n(t_{i_n + 1}) \to 0$, given $0 < \delta < \frac{1}{2}$, there exists an $N \in I^+$ such that $\Delta g_n(t_{i_n + 1}) < \delta$ if $n \geqslant N$.

$$\text{Thus } |f_n t_{i_n + 1} - f_n t_{i_n}| = \left| \frac{1 + \frac{1}{2}\Delta g_n t_{i_n + 1}}{1 - \frac{1}{2}\Delta g_n t_{i_n + 1}} - 1 \right| f_n t_{i_n} .$$

If $B > 0$ is as stated in Lemma 2.18,

$$\left| f_n t_{i_n+1} - f_n t_{i_n} \right| < \frac{\delta}{1 - \frac{1}{2}\delta} \quad B$$

Thus for $\varepsilon > 0$, there exists an $N \in I^+$ such that

$$\left| f_n t_{i_n+1} - f_n t_{i_n} \right| < \varepsilon \quad \text{if } n \geqslant N.$$

Using Lemma 2.17, we have that $\left| f_N t_{i_N+1} - f_N t_{i_N} \right| < \varepsilon$ implies that $\left| f_n t_{i_N+1} - f_n t_{i_N} \right| < \varepsilon$. Noting that

$$f_n t_{i_N+1} \geqslant f_n t_{i_n} \geqslant f_n t_{i_N} \quad ,$$

we have $\left| f_n t_{i_n} - f_n t_{i_N} \right| < \varepsilon$.

Since $f t_{i_N}$ exists and equals $\lim_{n\to\infty} f_n t_{i_N}$, given $\varepsilon > 0$ there exists an $N' \in I^+$ such that $\left| f_n t_{i_N} - f\, t_{i_N} \right| < \varepsilon$ if $n \geqslant N'$.

Let $N^* = \max \{ N, N' \}$. Then

$$\begin{aligned} \left| f_m t - f_n t \right| &= \left| f_m t_{i_m} - f_n t_{i_n} \right| \\ &\leqslant \left| f_m t_{i_m} - f_m t_{i_N} \right| + \left| f_m t_{i_N} - f t_{i_N} \right| + \left| f t_{i_N} - f_n t_{i_N} \right| \\ &\quad + \left| f_n t_{i_N} - f_n t_{i_n} \right| < 4\varepsilon \text{ , if } m,n \geqslant N^*. \end{aligned}$$

Thus $\{ f_n t \}_{n=1}^{\infty}$ is Cauchy and hence $\lim_{n\to\infty} f_n t$ exists. Define $ft = \lim_{n\to\infty} f_n t$.

Having shown that f is the pointwise limit of $\{ f_n \}_{n=1}^{\infty}$, it remains to be shown that $f \in QC_{TR}$.

We shall now show that the limit function $f \in QC_{TR}$ and then applying Lemma 2.11, we have that

$$f(t) = \int_0^t f(s)\, dg(s) \text{ if } 0 \leqslant t \leqslant T.$$

We then get that $\lambda = 1$ is an eigenvalue of U.

First let us note the following facts about f:

Remarks:

(1) $f\uparrow$.

(2) $f \geq 0$.

Proof: (1) If $f(t_2) < f(t_1)$ for $t_1 < t_2$, let $\mathcal{E} = f(t_1) - f(t_2)$. Then there exists an N_1 such that $|ft_1 - f_n t_1| < \mathcal{E}/4$ for all $n \geq N_1$ and there exists an N_2 such that $|ft_2 - f_n t_2| < \mathcal{E}/4$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then

$$f_N t_2 = f_N t_2 - ft_2 + ft_2 - ft_1 + ft_1 - f_N t_1 + f_N t_1$$
$$\leq \quad \mathcal{E}/4 - \mathcal{E} \quad + \mathcal{E}/4 + f_N t_1$$
$$f_N t_2 \leq f_N t_1 - \mathcal{E}/2 < f_N t_1 \ .$$

(2) $f \geq 0$ by Corollary 2.12.

Lemma 2.21: If $a \in (t_0, T)$ is such that a is not a point of discontinuity for any g_n, then f is continuous at a.

Proof: In the proof of Lemma 2.20, Case 2, we had the existence of an $N \in I^+$ such that $|f_n t_{i_N+1} - f_n t_{i_N}| < \mathcal{E}$ for all $n \geq N$, where $t_{i_N} < a < t_{i_N+1}$. If $t \in (t_{i_N}, t_{i_N+1})$, for each $n \geq N$, we have $|f_n t - f_n a| < \mathcal{E}$ using Lemma 2.17 and the fact that $f_n \uparrow$.

Since $f_n t \to ft$ and $f_n a \to fa$, there exists an N' such that $|f_n t - ft| < \mathcal{E}$ and $|f_n a - fa| < \mathcal{E}$ for all $n \geq N'$.

Let $N^* = \max\{N, N'\}$. Then

$$|ft - fa| \leq |ft - f_{N^*}t| + |f_{N^*}t - f_{N^*}a| + |f_{N^*}a - fa| < 3\mathcal{E}$$

for all $t \in (t_{i_N}, t_{i_N+1})$.

Thus if $\delta = \min\{a - t_{i_N}, t_{i_N+1} - a\}$, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|ft - fa| < 3\varepsilon \quad \text{if } |t - a| < \delta .$$

Lemma 2.22: Suppose $a \in [t_0, T)$ is such that for some g_n, $a = t_i$ (t_i as in the discussion on page 14). Then f is right continuous at a.

Proof: If $\{t_0, t_1, t_2, \ldots, t_i, t_{i+1}, \ldots, t_{m_n}\}$ is the set of points at which g_n is discontinuous, let $a = t_i$. We may assume $gt > ga$ for $t > a$, otherwise $gt = ga$ for $a \leqslant t < t^*$ implies that $ft = fa$ for $a \leqslant t < t^*$ by the way our f_n's are constructed.

Then at some point in our construction of $\{g_n\}_{n=1}^{\infty}$, there exists a g_m such that $a = t_i$ and t_{i+1} are the same successive points of discontinuity for both g_n and g_m but t' for g_{m+1} is such that $t_i < t' < t_{i+1}$.

Then by Lemma 2.19, for $\varepsilon > 0$, there exists an N such that $|f_{m+1}t' - f_{m+1}t_i| < \varepsilon$ if $m \geqslant N$. Thus, given $\varepsilon > 0$, picking g_n such that $n \geqslant N$, there exists $t' > a$ such that $|f_{m+1}t' - f_{m+1}a| < \varepsilon$ for some $m \geqslant n$.

By Lemma 2.17,

$$|f_j t' - f_j a| < \varepsilon \quad \text{for } j \geqslant m+1.$$

If $a \leqslant t < t'$ $\quad |ft - fa| \leqslant |ft' - fa|$.

Since $ft' = \lim_{n\to\infty} f_n t'$ and $fa = \lim_{n\to\infty} f_n a$, given $\varepsilon > 0$ there exists an N such that $|fa - f_n a| < \varepsilon$ and $|ft' - f_n t'| < \varepsilon$ for all $n \geqslant N$.

Let $n = \max\{N, m+1\}$. Then

$$|ft - fa| \leqslant |ft' - f_n t'| + |f_n t' - f_n a| + |f_n a - fa|$$
$$< 3\varepsilon .$$

Therefore, letting $\delta = t' - a$, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|ft - fa| < \varepsilon$ if $a \leqslant t < a + \delta$.

Lemma 2.23: If $a \in [t_0, T]$, $f(a-)$ exists.

Proof: Since $f\uparrow$, $0 \leqslant ft \leqslant fa$ if $0 \leqslant t < a$. Let $\mathcal{A} = \{ft \text{ such that } 0 \leqslant t < a\}$. $\mathcal{A}$ is bounded above by fa thus, there exists a $b = \text{l.u.b.}\ \mathcal{A}$.

Let $\varepsilon > 0$, there exists $ft^* \in \mathcal{A}$ such that $b - \varepsilon < ft^* \leqslant b$.

$f\uparrow$ implies that $b - \varepsilon < ft \leqslant b$ for each t such that $a > t > t^*$. Let $\delta = a - t^*$. Thus, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|b - ft| < \varepsilon \quad \text{for all } t \in (a - \delta, a) .$$

Thus $b = \lim_{t \to a-} ft = f(a-)$.

We have shown that for $\lambda = 1$, and for $g \in BV_{TR}$, the condition that $\Delta g(t) < 2$ for all t such that $T \geqslant t > t_0$ where $\Delta g(t_0) = 2$ is sufficient to guarantee that there exists an eigenfunction f of U such that $f \geqslant 0$ on $[0,T]$.

As noted on page 12, there are only a finite number of $t \in [0,T]$ such that $\Delta g(t) = 2$. Since we have picked t_0 to be the largest of these we need only consider what happens to f if $\Delta g(t) > 2$ for $t_0 < t \leqslant T$.

Let us assume there exists t, $t_0 < t \leqslant T$, such that $\Delta g(t) > 2$. We have

$$ft = \int_0^t fs\ dgs = \int_0^{t-} fs\ dgs + \int_{t-}^{t} fs\ dgs$$

or

$$ft = f(t-) + \tfrac{1}{2}\left[ft + f(t-)\right] \Delta g(t).$$

Solving for ft we obtain

$$ft = f(t-)\ \frac{1+\frac{1}{2}\Delta gt}{1-\frac{1}{2}\Delta gt}\ .$$

$$\Delta g(t) > 2 \text{ implies that } \frac{1+\frac{1}{2}\Delta gt}{1-\frac{1}{2}\Delta gt} < 0.$$

Thus either $ft < 0$ or $f(t-) < 0$. In either case this implies that there exists $\tilde{t} \in (t_0, T]$ such that $f\tilde{t} < 0$. This contradicts the fact that $f \geqslant 0$ on $[0,T]$. Thus our condition that $\Delta g(t) < 2$ for $t > t_0$ is both necessary and sufficient for f to be a nonnegative eigenfunction of U on $[0,T]$.

CHAPTER THREE: A GRONWALL INEQUALITY FOR THE WEIGHTED REFINEMENT INTEGRAL

In addition to the Gronwall inequality derived by Schmaedeke and Sell [1], Gronwall inequalities for other types of integrals have been established, one of the more recent being for the linear Stieltjes integral by J. V. Herod [4]. In this chapter we shall establish a Gronwall inequality for the weighted refinement integral introduced by Wright and Baker in [2].

We shall quote here for reference Definition 1.1 of [2].

<u>Definition 3.1</u>: Let $p \in I^+$ such that $p \geqslant 2$ and let $(w_1, w_2, \ldots, w_p)$ be an ordered p-tuple in $\mathbf{R}^p$ such that $w_1 + w_2 + \ldots + w_p = 1$. Let f and g be real valued functions on the closed interval $[a,b] \subset \mathbf{R}$. For a partition

$$P = \{a = x_0 < x_1 < \ldots < x_n = b\}$$

of $[a,b]$, choose for each $i = 1,2,\ldots,n$ a partition

$$\Delta_i = \{x_{i-1} = t_{1,i} < t_{2,i} < \ldots < t_{p,i} = x_i\}$$

of $[x_{i-1}, x_i]$ consisting of p points. Form the sum

$$S(P; \Delta_1, \ldots, \Delta_n) = \sum_{i=1}^{n} \left\{ \sum_{j=1}^{p} w_j \cdot f(t_{j,i}) \right\} [gx_i - gx_{i-1}]$$

If the refinement limit $\lim S(P; \Delta_1, \ldots, \Delta_n)$ exists and is finite, this limit will be denoted by

$$\int_a^b f(x)\, dg(x) ,$$

which is called the weighted refinement integral of f with respect to g on $[a,b]$.

In order to produce a result which is analogous to that in reference [1], we must impose the condition that

(i) g is right continuous on $[0,T]$.

We shall require that the functions f be bounded on $[0,T]$.

From [2] we have that if $\int_0^T ft\, dgt$ exists then

(ii) $f(c-)$ must exist for each $c \in (0,T]$ such that $g(c-) \neq g(c)$.

We shall state without proof the next two observations:

(iii) $\lim\limits_{b \to a+} \int_a^b f(s)\ dg(s) = 0$ and

(iv) $\lim\limits_{a \to b-} \int_a^b f(s)\ dg(s) = \left[\omega_p fb + (1-\omega_p) f(b-)\right] \ \Delta g(b)$.

Since it is possible to have both $g\uparrow$ and $f \geqslant 0$ and have $\int_a^b f(s)\ dg(s) < 0$, we shall impose the added restriction on $\omega_1, \omega_2, \ldots, \omega_p$ that

(v) $\omega_i \geqslant 0$ for $i = 1,2,\ldots,p$.

To summarize, we consider bounded functions $f \geqslant 0$ satisfying condition (ii); a function g of bounded variation and right continuous on $[0,T]$ such that $g\uparrow$ on $[0,T]$; and that $\omega_i \geqslant 0$ for $i = 1,2,\ldots,p$.

Finally we suppose that if $\varepsilon \geqslant 0$ we have

$$f(t) \leqslant \varepsilon + \int_0^t f(s)\ dg(s) \text{ for } 0 \leqslant t \leqslant T .$$

With these restrictions holding on $[0,T]$, both Lemma 1 and Lemma 2 of [1] hold and will be stated as 3.1 and 3.2 with proofs essentially those given in [1].

<u>Lemma 3.1</u>: If $ft \leqslant K\varepsilon$ for $0 \leqslant t \leqslant t_1$, then there are a $t_2 > t_1$ and a K' such that $ft \leqslant K'\varepsilon$ for $0 \leqslant t < t_2$.

Lemma 3.2: If $ft \leqslant K\varepsilon$ for $0 \leqslant t \leqslant t_1$ and $V_{t_1}^{t}(g) \leqslant \rho < 1$ for $t_1 \leqslant t < t_2$, then there exists a K' such that $ft < K'\varepsilon$ for $0 \leqslant t < t_2$.

While Lemmas 3.1 and 3.2 remain as stated in [1], Lemma 3 of [1] must be altered slightly.

Lemma 3.3: If $ft \leqslant K\varepsilon$ for $0 \leqslant t < t_1$ and either $W_p = 0$ or $\Delta g(t_1) < 1/W_p$, then there exists a $K' > 0$ such that $ft \leqslant K'\varepsilon$ for $0 \leqslant t \leqslant t_1$.

Proof: We wish to find a bound for ft_1. Consider

$$\begin{aligned} ft_1 &\leqslant \varepsilon + \int_0^{t_1} f(s)\,dg(s) \\ &\leqslant \varepsilon + \int_0^{t} f(s)\,dg(s) + \int_t^{t_1} f(s)\,dg(s) \\ &\leqslant \varepsilon + K\varepsilon V_0^t(g) + \int_t^{t_1} f(s)\,dg(s) \quad . \end{aligned}$$

Taking the limit as $t \to t_1$, we have

$$ft_1 \leqslant \varepsilon + K\varepsilon V_0^{t_1}(g) + [1 - W_p]\; K\varepsilon\Delta gt_1 + W_p ft_1\,\Delta gt_1$$

Note that if $W_p = 0$,

$$ft_1 \leqslant \left\{1 + K\,[V_0^{t_1}(g) + \Delta gt_1]\right\}\varepsilon$$

and we have a bound for ft_1.

If $W_p \neq 0$,

$$ft_1 = \frac{1 + K\,[V_0^{t_1}(g) + (1 - W_p)\,\Delta gt_1]}{1 - W_p\,\Delta g(t_1)}$$

and again we have a bound on ft_1.

The proof of the Gronwall inequality then follows the proof in [1] and here we shall only state the theorem.

Theorem 3.4: Let f be a bounded function on $[0,T]$ and g be of bounded variation on $[0,T]$ such that g is right continuous and

$f(t-)$ exists whenever $\Delta g(t) \neq 0$. Let $\varepsilon \geqslant 0$. Also let $f \geqslant 0$ and $g\uparrow$.
If

$$f(t) \leqslant \varepsilon + \int_0^t f(s)\, dg(s), \quad 0 \leqslant t \leqslant T$$

then there exist a T' and a K, depending only on g, such that $0 < T' \leqslant T$ and $0 \leqslant K$ and $f(t) \leqslant K\varepsilon$ for $0 \leqslant t < T'$.

Further, T' is maximal in the sense that, if $\omega_p = 0$, $T' = T$ or if $\omega_p \neq 0$ $\Delta g(T') \geqslant 1/\omega_p$. In the case where $\omega_p = 0$, by Lemma 3.3, $f(t) \leqslant K\varepsilon$ for $0 \leqslant t \leqslant T$.

The preceding discussion and the results of Chapter 2 suggest that the following theorem is true.

Let U be the operator on the set of all bounded, real valued functions on $[0,T]$ defined by

$$Uf(t) = \int_0^t f(s)\, dg(s), \quad 0 \leqslant t \leqslant T,$$

with g as in Theorem 3.4.

<u>Theorem 3.5</u>: Let $\lambda > 0$. λ is an eigenvalue of U, having a nonnegative eigenfunction f, iff.

(1) $\omega_p \neq 0$

and (2) there exists a $T' \in (0,T]$ such that $\lambda = \omega_p \Delta g(T')$ and $\lambda > \omega_p \Delta g(t)$ if $T' < t \leqslant T$.

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