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SOME RESULTS ON GENERALIZED LIMITS

A Thesis

Presented to the

Department of Mathematics

and the

Faculty of the Graduate College

University of Nebraska at Omaha

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

by

Suzanne R. Neu

June 1969

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Accepted for the faculty of The Graduate College of the University of Nebraska at Omaha, in partial fulfillment of the requirements for the degree Master of Arts.

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CHAPTER ONE: EQUIVALENCE OF LIMITS THROUGH NETS, FILTERS, AND DIRECTED FUNCTIONS

In 1915 E. H. Moore introduced the following extension of the calculus limit. If D is a set of general elements p and G is the set of all finite classes s of elements p, then a number-valued function $f \equiv (f(s)/s)$ on the domain S converges to a number a if e positive implies there exists s_e in G such that $s_e \subset s$ implies $|f(s) - a| < e$. This concept was later refined by E. H. Moore and H. L. Smith (5). Their refinement closely parallels the definition of limit through a net. They considered a general class B of elements p and a binary relation R on B which is transitive and compositive. A numerically valued function $f \equiv (f(p)/p)$, not always single valued, on the domain B converges with respect to the relation R to a number a if and only if for e positive there is an element p_e in B such that pRp_e implies $|f(p) - a| < e$. If f is not single valued, this inequality is understood to hold for all possible values of f(p).

We now consider three well-known extensions of Moore-Smith type convergence obtained by taking limits through nets, filters, and directions. Suppose henceforth in this chapter that $f:D \rightarrow Y$ where D is a set and Y is a topological space. Let $\mathcal{V}_a$ be the set of neighborhoods of the point a in Y.

First we give the following eight basic definitions relative to these three kinds of limits.

Defintion 1.1. A net is a pair (f,R) where f is a function and R is a relation in D satisfying: (1) for each m in D, mRm, (2) for each m and n in D there is a p in D such that pRm and pRn, and (3) for each m, n and p in D and mRn, nRp, then mRp.

Defintion 1.2. Let (f,R) be a net. Then $\lim_{R} f = a$ if and only if for V a neighborhood of a there exists m in D such that mRn implies fm is in V.

Definition 1.3. A filter is a nonempty set F of subsets of D such that $\emptyset$ is not in F, if X and Y are subsets of D, X is in F and $X \subset Y$, then Y is in F and if X and Y are in F, then $X \cap Y$ is in F.

Definition 1.4. Let F be a filter. Then $\lim_{F} f = a$ if and only if for V a neighborhood of a there exists X in F such that $fX \subset V$.

Definition 1.5. $\mathcal{B}$ is a base for the filter F if and only if each B in $\mathcal{B}$ is nonempty, $\mathcal{B} \subset F$, and X in F implies there exists B in $\mathcal{B}$ such that $B \subset X$.

Defintion 1.6. Let $\mathcal{B}$ be a base for the filter F. Then $\lim_{\mathcal{B}} f = a$ if and only if for V a neighborhood of a there exists B in $\mathcal{B}$ such that $fB \subset V$.

The above definitions are taken from Hildebrandt (3), and the following definitions of a directed function and the limit through it are given by McShane and Botts (4).

Definition 1.7. A directed function is a pair $(f, \mathcal{N})$ where f is a function and $\mathcal{N}$ is a nonempty family of nonempty subsets of D such that if N and N' are in $\mathcal{N}$, then there is an N" in $\mathcal{N}$ such that $N'' \subset N$ and $N'' \subset N'$.

Definition 1.8. Let $(f, \mathcal{N})$ be a directed function in D. Then

$\lim_{\mathcal{N}} f = a$ if and only if for each neighborhood V of a there is an N in $\mathcal{N}$ such that $fN \subset V$.

These three ways of considering limits are equivalent as we now show. We first prove a lemma showing the relationship between the limit through a filter and a base for the filter.

<u>Lemma 1.1</u>. $\lim_{\mathcal{B}} f = a$ if and only if $\lim_{F} f = a$.

Proof. Suppose F is a filter in D and $\mathcal{B}$ is a filter base for F. If $\lim_{\mathcal{B}} f = a$, then for V in $\mathcal{V}_a$ there is a B in $\mathcal{B}$ such that $fB \subset V$. Since B is in $\mathcal{B}$, it is also in F. Thus, V in $\mathcal{V}_a$ implies there is an element X in F such that $fX \subset V$. Conversely, if $\lim_{F} f = a$, then for V in $\mathcal{V}_a$ there is an X in F such that $fX \subset V$. Since $\mathcal{B}$ is a filter base for F, there is a B in $\mathcal{B}$ that is contained in X. Thus, $fB \subset V$ and therefore, $\lim_{\mathcal{B}} f = a$.

1. <u>Equivalence of limits through nets and filters</u>.

Suppose D is a nonempty directed set relative to the relation R. Then we show that D yields a filter F on D and that $\lim_{R} f = a$ if and only if $\lim_{F} f = a$. We define $B_q = \{ q' \text{ in } D \mid q'Rq \}$, $\mathcal{B} = \{ B_q \mid q \text{ is in } D \}$, and $F = \{ X \subset D \mid \text{there exists } B \text{ in } \mathcal{B} \text{ such that } B \subset X \}$. F is a nonempty collection because D is nonempty. The empty set is not in F since if X is in F there is a q in D such that $B_q \subset X$ and B_q is nonempty since q is in B_q. Let X and Y be subsets of D such that X is in F and $X \subset Y$. Then there is a B in $\mathcal{B}$ which is contained in X and hence in Y. Thus, Y is in F. Let X and Y be in F so that there are subsets of D, B_q and B_r in $\mathcal{B}$, so that $B_q \subset X$ and $B_r \subset Y$. Because D is a directed set, there is an element s in D such that

sRq and sRr. Since R is transitive, $B_s \subset B_q \cap B_r \subset X \cap Y$ and $X \cap Y$ is in F. Thus, F is a filter on D and $\mathcal{B}$ is a base for F.

Suppose $\lim_R f = a$, that is, for V in $\mathcal{V}_a$ there is an m in D such that nRm implies fn is in V. B_m is the set of all n for which nRm so that $fB_m \subset V$ and $\lim_{\mathcal{B}} f = a$. By lemma 1.1 $\lim_F f = a$.

Suppose that $\lim_F f = \lim_{\mathcal{B}} f = a$. Then V in $\mathcal{V}_a$ implies there is a B_q in $\mathcal{B}$ such the $fB_q \subset V$. That is, for q in D and $q'Rq$, fq' is in V. Thus, $\lim_R f = a$.

Conversely, let F be a filter in D. We show that F is also a directed set relative to the relation $\subset$ and there is a relation f^* such that $\lim_F f = a$ if and only if $\lim_{\subset} f^* = a$. For each X in F, $X = X$ so that $X \subset X$. For each X and Y in F, $X \cap Y$ is in F since F is a filter and $X \cap Y \subset X$ and $X \cap Y \subset Y$. If X, Y and Z are in F and $X \subset Y \subset Z$, then by the definition of "$\subset$", $X \subset Z$.

should be power set of Y

Suppose $\lim_F f = a$. Define a relation $f^*: F \to Y$ by $f^*X = fX$ for each X in F. V in $\mathcal{V}_a$ implies there is an X in F such that $fX \subset V$. If Y is in F and $Y \subset X$, $f^*Y = fY \subset fX \subset V$. Thus, $\lim_{\subset} f^* = a$.

Now if $\lim_{\subset} f^* = a$ and V is in $\mathcal{V}_a$, there is an X in F so that $Y \subset X$ implies $f^*Y \subset V$. Since $X \subset X$, $f^*X = fX \subset V$. Thus, $\lim_F f = a$.

2. <u>Equivalence of limits through directed functions and nets</u>.

Let $(f, \mathcal{N})$ be a directed function on D. Then we show that $\mathcal{N}$ is a directed set relative to the relation $\subset$ and there is a relation f^* such that $\lim_{\mathcal{N}} f = a$ if and only if $\lim_{\subset} f^* = a$. For each N in $\mathcal{N}$, $N \subset N$ since $N = N$. If N and N' are in $\mathcal{N}$, there is an N'' in $\mathcal{N}$ such that $N'' \subset N \cap N'$ and if N, N' and N'' are in $\mathcal{N}$ and $N \subset N' \subset N''$, then $N \subset N''$ by the definition of "$\subset$".

Suppose $\lim_{\mathcal{N}} f = a$. V in $\mathcal{V}_a$ implies there exists an N in $\mathcal{N}$ such that $fN \subset V$. Define the relation $f^*: \mathcal{N} \to Y$ by $f^*N = fN$ for each N in $\mathcal{N}$. Then for V in $\mathcal{V}_a$ there is an N in $\mathcal{N}$ such that $N' \subset N$ implies $f^*N' = fN' \subset fN \subset V$; hence, $\lim_{\subset} f^* = a$.

If $\lim_{\subset} f^* = a$ and V is in $\mathcal{V}_a$, then there is an N in $\mathcal{N}$ for which $N' \subset N$ implies $f^*N' \subset V$. Since $N \subset N$, $f^*N = fN \subset V$; thus, $\lim_{\mathcal{N}} f = a$.

Let (f,R) be a net in D. We show that D yields a direction $\mathcal{N}$ in D and that $\lim_{R} f = a$ if and only if $\lim_{\mathcal{N}} f = a$. For each q in D define $N_q = \{ q' \text{ in } D \mid q'Rq \}$ and define $\mathcal{N} = \{N_q \mid q \text{ is in } D\}$. $\mathcal{N}$ is nonempty since D is nonempty and N_q is nonempty since q in D implies qRq. For N_q and $N_{q'}$ in $\mathcal{N}$, there is a q" in D such that q"Rq and q"Rq'. Because R is transitive, $N_{q''} \subset N_q \cap N_{q'}$. Thus, $\mathcal{N}$ is a direction in D.

Suppose $\lim_{R} f = a$. Then V in $\mathcal{V}_a$ implies there exists q in D such that for q'Rq, fq' is in V. N_q is the set of all such q', so $fN_q \subset V$ and therefore, $\lim_{\mathcal{N}} f = a$.

If $\lim_{\mathcal{N}} f = a$ and V is in $\mathcal{V}_a$, then there is an N_q in $\mathcal{N}$ such that $fN_q \subset V$. If q'Rq, q' is in N_q and fq' is in V. Thus, $\lim_{R} f = a$.

3. <u>Equivalence of limits through directed functions and filters</u>.

Suppose $(f, \mathcal{N})$ is a directed function on D. We show that $\mathcal{N}$ yields a filter F in D and that $\lim_{\mathcal{N}} f = a$ if and only if $\lim_{F} f = a$. Let $F = \{X \subset D \mid \text{there exists N in } \mathcal{N} \text{ such that } N \subset X\}$. The empty set is not in F because it is not in $\mathcal{N}$. For X in F and $X \subset Y \subset D$, there is an N in $\mathcal{N}$ such that $N \subset X \subset Y$ which implies Y is in F, and for X and Y in F, there are N' and N" in $\mathcal{N}$ such that $N' \subset X$ and $N'' \subset Y$.

Since $\mathcal{N}$ is a direction, there exists an N in $\mathcal{N}$ such that $N \subset N'$ and $N \subset N''$. Thus, $N \subset N' \cap N'' \subset X \cap Y$ which implies $X \cap Y$ is in F. Then we see that F is a filter in D.

Suppose $\lim_{\mathcal{N}} f = a$. Then V in $\mathcal{V}_a$ implies there exists N in $\mathcal{N}$ such that $fN \subset V$. Since $N \subset N$, N is in F by the definition of F; thus $\lim_{F} f = a$.

Suppose $\lim_{F} f = a$ and V is in $\mathcal{V}_a$. Then there exists X in F such that $fX \subset V$. For X in F there is an N in $\mathcal{N}$ such that $N \subset X$ and $fN \subset fX \subset V$. Thus, $\lim_{\mathcal{N}} f = a$.

Let F be a filter in D. Then we show that F is a direction in D and that $\lim_{F} f = a$ if and only if $\lim_{\mathcal{N}} f = a$. F is nonempty since it is a filter and X in F is nonempty because the empty set is not in F. For X and Y in F, $X \cap Y$ is in F because F is a filter so that $X \cap Y \subset X$ and $X \cap Y \subset Y$. If $\mathcal{N} = F$, $\mathcal{N}$ is a direction in D.

Suppose $\lim_{F} f = a$ and V is in $\mathcal{V}_a$. Then there exists an X in F such that $fX \subset V$. There is an N in $\mathcal{N}$ for which $N = X$, so that $fN = fX \subset V$. Thus, $\lim_{\mathcal{N}} f = a$.

Suppose $\lim_{\mathcal{N}} f = a$. If V is in $\mathcal{V}_a$, there exists an N in $\mathcal{N}$ such that $fN \subset V$. $N = X$ for some X in F, so that $fX = fN \subset V$. Therefore, $\lim_{F} f = a$.

CHAPTER TWO: GENERALIZATION OF LIMITS OF FUNCTIONS MAPPING ONE TOPOLOGICAL SPACE INTO ANOTHER

We now consider a funtion f which maps a topological space into a topological space. Let X and Y be topological spaces, $D \subset X$, $f:D \rightarrow Y$ and $\mathcal{N}_x$ be the set of neighborhoods of the point x. In this setting T. E. Frayne (2) gave the following definition of the limit through sets in both the domain and range of the function considered, thus giving a generalization of the concept of the limit of a function mapping one topological space into another.

Definition 2.1. $\lim_{\substack{x \rightarrow a \\ S}} fx \underset{T}{=} b$ if and only if for each N in $\mathcal{N}_b$ there is an M in $\mathcal{N}_a$ such that $f(M \cap S) \subset N \cap T$.

We further extend this definition by using generalized neighborhoods which we call vicinities. H. L. Bentley in (1) defined a colander G to be a set such that if x and y are in G there is a z in G such that $z \subset x \cap y$. We shall call a set X a space if and only if each point x of X has a collection of vicinities where we define the collection of vicinities of a point x, denoted by $\mathcal{V}_x$, to be a colander such that each vicinity of x is a subset of X containing x. Let X and Y be spaces, $D \subset X$ and $f:D \rightarrow Y$. To make the limit in definition 2.1 meaningful, we shall make some restrictions on the sets S and T and the vicinities. We require S to be a subset of D and T to be a subset of Y. If each vicinity of a does not contain a point of S, there is an N in $\mathcal{V}_a$ such that $N \cap S = \emptyset$ so that $\lim_{\substack{x \rightarrow a \\ S}} fx \underset{T}{=} b$ for all b in Y. Hence, we shall assume that

Restriction 1. Each vicinity of a contains a point of S.

For each b in Y if each vicinity of b does not contain a point of $T \cap \text{range } f$, there is an N in $\mathcal{V}_b$ such that $N \cap T \cap \text{range } f = \emptyset$ so that there is no a in D for which $\lim_{\substack{x \to a \\ S}} fx \underset{T}{=} b$. We thus assume that

Restriction 2. Each vicinity of b contains a point of $T \cap \text{range } f$.

Then the following definition will be used for the remainder of the chapter and the context set by its hypothesis will be assumed to hold whenever we refer to the limit.

Definition 2.2. Suppose that X and Y are spaces, $D \subset X$, $f:D \to Y$, a is in X and b is in Y. Also suppose that $S \subset D$, $T \subset Y$, each vicinity of a contains a point of S and each vicinity of b contains a point of $T \cap \text{range } f$. Then $\lim_{\substack{x \to a \\ S}} fx \underset{T}{=} b$ if and only if for each N in $\mathcal{V}_b$ there is an M in $\mathcal{V}_a$ such that $f(M \cap S) \subset N \cap T$.

To provide an illustration of how this limit can be applied, consider the following example.

Example 2.1. Let $f:[a,b] \to R$, $\mathcal{B}$ be the set of all ascending sequences $\{x_i\}_{i=0}^{2n}$ such that $x_0 = a$ and $x_{2n} = b$, and $\mathcal{V}_L$ be the set of basic neighborhoods of the point L in R. Define

$$S_f(\sigma) = \sum_{i=0}^{n-1} fx_{2i+1}\,(x_{2i+2} - x_{2i})$$

where $\sigma = \{x_i\}_{i=0}^{2n}$ and for e positive define $M_e = \{\sigma \text{ in } \mathcal{B} \mid$ for each x in $[a,b]$ there is an x_k in σ such that $|x_k - x| < e\}$. Then $\lim_{\substack{\sigma \to [a,b] \\ \mathcal{B}}} S_f(\sigma) \underset{R}{=} L$ if and only if for N in $\mathcal{V}_L$ there exists M_e in $\mathcal{V}_{[a,b]}$ such that $S_f(M_e \cap \mathcal{B}) \subset N \cap R$. Thus, $\lim_{\substack{\sigma \to [a,b] \\ \mathcal{B}}} S_f(\sigma) \underset{R}{=} L$ if and only if $\int_a^b fx\, dx = L$.

Suppose W and Z are spaces, $E \subset W$ and $g:E \to Z$. We now cite the first theorem from Frayne's paper (2) and give our proof of it.

Theorem 2.1. Suppose X,Y,W and Z are spaces, $f:D\to Y$ where $D\subset X$ and $g:E\to Z$ where $E\subset W$. If $\lim_{\underset{T}{u\to b}} gu \underset{U}{=} c$ and $\lim_{\underset{S}{x\to a}} fx \underset{T}{=} b$, then $\lim_{\underset{S}{x\to a}} gfx \underset{U}{=} c$.

Proof. For N in $\mathcal{V}_c$ there is a P in $\mathcal{V}_b$ such that $g(P\cap T)\subset N\cap U$. For P there exists M in $\mathcal{V}_a$ such that $f(M\cap S)\subset P\cap T$. Thus, $gf(M\cap S) \subset g(P\cap T)\subset N\cap U$.

The following theorem is the calculus counterpart of theorem 2.1.

Theorem 2.2. Let f and g be real valued functions of a real variable such that there is a deleted basic neighborhood α of a in D and a deleted basic neighborhood β of b in E. If there exists a deleted basic neighborhood μ of a relative to the domain of f such that b is not in $f\mu$, there exists a deleted basic neighborhood δ of a relative to the domain of f such that $f\delta\subset$ domain g, $\lim_{u\to b} gu = c$ and $\lim_{x\to a} fx = b$, then $\lim_{x\to a} gfx = c$.

Proof. Let N be a basic neighborhood of c. Then there is a deleted basic neighborhood P of b such that $gP\subset N$ where $P\subset P'\cap\beta$ for some deleted basic neighborhood P' of b. Let W be a basic neighborhood of b. Then there exists a deleted basic neighborhood M' of a such that $fM'\subset W$ where $M'\subset M''\cap\alpha$ for some deleted basic neighborhood M'' of a.

If b is a real number, then $P\cup\{b\}$ is a basic neighborhood of b. Thus, there exists a deleted basic neighborhood M of a such that $fM\subset P\cup\{b\}$ where $M\subset M_1\cap\alpha\cap\mu\cap\delta$ for some deleted basic neighborhood M_1 of a. Because M is contained in μ, b is not in fM. M contained in δ implies $fM\subset$ domain g. Thus, $fM\subset P$.

If b is $+\infty$ or $-\infty$, then P is a basic neighborhood of b, and for it there is a deleted basic neighborhood M of a such that $fM \subset P$ where $M \subset M_1 \cap \alpha \cap \mu \cap \delta$ for some deleted basic neighborhood M_1 of a.

Thus, in either case, $gfM \subset gP \subset N$ and $\lim_{x \to a} gfx = c$.

We now show that theorem 2.2 can be obtained as a special case of theorem 2.1. Suppose $U = R$, $T = (\text{range } f \cup \beta) - \{b\}$, $S = \text{domain } f - \{a\}$ and $\mathcal{V}_x$ is the set of basic neighborhoods of the point x. Then the vicinities of c intersected with U are the basic neighborhoods of c, the vicinities of b intersected with T are deleted basic neighborhoods of b relativized to a subset of the domain g and the vicinities of a intersected with S are deleted basic neighborhoods relativized to the domain of f. Thus, $\lim_{\substack{u \to b \\ T}} gu \underset{U}{=} c$ and $\lim_{\substack{x \to a \\ S}} fx \underset{T}{=} b$ reduce to the limits considered in theorem 2.2. That there is a deleted basic neighborhood M of a relativized to the domain of f such that b is not in fM follows from the hypothesis that $\lim_{\substack{x \to a \\ S}} fx \underset{T}{=} b$ since if P is in $\mathcal{V}_b$ there is a vicinity M of a such that $f(M \cap S) \subset P \cap T$ and b is not in $f(M \cap S)$ since b is not in T. Then $M \cap S$ is a deleted basic neighborhood of a that does not contain b. That there is a deleted basic neighborhood M' of a relativized to the domain of f such that fM' is contained in the domain of g follows from the hypotheses that T is contained in the domain of g and that $\lim_{\substack{x \to a \\ S}} fx \underset{T}{=} b$ since if P is in $\mathcal{V}_b$ there is an M' in $\mathcal{V}_a$ such that $f(M' \cap S) \subset P \cap T \subset T \subset \text{domain } g$. Then $M' \cap S$ is a deleted basic neighborhood of a such that $f(M' \cap S) \subset \text{domain } g$.

An application of theorem 2.1 is to the change of variable rules which are often used in calculus.

Example 2.2. Suppose $\lim_{z \xrightarrow[R]{} 0} gz \underset{R}{=} c$. We know that $\lim_{x \xrightarrow[R]{} +\infty} fx \underset{R^+}{=} 0$ where $fx = 1/x$ so that applying theorem 2.1 we have that $\lim_{x \xrightarrow[R]{} +\infty} gfx \underset{R}{=} c$ or $\lim_{x \xrightarrow[R]{} +\infty} g(1/x) \underset{R}{=} c$. Similarly, suppose that $\lim_{x \xrightarrow[R]{} a} gx \underset{R}{=} c$. We know that $\lim_{h \xrightarrow[R]{} 0} fh \underset{R}{=} a$ where $fh = a + h$, so that applying theorem 2.1 we have that $\lim_{h \xrightarrow[R]{} 0} gfh \underset{R}{=} c$ or $\lim_{h \xrightarrow[R]{} 0} g(a+h) \underset{R}{=} c$.

Given that $\lim_{x \xrightarrow[S]{} a} fx \underset{T}{=} b$ we inquire if we can alter S and T without affecting the value of the limit. The next several theorems give a partial answer to this question. Theorems 2.3 and 2.4 are cited from Frayne's paper (2) and given with our proofs.

Theorem 2.3. Suppose X and Y are spaces, $D \subset X$ and $f: D \rightarrow Y$. If $\lim_{x \xrightarrow[S]{} a} fx \underset{T}{=} b$, $T \subset T'$, $S' \subset S$ and each vicinity of a contains a point of S', then $\lim_{x \xrightarrow[S']{} a} fx \underset{T'}{=} b$.

Proof. For N in $\mathcal{V}_b$ there exists M in $\mathcal{V}_a$ such that $f(M \cap S) \subset N \cap T$. Thus, $f(M \cap S') \subset f(M \cap S) \subset N \cap T \subset N \cap T'$ and $\lim_{x \xrightarrow[S']{} a} fx \underset{T'}{=} b$.

Theorem 2.4. Suppose X and Y are spaces, $D \subset X$ and $f: D \rightarrow Y$. If $S = S' \cup S''$, $\lim_{x \xrightarrow[S']{} a} fx \underset{T}{=} b$ and $\lim_{x \xrightarrow[S'']{} a} fx \underset{T}{=} b$, then $\lim_{x \xrightarrow[S]{} a} fx \underset{T}{=} b$.

Proof. For N in $\mathcal{V}_b$ there is an M' in $\mathcal{V}_a$ such that $f(M' \cap S') \subset N \cap T$ and there is an M'' in $\mathcal{V}_a$ such that $f(M'' \cap S'') \subset N \cap T$. There exists an M in $\mathcal{V}_a$ such that $M \subset M' \cap M''$. Then $f(M \cap S) = f(M \cap (S' \cup S'')) \subset f(M' \cap S') \cup f(M'' \cap S'') \subset N \cap T$. Thus, $\lim_{x \xrightarrow[S]{} a} fx \underset{T}{=} b$.

Theorem 2.3 shows that if $\lim_{x \xrightarrow[S]{} a} fx \underset{T}{=} b$ we can replace S by a subset S' without changing the value of the limit; concomitantly, we consider when we can keep the limit the same using a set which contains S. Theorem 2.5 answers this in part.

Theorem 2.5. Suppose X and Y are spaces, $D \subset X$, $f:D \to Y$, and $\{y_n\}_{n=1}^{\infty}$ is a sequence in D - S such that $\lim_{n\to\infty} y_n \underset{S'}{=} a$ where $S' = \{y_n \mid n \text{ is in } I^+\}$, $\lim_{n\to\infty} fy_n \underset{T}{=} b$ and i in I^+ implies that there exists M_i in $\mathcal{V}_a$ such that y_i is not in M_i. Then $\lim_{\substack{x\to a\\ S}} fx \underset{T}{=} b$ if and only if $\lim_{\substack{x\to a\\ S\cup S'}} fx \underset{T}{=} b$.

Proof. N in $\mathcal{V}_b$ implies there is a positive integer p such that $fy(I_p^+) \subset N \cap T$. M' in $\mathcal{V}_a$ implies there is a positive integer q such that $y(I_q^+) \subset M' \cap S'$. Let r be the maximum of p and q. Then there exists an M in $\mathcal{V}_a$ such that $M \subset M' \cap (\bigcap_{i=1}^{r-1} M_i)$. Then $M \cap S' \subset y(I_r^+)$ so that $f(M \cap S') \subset fy(I_r^+) \subset N \cap T$. Thus, $\lim_{\substack{x\to a\\ S'}} fx \underset{T}{=} b$. By hypothesis, $\lim_{\substack{x\to a\\ S}} fx \underset{T}{=} b$, so applying theorem 2.4 we have that $\lim_{\substack{x\to a\\ S\cup S'}} fx \underset{T}{=} b$.

Conversely, if $\lim_{\substack{x\to a\\ S\cup S'}} fx \underset{T}{=} b$, then we can use theorem 2.3 to show that $\lim_{\substack{x\to a\\ S}} fx \underset{T}{=} b$ since $S \subset S \cup S'$.

According to theorem 2.5 if $\lim_{\substack{x\to a\\ S}} fx \underset{T}{=} b$, we can add a sequence of points in the complement of S to S without changing the value of the limit. An analogous question is under what conditions we can delete points from T. First we consider the problem of subtracting a finite number of points from T.

Lemma 2.1. Suppose X and Y are spaces, $D \subset X$, $f:D \to Y$, b is in Y, y' is in T and there exists N' in $\mathcal{V}_b$ such that y' is not in N'. Then, $\lim_{\substack{x\to a\\ S}} fx \underset{T}{=} b$ if and only if $\lim_{\substack{x\to a\\ S}} fx \underset{T-\{y'\}}{=} b$.

Proof. Suppose that $\lim_{\substack{x\to a\\ S}} fx \underset{T}{=} b$ and that for y' in T there exists N' in $\mathcal{V}_b$ such that y' is not in N'. For N in $\mathcal{V}_b$ there exists N" in

$\mathcal{V}_b$ such that $N'' \subset N \cap N'$. Then for N'' there is an M in $\mathcal{V}_a$ such that $f(M \cap S) \subset N'' \cap T = N'' \cap (T - \{y'\}) \subset N \cap (T - \{y'\})$. Thus, $\lim_{x \xrightarrow[S]{} a} fx \underset{T-\{y'\}}{=} b$.

Conversely, if $\lim_{x \xrightarrow[S]{} a} fx \underset{T-\{y'\}}{=} b$, we can apply theorem 2.3 to show that $\lim_{x \xrightarrow[S]{} a} fx \underset{T}{=} b$ since $T - \{y'\} \subset T$.

By induction this result can be extended to give the following theorem.

<u>Theorem 2.6</u>. Suppose X and Y are spaces, $D \subset X$, $f : D \to Y$, b is in Y, y_i is in T for $i = 1,2,\ldots,n$ and there exists N_i in $\mathcal{V}_b$ for $i = 1,2,\ldots n$ such that y_i is not in N_i. Then $\lim_{x \xrightarrow[S]{} a} fx \underset{T}{=} b$ if and only if $\lim_{x \xrightarrow[S]{} a} fx \underset{T-\{y_i\}_{i=1}^{n}}{=} b$.

Now suppose that $\lim_{x \xrightarrow[S]{} a} fx \underset{T}{=} b$ and consider the problem of deleting from T a sequence of points which contains <u>infinitely many</u> different terms. Suppose $T \subset$ range f, $\{z_n\}_{n=1}^{\infty}$ is a sequence in T containing infinitely many different z_n, $\lim_{n \to \infty} z_n \underset{T}{=} b$ and if N is in $\mathcal{V}_b$ then N contains an infinite number of points of $T - \{z_i\}_{i=1}^{\infty}$. These suppositions are not sufficient to show the $\lim_{x \xrightarrow[S]{} a} fx \underset{T-\{z_n\}_{n=1}^{\infty}}{=} b$ as the following example illustrates.

<u>Example 2.3</u>. Let $f:(0,1) \to (0,1)$ by $fx = 0$ if x is irrational and $fx = 1/n$ is x is the rational m/n in reduced form where n is positive. Let S and T each be the segment (0,1). Then $\lim_{x \xrightarrow[S]{} \frac{1}{2}} fx \underset{T}{=} 0$. Let $z_n = 1/n$ for n in I^+. Since each vicinity of ½ intersected with S contains points which are mapped into $\{z_n\}_{n=1}^{\infty}$, $\lim_{x \xrightarrow[S]{} \frac{1}{2}} fx \underset{T-\{z_n\}_{n=1}^{\infty}}{\neq} b$.

Another possibility, given that $\lim_{x \xrightarrow[S]{} a} fx \underset{T}{=} b$, is to look for a condition on the function f so that we can subtract a sequence of points in T from T and keep the limit the same. The following example shows that having f a continuous function is not sufficient.

Example 2.4. Let f be the identity function on the set of real numbers, S and T each be the set of real numbers, and $\mathcal{V}_x$ be the set of basic neighborhoods of x. Then $\lim_{\substack{x\to 1\\ S}} fx \underset{T}{=} 1$. Let $\{z_n\}_{n=1}^{\infty}$ be the rationals in (1,2), then any vicinity of one intersected with S contains infinitely many points of $\{z_n\}_{n=1}^{\infty}$ so that $\lim_{\substack{x\to 1\\ S}} fx \underset{T-\{z_n\}_{n=1}^{\infty}}{\neq} 1$.

If $\lim_{\substack{x\to a\\ S}} fx \underset{T}{=} b$ and we do not keep S fixed as we have in our discussion thus far, we can prove the following theorem.

Theorem 2.7. Suppose X and Y are spaces, $D\subset X$ and $f:D\to Y$. If $\lim_{\substack{x\to a\\ S}} fx \underset{T}{=} b$, $\emptyset \neq T'\subset T\cap \text{range } f$, $S\cap f^{-1}(T')$ has a nonempty intersection with each vicinity of a and $T'\cap$ range f has a nonempty intersection with each vicinity of b, then $\lim_{\substack{x\to a\\ S\cap f^{-1}(T')}} fx \underset{T'}{=} b$.

Proof. For N in $\mathcal{V}_b$ there exists an M in $\mathcal{V}_a$ such that $f(M\cap S)\subset N\cap T$. Then $f(M\cap(S\cap f^{-1}(T')))\subset f(M\cap S)\cap f(f^{-1}(T'))\subset N\cap T\cap T' = N\cap T'$.

Another interesting question involves conditions for the existence of this limit. For a fixed S and T we can prove existence theorems analogous to the ones for functions in a topological space. Suppose, however, that S and T are not fixed. We have several conjectured existence conditions and counterexamples to them.

Conjecture 2.1. Suppose X and Y are spaces, $D\subset X$ and $f:D\to Y$. It is sufficient for $\lim_{\substack{x\to a\\ S}} fx \underset{T}{=} b$ that if M is in $\mathcal{V}_a$, then there is an a' in M different from a such that $\lim_{\substack{x\to a'\\ S}} fx$ exists through T.

Example 2.5. Define $f:R\to R$ by $fx = -1$ for x negative, $fx = 0$ for $x = 0$ and $fx = 1$ for x positive. Let S and T each be the set of real numbers and let $\mathcal{V}_x$ be the set of basic neighborhoods of x. Then for any vicinity of zero and a' different from zero in the vicinity, $\lim_{\substack{x\to a'\\ S}} fx \underset{T}{=} 1$ if a' is positive and $\lim_{\substack{x\to a'\\ S}} fx \underset{T}{=} -1$ if a' is negative. However, $\lim_{\substack{x\to 0\\ S}} fx$ does not exist through T.

Conjecture 2.2. Suppose X and Y are spaces, $D\subset X$ and $f:D\to Y$. It is sufficient for $\lim_{\substack{x\to a\\ S}} fx \underset{T}{=} b$ that if M is in $\mathcal{V}_a$, then for all a' in M different from a $\lim_{\substack{x\to a'\\ S}} fx$ exists through T.

The example given for conjecture 2.1 also shows conjecture 2.2 is false.

Conjecture 2.3. Let M be in $\mathcal{V}_a$ and $\lim_{\substack{x\to a\\ S}} fx \underset{T}{=} b$. Then there exists an a' in M different from a such that $\lim_{\substack{x\to a'\\ S}} fx$ exists through T.

Example 2.6. Define $f:R\to R$ by $fx = 0$ if x is irrational and $fx = |x|$ if x is rational. Let S and T each be the set of real numbers and let $\mathcal{V}_x$ be the set of basic neighborhoods of x. Then $\lim_{\substack{x\to 0\\ S}} fx \underset{T}{=} 0$, but if a is in $T-\{0\}$, $\lim_{\substack{x\to a\\ S}} fx$ does not exist through T.

In the special case where $\mathcal{V}_a$ is a countable collection, we can prove the following existence theorem.

Theorem 2.8. Let X and Y be spaces, $D\subset X$ and $f:D\to Y$. For $\lim_{\substack{x\to a\\ S}} fx \underset{T}{=} b$ it is necessary and sufficient that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in $S\cap f^{-1}(T)$ and $\lim_{n\to\infty} x_n \underset{S}{=} a$, then $\lim_{n\to\infty} fx_n \underset{T}{=} b$.

Proof. Suppose $\lim_{n\to\infty} x_n \underset{S}{=} a$ implies that $\lim_{n\to\infty} fx_n \underset{T}{=} b$ and suppose

$\lim_{\substack{x \to a \\ S}} fx \underset{T}{\neq} b$. Then there exists N' in $\mathcal{V}_b$ such that for all M in $\mathcal{V}_a$ there is a point x in M for which fx is not in $N' \cap T$. Since $\mathcal{V}_a$ is a countable collection, we can list the vicinities of a as M_1, M_2, $M_3, \ldots$. We construct a nested sequence of vicinities in the following way: $M_1' = M_1$ and for k a positive integer greater than one there is an M_k' in $\mathcal{V}_a$ such that $M_k' \subset M_k \cap M_{k-1}'$. For each positive integer i there is an x_i in $M_i' \cap S$ such that fx_i is not in $N' \cap T$. $\lim_{n \to \infty} x_n \underset{S}{=} a$ because M_k in $\mathcal{V}_a$ implies $M_k' \subset M_k$ and x_n is in M_k' for n greater than $k-1$. Then, by hypothesis, $\lim_{n \to \infty} fx_n \underset{T}{=} b$. Now for N' in $\mathcal{V}_b$, fx_n is not in $N' \cap T$ for all positive integers n so that $\lim_{n \to \infty} fx_n \underset{T}{\neq} b$. Thus, our assumption that $\lim_{\substack{x \to a \\ S}} fx \underset{T}{\neq} b$ is false.

Conversely, suppose $\lim_{\substack{x \to a \\ S}} fx \underset{T}{=} b$ and $\lim_{n \to \infty} x_n \underset{S}{=} a$. Applying theorem 2.1, $\lim_{n \to \infty} fx_n \underset{T}{=} b$.

CHAPTER THREE: GENERALIZED LIMITS THROUGH DIRECTED FUNCTIONS

In this chapter we consider functions mapping a set D into a space Y. Let $\mathcal{V}_x$ be the set of vicinities of a point x in Y. Then we can apply the notion of limits through sets in the domain and range of the function f mapping D into Y to limits through directed functions giving the following definition.

Definition 3.1. Suppose Y is a space, $f:D\rightarrow Y$, b is in Y, $(f,\mathcal{N})$ is a directed function in D, $S\subset D$, $T\subset Y$, each vicinity of b contains a point of $T\cap$ range f, and each set in the direction $\mathcal{N}$ contains a point of S. Then $\lim_{\mathcal{N},S} f \underset{T}{=} b$ if and only if V in $\mathcal{V}_b$ implies there is an N in $\mathcal{N}$ such that $f(N\cap S)\subset V\cap T$.

We have shown in chapter one that limits through nets, filters and directed functions are equivalent, so that we shall use directed functions as a representative of the three. The results we derive then hold for all three kinds of limits. The limit of the composition of two functions is considered in this chapter with an added restriction, but the proof of the following theorem is similar to the one given for theorem 2.1.

Theorem 3.1. Let Y and Z be spaces, $f:D\rightarrow Y$, $g:E\rightarrow Z$, $(f,\mathcal{M})$ be a directed function in D and $(g,\mathcal{N})$ be a directed function in E. If $\lim_{\mathcal{N},T} g \underset{U}{=} c$, $\lim_{\mathcal{M},S} f \underset{T}{=} b$ and N in $\mathcal{N}$ implies there is a V in $\mathcal{V}_b$ such that $V\cap T\subset N\cap T$, then $\lim_{\mathcal{M},S} gf \underset{U}{=} c$.

Proof. W in $\mathcal{V}_c$ implies there exists N in $\mathcal{N}$ such that $g(N\cap T)\subset W\cap U$. N in $\mathcal{N}$ implies there exists V in $\mathcal{V}_b$ such that $V\cap T\subset N\cap T$. V in $\mathcal{V}_b$

implies there exists M in $\mathcal{M}$ such that $f(M\cap S)\subset V\cap T$. Thus, we have $gf(M\cap S)\subset g(V\cap T)\subset g(N\cap T)\subset W\cap U$.

We concern ourselves with the problem of altering S and T given that $\lim_{\mathcal{N},S} f \underset{T}{=} b$ after proving a theorem involving subsets of S.

Theorem 3.2. Suppose Y is a space and $f:D\rightarrow Y$. If $S = S'\cup S''$, $\lim_{\mathcal{N},S'} f \underset{T}{=} b$ and $\lim_{\mathcal{N},S''} f \underset{T}{=} b$, then $\lim_{\mathcal{N},S} f \underset{T}{=} b$.

Proof. V in $\mathcal{V}_b$ implies there exist N' and N'' in $\mathcal{N}$ such that $f(N'\cap S')\subset V\cap T$ and $f(N''\cap S'')\subset V\cap T$. There is an N in $\mathcal{N}$ such that $N\subset N'\cap N''$. Thus, $f(N\cap S)\subset f(N\cap(S'\cup S''))\subset f(N\cap S')\cup f(N\cap S'')\subset V\cap T$.

Theorem 3.3. Suppose Y is a space and $f:D\rightarrow Y$. If $\lim_{\mathcal{N},S} f \underset{T}{=} b$, $T\subset T'$, $S'\subset S$, and $S'\cap N$ is nonempty for each N in $\mathcal{N}$, then $\lim_{\mathcal{N},S'} f \underset{T'}{=} b$.

Proof. V in $\mathcal{V}_b$ implies there exists N in $\mathcal{N}$ such that $f(N\cap S)\subset V\cap T$. Thus, $f(N\cap S')\subset f(N\cap S)\subset V\cap T\subset V\cap T'$.

The following theorem is a generalization of theorem 2.5, and it has a more elegant proof in this generalized form.

Theorem 3.4. Suppose Y is a space, $f:D\rightarrow Y$, $\{y_n\}_{n=1}^{\infty}$ is a sequence in $D - S$ such that $\lim_{n\rightarrow\infty} fy_n = b$ and i in I^+ implies there exists N_i in $\mathcal{N}$ such that y_i is not in N_i, and $S' = S\cup\{y_n\}_{n=1}^{\infty}$. Then $\lim_{\mathcal{N},S} f \underset{T}{=} b$ if and only if $\lim_{\mathcal{N},S'} f \underset{T}{=} b$.

Proof. Suppose that $\lim_{\mathcal{N},S} f \underset{T}{=} b$. V in $\mathcal{V}_b$ implies that there exists N' in $\mathcal{N}$ such that $f(N'\cap S)\subset V\cap T$ and that there exists a positive integer k such that $fy(I_k^+)\subset V\cap T$. There is an N in $\mathcal{N}$ such that $N\subset N'\cap(\bigcap_{i=1}^{k-1} N_i)$. Thus, $f(N\cap S')\subset f(N\cap(S\cup\{y_n\}_{n=1}^{\infty}))$ and $f(N\cap(S\cup\{y_n\}_{n=1}^{\infty}))\subset f(N\cap S)\cup f(N\cap\{y_n\})\subset f(N'\cap S)\cup fy(I_k^+)\subset V\cap T$.

Conversely, suppose $\lim_{\mathcal{N},S'} f \underset{T}{=} b$. By theorem 3.3 $\lim_{\mathcal{N},S} f \underset{T}{=} b$.

Lemma 3.1. Suppose Y is a space, $f:D\rightarrow Y$, y' is in T, b is in Y, and there exists a V' in $\mathcal{V}_b$ such that y' is not in V'. Then, $\lim_{\mathcal{N},S} f \underset{T}{=} b$ if and only if $\lim_{\mathcal{N},S} f \underset{T-\{y'\}}{=} b$.

Proof. Suppose $\lim_{\mathcal{N},S} f \underset{T}{=} b$ and V'' is in $\mathcal{V}_b$. There exists V in $\mathcal{V}_b$ such that $V \subset V' \cap V''$. For V there is an N in $\mathcal{N}$ such that $f(N \cap S) \subset V \cap T = V \cap (T-\{y'\}) \subset V'' \cap (T-\{y'\})$.

Conversely, suppose $\lim_{\mathcal{N},S} f \underset{T-\{y'\}}{=} b$. By theorem 3.3 $\lim_{\mathcal{N},S} f \underset{T}{=} b$.

As before this result can be extended by induction to give the following theorem.

Theorem 3.5. Suppose Y is a space, $f:D\rightarrow Y$, b is in Y, y_i is in T for $i = 1,2,\ldots,n$ and there exists V_i in $\mathcal{V}_b$ such that y_i is not in V_i for $i=1,2,\ldots,n$. Then $\lim_{\mathcal{N},S} f \underset{T}{=} b$ if and only if $\lim_{\mathcal{N},S} f \underset{T-\{y_i\}}{=} b$.

Given that $\lim_{\mathcal{N},S} f \underset{T}{=} b$, we can prove a theorem similar to theorem 2.8 in which we changed T and made a corresponding change in S so that the value of the new limit was b.

Theorem 3.6. Suppose Y is a space and $f:D\rightarrow Y$. If $\lim_{\mathcal{N},S} f \underset{T}{=} b$, $\emptyset \neq T' \subset T \cap \text{range } f$, V in $\mathcal{V}_b$ implies $V \cap T'$ is nonempty and N in $\mathcal{N}$ implies $N \cap S \cap f^{-1}(T')$ is nonempty, then $\lim_{\mathcal{N},S\cap f^{-1}(T')} f \underset{T'}{=} b$.

Proof. V in $\mathcal{V}_b$ implies there is an N in $\mathcal{N}$ such that $f(N \cap S) \subset V \cap T$. Then, $f(N \cap (S \cap f^{-1}(T'))) \subset f(N \cap S) \cap f(f^{-1}(T')) \subset V \cap T \cap T' = V \cap T'$.

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TABLE I

TABLE OF SYMBOLS

R,	the set of real numbers.
R^+,	the set of positive real numbers.
I^+,	the set of positive integers.
I_r^+,	the set of positive integers greater than or equal to r.
$f:D\rightarrow Y$,	f is a function which maps a set D into a set Y.