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Investigation of sufficient conditions for a non-regular problem in the calculus of variations.

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INVESTIGATION OF SUFFICIENT CONDITIONS FOR A NON-REGULAR PROBLEM IN THE CALCULUS OF VARIATIONS

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CHAPTER ONE: INTRODUCTION

In his paper [1], Ewing has established sufficient conditions for a non-regular problem in the calculus of variations. In chapter 2, we shall discuss his method. In chapter 3, we will provide an example in which the main result of [1] will apply. In this chapter, we shall state for your convenience some definitions, theorems and conditions from [2].

We will suppose that there is a region R of xyz space in which the integrand function f(x,y,z) has continuous partial derivatives up to and including those of the fourth order. The general problem of the calculus of variations is finding in a class of arcs E:y(x), $x_1 \leq x \leq x_2$ joining two fixed points 1 and 2 in xy space, one which minimizes an integral of the form

$$1.1 \qquad I(E)=\int_{x_1}^{x_2} f(x,y(x),y'(x))dx.$$

<u>Definition 1.2</u>: A point (x,y(x),y'(x)) interior to the region R is called an admissible point.

<u>Definition 1.3</u>: An arc E given by (x,y(x)) where $x_1 \leq x \leq x_2$ is said to be regular if and only if y and y' are continuous.

Definition 1.4: Let $h: [a,b] \longrightarrow E_1$, then h is said to be piecewise continuous if and only if:

a) h is bounded on $[a,b]$

b) there exists points $p_0 = a < p_1 < p_2 < \dots < p_n = b$ such that h is continuous on (p_k, p_{k+1}), $k=0,1,\dots,n-1$

c) the left hand limit exists on $(a,b]$

d) the right hand limit exists on $[a,b)$.

Definition 1.5: An arc E, given by $(x,y(x))$ where $x_1 \leq x \leq x_2$, is said to be admissible if and only if:

a) $\{(x,y(x),y'(x)): x_1 \leq x \leq x_2\}$ is an admissible set

b) y is continuous on $[x_1, x_2]$

c) y' is piecewise continuous on $[x_1, x_2]$.

Condition I (Euler): An admissible arc E is said to satisfy condition I if there exists a constant c such that $f_{y'} = \int_{x_1}^{x} f_y + c$ holds along E.

Definition 1.6: Let E be an admissible arc, then E is said to be:

a) non-singular if $f_{y'y'}(x,y(x),y'(x)) \neq 0$ for $x_1 \leq x \leq x_2$

b) an extremal if E has continuous first and second derivatives and the equation $f_{y'y}y' + f_{y'x} + f_{y'y'}y'' - f_y = 0$ holds for all points $(x,y(x),y'(x))$, $x_1 \leq x \leq x_2$.

Theorem 1.7 (The Imbedding Theorem): Every non-singular extremal arc is imbedded in a two-parameter family of extremals $y(x,a,b)$, where E is given by $y(x,a_0,b_0)$ for $x_1 \leq x \leq x_2$, whose functions y, y_x have continuous partial derivatives of at least the second order in a neighborhood of the sets (x,a,b) belonging to E. Furthermore the determinant $\begin{vmatrix} y_a & y_b \\ y_{ax} & y_{bx} \end{vmatrix}$ is different from zero along E.

Note that $y_a = \dfrac{\partial y}{\partial a}$, $y_b = \dfrac{\partial y}{\partial b}$, $y_{ax} = \dfrac{\partial^2 y}{\partial a \partial x}$ and $y_{bx} = \dfrac{\partial^2 y}{\partial b \partial x}$.

We use the notation

1.8 $E(x,y,y',Y') = f(x,y,Y') - f(x,y,y') - (Y'-y')f_{y'}(x,y,y')$

in the Weierstrass' necessary condition.

Condition II(Weierstrass): An admissible arc E is said to satisfy condition II if at every element (x,y,y') of E the condition $E(x,y,y',Y') \geq 0$ is satisfied for every admissible element (x,y,Y') with $Y' \neq y'$.

Condition III(Legendre): An admissible arc E is said to satisfy condition III if at each element (x,y,y') of E $f_{y'y'} \geq 0$.

Let E_{12} denote an arc E from point 1 to point 2.

Bliss' Corollary 12.1 we shall call Corollary 1.9.

Corollary 1.9: On a member E of a two-parameter family of extremals $y(x,a,b)$ the points conjugate to a point 1 are determined by the zeros of the determinant

$$D(x,x_1,a,b)=\begin{vmatrix} y_a(x) & y_b(x) \\ y_a(x_1) & y_b(x_1) \end{vmatrix},$$

provided that this determinant is not identically zero along E.

Condition IV (Jacobi): A non-singular extremal arc E_{12} is said to satisfy condition IV if it has on it between 1 and 2 no point 6 conjugate to 1. Every non-singular minimizing arc E_{12} without corners is an extremal arc satisfying this condition.

Consider an admissible arc E, joinging the points 1 and 2 and defined by the function y(x), where $x_1 \leq x \leq x_2$, whose minimizing properties are to be tested. Let a be an arbitrary constant and if $\eta(x)$ is a function vanishing at x_1 and x_2 and having continuity properties similar to those of y(x), then every arc of the one-parameter family

$$1.10 \quad y(x)+a\,\eta(x) \qquad (x_1 \leq x \leq x_2)$$

passes through the end-points 1 and 2 of E and the family contains E for a=0. The arcs of the family, for sufficiently small values of a, are all admissible, since the elements (x,y,y') of E are all interior to the region R and the corresponding elements of the arc 1.10 will also be interior to R when a is small.

When the function 1.10 is substituted in the integrand of the integral I, a function of the parameter a of the form $I(a)=\int_{x_1}^{x_2} f(x,y(x)+a\,\eta(x),y'(x)+a\,\eta'(x))dx$ is obtained. If the arc E furnishes a minimum value for the integral I, then I(a) must have a minimum at a=0, and $I'(0)=0$, $I''(0)\geq 0$ must be satisfied. The values of these derivatives are:

$$I'(0)=\int_{x_1}^{x_2}(f_y\,\eta+f_{y'}\,\eta')dx$$

$$I''(0)=\int_{x_1}^{x_2}2w(x,\eta,\eta')dx \text{ where}$$

1.11 $\quad 2w(x,\eta,\eta')=f_{yy}\,\eta^2+2f_{yy'}\,\eta\eta'+f_{y'y'}\,\eta'^2.$

$I'(0)$ and $I''(0)$ are called the first and second variations of the integral I along the arc E. When the two variations of I are dependent upon the function η, they will be denoted by $I_1(\eta)$ and $I_2(\eta)$ respectively.

Definition 1.12: The accessory minimum problem is that of finding in a class of admissible variations $\eta(x)$ vanishing at x_1 and x_2 one which minimizes $I_2(\eta)$. The Euler equation of this problem, $\frac{d}{dx}w_{\eta'}=w_\eta$ is called the "accessory differential equation". Since it was first used by Jacobi, it is often called the "Jacobi equation".

Definition 1.13: A point x_6 is said to be conjugate to the point 1 on an extremal arc E_{12} if there exists a solution $\eta=u$ of the accessory equation with the element u, vanishing at x_1 and x_6, where $x_1 < x_6 < x_2$, but not identically zero between x_1 and x_6.

The Roman numerals I, II, III, and IV denote necessary conditions for a minimum arc. The symbols II' and III' are used to denote the necessary conditions of Weierstrass and Legendre with the equality signs excluded in II and III. Similarly IV' is Jacobi's condition IV strengthened to exclude points 6 conjugate to 1 from the end-point 2 of an extremal arc E_{12} as well as from the interior of the arc. It is understood that C_{12} is an arc with the points 1 and 2 as end-points and that $I(C_{12})$ is the value of the integral I taken along this arc.

<u>Definition 1.14</u>: If an arc E_{12} gives I a minimum value relative to the class of admissible arcs C_{12} in a sufficiently small neighborhood of elements (x,y,y') on E_{12} then $I(E_{12})$ is said to be a weak relative minimum.

<u>Definition 1.15</u>: A minimum provided by E_{12} relative to the class of admissible arcs C_{12}, restricted only to have their points (x,y) in a sufficiently small neighborhood F of E_{12} in xy space, is called a strong relative minimum.

An arc E_{12} is said to satisfy condition II_N if there is a neighborhood N of the elements (x,y,y') on E_{12} such that the condition $E(x,y,y',Y')$ 0 holds for all sets (x,y,y',Y') with (x,y,y') admissible and in N and with (x,y,Y') admissible and having $Y' \neq y'$. The condition II'_N is this condition with the equality excluded.

Bliss' theorem 16.2 we shall call theorem 1.16.

Theorem 1.16 (Sufficient Conditions for a Strong Relative Minimum): If an admissible arc E_{12} without corners is non-singular and satisfies the conditions I, II_N', IV', then there is a neighborhood F of E_{12} in xy space such that the relation $I(C_{12}) > I(E_{12})$ holds for every admissible arc C_{12} in F not identical with E_{12}.

Evidently the conditions I, II_N', III', IV' also insure a strong relative minimum since III' implies the non-singularity of E_{12} and since the remaining hypothesis of theorem 1.16 are immediate consequences of I, II_N' and IV'.

In special cases the region R may have the property that when two elements (x,y,y_1'), (x,y,y_2') belong to it so do all the elements (x,y,y') with $y_1' \leq y' \leq y_2'$. In this case the region is said to be convex in the variable y'. The notation III_F designates the property that the inequality $f_{y'y'} \geq 0$ holds for all admissible elements (x,y,y') with projections (x,y) in a neighborhood F of the arc E_{12}.

Bliss' corollary 16.1 we shall call corollary 1.17.

Corollary 1.17: If the region R has the convexity property just described, then an admissible arc E_{12} without corners and satisfying the conditions I, III_F', IV', will make $I(E_{12})$ a strong relative minimum.

CHAPTER TWO: INVESTIGATION OF SUFFICIENT CONDITIONS

Our major investigation in this chapter, as in Ewing's paper [1], is that of obtaining a set of sufficient conditions for a minimizing arc without corners along which $f_{y'y'}$ may have zeros. In order to be consistant with [1], we shall change our notation of the integral I from I(E) to J(E).

Definition 2.1: $J=\int_{x_1}^{x_2} f(x,y,y')dx$. Let E_{12} be an arc from point 1 to point 2 which minimizes J and let C be any other arc from point 1 to point 2. E_{12} is proper if $J(C) > J(E_{12})$ for all $C \neq E_{12}$. E_{12} is improper if $J(C) \geq J(E_{12})$ for all $C \neq E_{12}$ and there exists a $C \neq E_{12}$ for which equality holds.

In order to obtain a Jacobi condition since $f_{y'y'}$ may have zeros, Ewing introduced the integral

$$L \equiv \int_{x_1}^{x_2} \varphi(x,y,y')dx \text{ where } \varphi(x,y,y') \equiv f(x,y,y')+k^2(y'-e'(x))^2,$$

$x_1 \leq x \leq x_2$, $k \gtrless 0$, and e(x) is the minimizing arc for J.

We shall now prove Ewing's statement that if E:y=e(x) furnishes at least an improper strong relative minimum for J, it furnishes a proper strong relative minimum for L.

Proof: E:y=e(x) furnishes at least an improper strong relative minimum for J. Therefore $J(C) \geq J(E)$ for all $C \neq E$. Let us look at an arc $\bar{C} \neq E$ such that $J(\bar{C})=J(E)$.

$$L(E)=\int_{x_1}^{x_2} \left[f(x,e(x),e'(x))+k^2(e'(x)-e'(x))^2\right] dx=J(E)$$

$$L(\bar{C})=J(\bar{C})+\int_{x_1}^{x_2} k^2(y'(x)-e'(x))^2\, dx$$

Since $k^2(y'(x)-e'(x))^2 > 0$ on some neighborhood, $L(\bar{C}) > J(\bar{C})=J(E)=L(E)$. Therefore $L(\bar{C}) > L(E)$ for $\bar{C}\neq E$. Therefore for all arcs, say C, from point 1 to point 2 where $C\neq E$, $L(C) > L(E)$ and therefore E furnishes a proper strong relative minimum for L.

<u>Definition 2.2</u>: If E satisfies condition IV (or IV') for every $k\neq 0$, we shall say that it satisfies condition IV_L (or IV_L' respectively) for J.

The "Jacobi differential equation" is

2.3 $\frac{d}{dx}w_{u'}-w_u=0$ where $w=\frac{1}{2}f_{yy}u^2+\frac{1}{2}f_{y'y'}u'^2+f_{yy'}uu'$

$u\equiv u(x)$ and where $f_{yy'}$, f_{yy}, and $f_{y'y'}$ are evaluated at $(x,e(x),e'(x))$.

We will now discuss a Jacobi nesessary condition of [1]. In [1] the parameter in L is written in the form $k^2=(\tilde{a}^2+\alpha)/2$, $\tilde{a}\neq 0$, $\alpha > -\tilde{a}^2$ and the Jacobi differential equations

2.4 $qu''+ru'+su=0$

2.5 $(q+\tilde{a}^2+\alpha)u''+ru'+su=0$

for J and L, where $q=f_{y'y'}(x,e(x),e'(x))\geq 0$ in the closed interval $[x_1,x_2]$ from condition III,

$r=\frac{d}{dx}f_{y'y'}(x,e(x),e'(x))$ and

$s=\frac{d}{dx}f_{yy'}(x,e(x),e'(x))-f_{yy}(x,e(x),e'(x))$.

Since q may vanish in $[x_1, x_2]$, the usual existence theorems can not be applied for 2.4. They do apply to 2.5 however, and the general solution with $\alpha=0$ is $u=c_1u_1(x)+c_2u_2(x)$, where the u's constitute a fundamental system and are of class C" in $[x_1, x_2]$ where the class C" is the class of functions having continuous second derivatives. By hypothesis, $E: y=e(x)$ is a minimizing curve satisfying IV_L so that, by proper choice of c_1 and c_2, $u(x)=\Delta(x,x_1)$ is positive in the interval $x_1 < x < x_2$.

We shall prove that for every admissible α (that is $\alpha > -\tilde{a}^2$) there exists a solution $\Delta(x,x_1,\alpha)$ of 2.5 vanishing at $x=x_1$, and such that $\Delta'(x_1,x_1,\alpha)=\Delta'(x_1,x_1)$, where $\Delta''(x,x_1,\alpha)$ is continuous in x and of class C' in α.

<u>Proof</u>: We have to show for $\alpha > -\tilde{a}^2$ there exists $\Delta(x,x_1,\alpha)$ which is a solution of 2.5 with boundary conditions $\Delta(x_1,x_1,\alpha)=0$ and $\Delta'(x_1,x_1,\alpha)=\Delta'(x_1,x_1)$. Let $\alpha > -\tilde{a}^2$. From 2.5 $u''=(1/(q+\tilde{a}^2+\alpha))\,(-su-ru')$

$$=f(x,u,u',\alpha).$$

This can be written as a system of equations by setting $y_1=u$ and $y_2=u'$ then $y_1'=u'=g(x,y_1,y_2,\alpha)$ and $y_2'=u''=f(x,y_1,y_2,\alpha)$ or $y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $h=\begin{pmatrix} g \\ f \end{pmatrix}$.

$y'=h(x,y,\alpha)$ since $\dfrac{\partial h}{\partial \alpha}$ is continuous for $\alpha > -\tilde{a}^2$. y and y' have continuous partials with respect to α.

i.e. $\begin{pmatrix} u \\ u' \end{pmatrix}$, $\begin{pmatrix} u' \\ u'' \end{pmatrix}$ have continuous partials with respect to α.

Therefore $\begin{pmatrix} \Delta(x,x_1,\alpha) \\ \Delta'(x,x_1,\alpha) \end{pmatrix}$, $\begin{pmatrix} \Delta'(x,x_1,\alpha) \\ \Delta''(x,x_1,\alpha) \end{pmatrix}$ have continuous partials with respect to α.

Ewing next studies the related equation

2.6 $(q+\tilde{a}^2)u''+ru'+su=-\alpha\Delta''(x,x_1,\alpha)$.

Suppose u_1 and u_2 are linearly independent solutions to the corresponding homogeneous equation of 2.6. Then the complementary function of 2.6 is $u_c=c_1u_1(x)+c_2u_2(x)$ where c_1 and c_2 are arbitrary constants. We replace c_1 and c_2 respectively by functions $v_1(x)$ and $v_2(x)$ which will be determined so that the resulting function $u_p=v_1u_1+v_2u_2$ will be a particular solution of 2.6 and that $v_1'u_1+v_2'u_2=0$. Differentiating u_p we have $u_p'=v_1u_1'+v_2u_2'+v_1'u_1+v_2'u_2=v_1u_1'+v_2u_2'$. Differentiating u_p' we have $u_p''=v_1u_1''+v_2u_2''+v_1'u_1'+v_2'u_2'$.

Therefore $(q+\tilde{a}^2)(v_1u_1''+v_2u_2''+v_1'u_1'+v_2'u_2')+r(v_1u_1'+v_2u_2')$

$$+s(v_1u_1+v_2u_2)=-\alpha\Delta''(x,x_1,\alpha)=\alpha(-\Delta''(x,x_1,\alpha)).$$

$v_1((q+\tilde{a}^2)u_1''+ru_1'+su_1)+v_2((q+\tilde{a}^2)u_2''+ru_2'+su_2)+(q+\tilde{a}^2)(v_1'u_1'+v_2'u_2')$

$=\alpha(-\Delta''(x,x_1,\alpha))$

$(q+\tilde{a}^2)(v_1'u'_1+v_2'u_2')=\alpha(-\Delta''(x,x_1,\alpha))$

$v_1'u_1'+v_2'u_2'=\alpha(-\Delta''(x,x_1,\alpha))/(q+\tilde{a}^2)$

2.7
$$v_1'u_1+v_2'u_2=0$$
$$v_1'u_1'+v_2'u_2'=\alpha(-\Delta''(x,x_1,\alpha))/(q+\tilde{a}^2)$$

The determinant of the coefficients of 2.7 is

$$D(x)=\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} \neq 0$$

Therefore the system 2.7 has a unique solution

$$v_1'=\frac{\begin{vmatrix} 0 & u_2 \\ \alpha(-\Delta''(x,x_1,\alpha))/(q+\tilde{a}^2) & u_2' \end{vmatrix}}{D(x)}$$

$$=\alpha(\Delta''(x,x_1,\alpha))u_2/((q+\tilde{a}^2)D(x))$$

$$v_2'=\frac{\begin{vmatrix} u_1 & 0 \\ u_1' & \alpha(-\Delta''(x,x_1,\alpha))/(q+\tilde{a}^2) \end{vmatrix}}{D(x)}$$

$$=\alpha(-\Delta''(x,x_1,\alpha))u_1/((q+\tilde{a}^2)D(x))$$

Thus we obtain the functions $v_1(x)$ and $v_2(x)$ given by

$$v_1(x)=\int_{x_1}^{x}\frac{\alpha(\Delta''(t,x_1,\alpha))u_2(t)dt}{(q+\tilde{a}^2)D(t)}$$

$$v_2(x)=\int_{x_1}^{x}\frac{\alpha(-\Delta''(t,x_1,\alpha))u_1(t)dt}{(q+\tilde{a}^2)D(t)}$$
$$=-\int_{x_1}^{x}\frac{\alpha(\Delta''(t,x_1,\alpha))u_1(t)dt}{(q+\tilde{a}^2)D(t)}$$

Therfore the solution of 2.6 is $u=u_c+u_p$ or

2.8 $u(x)=c_1u_1(x)+c_2u_2(x)+\alpha A(x,\alpha)$, where

$$A(x,\alpha)=u_1(x)\int_{X_1}^{X} \frac{\Delta''(t,x_1,\alpha)u_2(t)dt}{(q+\tilde{a}^2)D(t)} - u_2(x)\int_{X_1}^{X} \frac{\Delta''(t,x_1,\alpha)u_1(t)dt}{(q+\tilde{a}^2)D(t)} .$$

$\Delta(x,x_1,\alpha)$ is a particular solution of 2.6 and can be represented in the form 2.8. Since it vanishes for $x=x_1$, we obtain

2.9 $\Delta(x,x_1,\alpha)=\lambda\Delta(x,x_1)+\alpha A(x,\alpha)$,

where in general λ is a function of α.

We shall now show that λ of 2.9 is a function of α.

<u>Proof</u>: Let Hom(x) be the solution of the homogeneous equation. From differential equations we have

$\Delta(x,x_1,\alpha)=\mathrm{Hom}(x)+\alpha A(x,\alpha)$.

$\Delta(x_1,x_1,\alpha)=\mathrm{Hom}(x_1)+\alpha A(x_1,\alpha)$

$0=\mathrm{Hom}(x_1)+0$

$\mathrm{Hom}(x_1)=0$

Set $G(x)=\mathrm{Hom}(x)-\lambda\Delta(x,x_1)$, $G(x_1)=0$.

$G'(x_1)=\mathrm{Hom}'(x_1)-\lambda\Delta'(x_1,x_1)$

Therefore $G'(x_1)=0$ if $\lambda=\mathrm{Hom}'(x_1)/\Delta'(x_1,x_1)$.

Therefore $\Delta(x,x_1,\alpha)=\lambda\Delta(x,x_1)+\alpha A(x,\alpha)$ and

$\lambda\Delta'(x_1,x_1)=\Delta'(x_1,x_1,\alpha)-\alpha A'(x_1,\alpha)$.

Therefore $\lambda(\alpha)=(\Delta'(x_1,x_1,\alpha)-\alpha A'(x_1,\alpha))/\Delta'(x_1,x_1)$.

Next Ewing tries to prove that there is at most one L and we may assume that for this L, $\alpha=0$, for which E fails to satisfy IV_L'. The method of proof is as follows.

By hypothesis E satisfies IV_L. If it fails to satisfy IV_L' for L corresponding to $\alpha=0$, then $\Delta(x_2,x_1,0)=\gamma(0)\Delta(x_2,x_1)=\Delta(x_2,x_1)=0$ since $\gamma(0)=1$. If $\alpha\neq 0$ has the same property, then

2.10 $\Delta(x_2,x_1,\alpha)=\gamma(\alpha)\Delta(x_2,x_1)+\alpha A(x_2,\alpha)=\alpha A(x_2,\alpha)=0.$

Therefore $A(x_2,\alpha)=\int_{x_2}^{x_1}\frac{\Delta''(x,x_1,\alpha)\Delta(x,x_2)}{(q+\tilde{a}^2)\,D(x)}\,dx$, where

$\Delta(x,x_2)$ is written for $u_2(x_2)u_1(x)-u_1(x_1)u_2(x)$.

Ewing apparently uses the generalized first mean value theorem and has $A(x_2,\alpha)=\frac{\Delta(\bar{x},x_2)}{(\bar{q}+\tilde{a}^2)\,D(\bar{x})}\int_{x_2}^{x_1}\Delta''(x,x_1,\alpha)dx$,

where $x_1\leq\bar{x}\leq\bar{x}_2$; $\bar{q}=q(\bar{x})$. However the generalized first mean value theorem's hypothesis states that $\Delta''(x,x_1,\alpha)$ does not change sign in $[x_1,x_2]$, but in his paper Ewing does not state or prove that $\Delta''(x,x_1,\alpha)$ does not change sign in $[x_1,x_2]$. Therefore for this proof we shall assume that $\Delta''(x,x_1,\alpha)$ does not change sign in $[x_1,x_2]$.

$$A(x_2,\alpha)=\frac{\Delta(\bar{x},x_2)\left[\Delta'(x_1,x_1,\alpha)-\Delta'(x_2,x_2,\alpha)\right]}{(\bar{q}+\tilde{a}^2)\,D(\bar{x})}$$

The fraction can not vanish since $\Delta(x,x_2)$ is different from zero by IV_L and $\left[\Delta'(x_1,x_1,\alpha)-\Delta'(x_2,x_1,\alpha)\right]$ being the difference between two terms of opposite sign. Therefore 2.10 is false and there exists at most one L, the one for $\alpha=0$, for which E fails to satisfy IV_L'.

We will now discuss the sufficient conditions of [1] for an improper strong relative minimum for J.

At this point we shall assume that E:y=e(x) satisfies condition III for J. We shall state a definition and two theorems from [2] that will be needed in the course of obtaining our results.

Definition 2.11: A field is a region F of xy-space with a slope function p(x,y) having the following properties:

a) p(x,y) is single valued and has continuous first partial derivatives in F,

b) the elements (x,y,p(x,y)) defined by the points (x,y) in F are all admissible,

c) the Hilbert integral, $J^*=\int (fdx+(dy-y'dx)f_{y'})$, is independent of the path in F.

Theorem 2.12: If a one parameter family of extremals y(x,a), where $a_1 \leq a \leq a_2$, $x_1(a) \leq x(a) \leq x_2(a)$, is cut by a curve C defined on the family by a function $x=\mathcal{E}(a)$ where $a_1 \leq a \leq a_2$, $\mathcal{E}(a)$ is single valued, continuous and has continuous partial derivatives of at least the first order and y(x,a) is continuous and has continuous partial derivatives of at least the second order, then every region F of the xy-plane which is simply covered by the extremals is a field with the slope function p(x,y) of the family, provided that the derivative $y_a(x,a)$ is different from zero at each set of values (x,a) corresponding to a point (x,y) in F.

Theorem 2.13 (The Fundamental Sufficiency Theorem): If E_{12} is an extremal arc of a field F and if at each point of the field the condition

2.14 $E(x,y,p(x,y),y') \geq 0$

holds for every admissible set (x,y,y') with $y' \neq p$, then the relation $J(C_{12}) \geq J(E_{12})$ is true for every admissible arc C_{12} in the field which joins the end-points 1 and 2 of E_{12}. If condition 2.14 holds without the equality sign then $J(C_{12}) > J(E_{12})$ unless C_{12} is identical with E_{12}.

To find how to strengthen our conditions so as to insure a field F which is independent of k, consider the line Λ : $x=x_1$, $y=n\tau+y_1$ together with a slope function $p(\tau)=m\tau+e'(x_1)$. The extremals for L are $y=y(x,a,b,\alpha)$ and the equations

2.15 $n\tau+y_1-y(x_1,a,b,\alpha)=0$,

2.16 $m\tau+e'(x_1)-y'(x_1,a,b,\alpha)=0$,

define $a=a(\tau,\alpha)=\bar{a}(y,\alpha)$ and $b=b(\tau,\alpha)=\bar{b}(y,\alpha)$ for any admissible α and for every y for which (x_1,y) is in the domain of f. a, b, $\bar{a}$, and $\bar{b}$ also depend on m and n, which are omitted in the notation. These implicit functions are of at least class C' in their respective variables. We have a family of extremals of parameter τ for each admissible α , $y=\varphi(x,\tau,\alpha)\equiv y(x,a(\tau,\alpha),b(\tau,\alpha),\alpha)$, intersecting Λ and including E for $\tau=0$. We shall denote $y_a(x,a(\tau,\alpha),b(\tau,\alpha),\alpha)$ as $y_a(x)$, $y_a(x_1,a(\tau,\alpha),b(\tau,\alpha),\alpha)$ as $y_a(x_1)$ and similarly for y_b,

y_a' and y_b'. Ewing attempts to show that this family furnishes a field and the field is independent of k. We shall state his proof until 2.18 and then we shall continue the proof in an alternate manner.

Proof: Suppose there exists an x, $x_1 < x \leq x_2$, such that $\varphi(x, \tau_1, \alpha) - \varphi(x, \tau_2, \alpha) = 0$. By the Mean Value Theorem, $\varphi_\tau(x, \bar{\tau}, \alpha) = 0$ for some $\tau_1 < \bar{\tau} < \tau_2$. Thus

2.17 $$\varphi_\tau(x, \bar{\tau}, \alpha) = y_a \frac{\partial a}{\partial \tau} + y_b \frac{\partial b}{\partial \tau} \Bigg|_{\tau = \bar{\tau}} = 0,$$

but from the differentiation with respect to τ of 2.15 and 2.16 we have

$$n - y_a \frac{\partial a}{\partial \tau} - y_b \frac{\partial b}{\partial \tau} = 0 \qquad \text{or} \qquad y_a(x_1) \frac{\partial a}{\partial \tau} + y_b(x_1) \frac{\partial b}{\partial \tau} = n$$

$$m - y_a' \frac{\partial a}{\partial \tau} - y_b' \frac{\partial b}{\partial \tau} = 0 \qquad\qquad y_a'(x_1) \frac{\partial a}{\partial \tau} + y_b'(x_1) \frac{\partial b}{\partial \tau} = m.$$

Let $D_1 = \begin{vmatrix} y_a(x_1) & y_b(x_1) \\ y_a'(x_1) & y_b'(x_1) \end{vmatrix}$ then by theorem 1,7 $D_1 \neq 0$ and

$$D_1 \frac{\partial a}{\partial \tau} = n y_b'(x_1) - m y_b(x_1), \quad D_1 \frac{\partial b}{\partial \tau} = -n y_a'(x_1) + m y_a(x_1).$$

Hence we have from 2.17

$$\frac{1}{D_1} \left(n y_a(x) y_b'(x_1) - m y_a(x) y_b(x_1) + m y_a(x_1) y_b(x) - n y_b(x) y_a'(x_1) \right) = 0$$

or

$$\frac{n}{D_1} \begin{vmatrix} y_a(x) & y_b(x) \\ y_a'(x_1) & y_b'(x_1) \end{vmatrix} - \frac{m}{D_1} \begin{vmatrix} y_a(x) & y_b(x) \\ y_a(x_1) & y_b(x_1) \end{vmatrix} = 0.$$

2.18 At this point in his proof (page 375-last paragraph), Ewing states that E satisfies the condition IV'_{LN} if constants $\delta > 0$, $\eta > 0$, and A exist such that

$$\Delta(x,x_1,y,\alpha) \equiv \begin{vmatrix} \bar{y}_a(x) & \bar{y}_b(x) \\ \bar{y}_a(x_1) & \bar{y}_b(x_1) \end{vmatrix}$$ is, in absolute value,

greater than δ in the region $x_1 < x \leq x_2$, $|y-y_1| \leq \eta$, $A \geq \alpha > -a^2$, where $\bar{y}_a(x)$ denotes $y_a(x,\bar{a}(y,\alpha),\bar{b}(y,\alpha),\alpha)$ and similarly for $\bar{y}_a(x_1)$, $\bar{y}_b(x)$, and $\bar{y}_b(x_1)$. Since $\bar{y}_a$ and $\bar{y}_b$ are continuous, $\Delta(x,x_1,y,\alpha)$ is continuous. Therefore $\lim_{x \to x_1+} \Delta(x,x_1,y,\alpha)$ exists and is greater than or equal to $\delta > 0$. However $\lim_{x \to x_1+} \Delta(x,x_1,y,\alpha) = \begin{vmatrix} \bar{y}_a(x_1) & \bar{y}_b(x_1) \\ \bar{y}_a(x_1) & \bar{y}_b(x_1) \end{vmatrix} = 0$

which is less than δ and we have a contradiction. To avoid this difficulty we will introduce Property I.

Let $Q = \begin{vmatrix} y_a(x) & y_b(x) \\ y'_a(x_1) & y'_b(x_1) \end{vmatrix}$ and

$$\Delta(x,x_1,y,\alpha) = \begin{vmatrix} y_a(x) & y_b(x) \\ y_a(x_1) & y_b(x_1) \end{vmatrix}.$$ Note that if

$y=y(x,a(0,\alpha),b(0,\alpha),\alpha)$ then $y=e$ and $\Delta(x,x_1,y,\alpha) \neq 0$ because of IV_L.

Now the solutions $y(x,a,b,\alpha)$ contain e for $a=a_0$, $b=b_0$, $\tau=0$ and thus $y(x,a_0,b_0,\alpha)=e(x)$, for all $\alpha > -a^2$. By continuity of partials, $y_a(x,a(\tau,\alpha),b(\tau,\alpha),\alpha)$ converges to $y_a(x,a_0,b_0,\alpha)$ a solution of Jacobi's equation.

Similarly for y_b, y'_a, y'_b. $Q(x)\neq 0$ for $x=x_1$ as y_a and y_b are linearly independent solutions of the accessory equations.

<u>Property I</u>: There exist an A, γ, n, $m \geq 0$ such that $H(x)=nQ-m\Delta(x,x_1,y,\alpha)\neq 0$ for all $x_1 \leq x \leq x_2$, $-\tilde{a}^2 < \alpha \leq A$ and $|\tau| < \gamma$.

Under these conditions $\varphi_\tau(x,\bar{\tau},\alpha)\neq 0$. i.e. For a fixed x and $-\tilde{a}^2 < \alpha \leq A$, φ is a monotone function in τ.

Now with values of m, n fixed a, b are strictly functions of τ and α. We can solve for a, b as functions of τ and α if $-\tilde{a}^2 < \alpha \leq A$ and $|\tau| < \gamma$. Each choice of τ and α gives an extremal satisfying:

2.19 $y(x_1,a(\tau,\alpha),b(\tau,\alpha),\alpha)=y_1+n\tau$

2.20 $y'(x_1,a(\tau,\alpha),b(\tau,\alpha),\alpha)=e'(x_1)+m\tau$.

Since $e(x)$ is a solution of $\frac{d}{dx} f_{y'}=f_y$, $e(x)$ is a solution of $\frac{d}{dx} \varphi_{y'}=\varphi_y$ for every k because

$$\varphi(x,y,y')=f(x,y,y')+k^2(y'-e')^2$$

$$\varphi_{y'}=f_{y'}+2k^2(y'-e') \text{ and } \varphi_y=f_y .$$

Furthermore differentiating we have

$$f_{y'x}+f_{y'y}y'+f_{y'y'}y''+2k^2y''-2k^2e''=f_y \text{ or}$$

$$y''=\frac{1}{2k^2+f_{y'y'}}(f_y+2k^2e''-f_{y'x}-f_{y'y}y')=g(x,y,y',k).$$

Set $z_1=y$, $z_2=y'$ then $z_1'=0(z_1)+1(z_2)=\hat{f}(x,z,k)$ and $z_2'=g(x,z,k)$ where $z=(z_1,z_2)$. Hence we are concerned with the differential equation

$$2.21 \quad z'=q(x,z,k) \text{ where } q=\begin{pmatrix}\hat{f}\\ g\end{pmatrix}.$$

Property II: There exists a neighborhood F of the arc $E_{12}:y=e(x)$ in the plane which is independent of α, for $-\tilde{a}^2 < \alpha \leq A$, such that through every point (x,y) of F there passes at least one member of the family of extremals given by 2.19.

We can now prove the following theorem.

Theorem 2.22: If conditions I, III, and II_N hold for J and properties I and II hold for L then $y=e(x)$ furnishes a strong improper relative minimum for J.

Proof: Since condition III holds for J we have the family of extremals 2.19. Through each point of the set F there passes a unique member of the family 2.19 since property I guarantees that the family $y(x,a(\tau,\alpha),b(\tau,\alpha))=\varphi(x,\tau,\alpha)$ is monotone in τ. $\varphi_\tau \neq 0$ at every point of F by property I, hence F is a field by theorem 2.12. We are free to assume that II_N holds on F, hence II_N' holds on F for every $-\tilde{a}^2 < \alpha \leq A$. For each $-\tilde{a}^2 < \alpha \leq A$, theorem 2.13 gives $L(C_{12}) > L(E_{12})$ for every arc $C_{12}:y=y(x)$ lying in F and joining the points 1 and 2. Thus for all such arcs C_{12} we have $J(C_{12})-J(E_{12}) > \int_{x_1}^{x_2} k^2(y'(x)-e'(x))^2\, dx$. Letting $k^2 \longrightarrow 0$ we obtain $J(C_{12}) \geq J(E_{12})$.

It was hoped that we would not need property II; however two difficulties arise. If we consider the arc $E_{12}: y=e(x)$ in three dimensional space, then for each fixed α there exists a neighborhood G_α of E_{12} such that through each point of G_α there passes an extremal defined on $[x_1, x_2]$. III requires only that $f_{y'y'}(x, e(x), e'(x)) \geq 0$, for all $x_1 \leq x \leq x_2$, thus on every neighborhood of $(x_1, e(x_1), e'(x_1))$ it may be that $f_{y'y'} < 0$ at some points in which case G_α will shrink, possibly to E_{12}, as $\alpha \to -\tilde{a}^2$. III_F is a stronger condition and would guarantee a neighborhood G of E_{12}, independent of α, such that through each point (x, y, y') of G there passes a unique solution of 2.21. Unfortunately, this solution may not be defined on all of $[x_1, x_2]$. If G is made small enough this can be insured but then G will again depend on α. This is so, as an examination of 2.21 shows that the Lipschitz constant gets larger, and may tend to ∞, as $k^2 \to 0$.

In the next chapter we shall look at an example in which R is three dimensional space, condition II_N holds everywhere, $f_{y'y'} \equiv 0$ everywhere, and property I holds independent of γ. In this example, the set F of property II may be taken as $F = \{(x,y): x_1 - \epsilon < x < x_2 + \epsilon, -\infty < y < \infty\}$ for any $\epsilon > 0$. In this case $y=e(x)$ is a strong minimum i.e. it is in fact the minimum in the class of arcs considered.

CHAPTER THREE: APPLICATION AND CONCLUDING REMARKS

In section 5, Ewing gives three applications to which his results would apply. We shall apply our results to the second problem in which $f \equiv x^2+y^2+yy'$, and $(x_1,y_1)=(x_1,0)$ and $(x_2,y_2)=(x_2,0)$. We shall show that:

1) the line $y=0$ is an extremal,

2) the sufficient conditions for an improper minimum are met by $y=0$, and

3) III' is not met.

Proof:

1) $y=0$ has continuous first and second derivatives.

$f_{y'y}y'+f_{y'x}+f_{y'y'}y''-f_y=y'+0+0-(2y+y')=-2y$

Substituting $y=0$ we get $f_{y'y}y'+f_{y'x}+f_{y'y'}y''-f_y=0$ and therefore by definition 1.6 $y=0$ is an extremal for this application.

3) $f_{y'}=y$, and $f_{y'y'}=0$

Therefore condition III' is not met.

2) $e(x)=0$ and hence the extremals are solutions of

$\frac{d}{dx}\varphi_{y'}=\varphi_y$ which is

$y'+2k^2y''=2y+y'$ or

$2k^2y''-2y=0$ or

$(\tilde{a}^2+\alpha)y''-2y=0.$

Let $\beta=\sqrt{\frac{2}{a^2+\alpha}}$

Therefore the family of extremals are

$y(x,a,b,\alpha)=ae^{\beta x}+be^{-\beta x}$.

$y'(x,a,b,\alpha)=\beta ae^{\beta x}-\beta be^{-\beta x}$

$y_a=e^{\beta x}$, $y_b=e^{-\beta x}$, $y_a'=\beta e^{\beta x}$, $y_b'=-\beta e^{-\beta x}$

$$Q(x)=\begin{vmatrix} e^{\beta x} & e^{-\beta x} \\ \beta e^{\beta x_1} & -\beta e^{-\beta x_1} \end{vmatrix}=-\beta e^{\beta(x-x_1)}-\beta e^{\beta(x_1-x)}$$

$Q(x_1)=-2\beta\neq 0$ for all $\alpha>-\tilde{a}^2$.

$$\Delta(x,x_1,y,\alpha)=\begin{vmatrix} e^{\beta x} & e^{-\beta x} \\ e^{\beta x_1} & e^{-\beta x_1} \end{vmatrix}=e^{\beta(x-x_1)}-e^{-\beta(x-x_1)}$$

$$nQ-m\Delta=(-n\beta-m)e^{\beta(x-x_1)}+(-n\beta+m)e^{-\beta(x-x_1)}$$

We must show that y=0 satisfies property I, conditions I and II_N'.

<u>Property I</u> Note that $e^j>0$ for all $-\infty<j<\infty$. For this application y is independent of γ therefore we can select any $\gamma>0$.

Suppose for some x that $nQ-m\Delta=0$. Then

$(-n\beta-m)e^{\beta(x-x_1)}=(n\beta-m)e^{-\beta(x-x_1)}$, and

$e^{2\beta(x-x_1)}=-\frac{n\beta-m}{n\beta+m}$. This is impossible if $n\beta-m>0$ or $0\leq m<n\beta$. This condition can be forced by choosing m, n such that $0\leq m<n\sqrt{\frac{2}{A+\tilde{a}^2}}$.

For $0<\tilde{a}^2+\alpha \leq A+\tilde{a}^2$, $\frac{1}{A+\tilde{a}^2} \leq \frac{1}{\tilde{a}^2+\alpha}$ implies that

$\sqrt{\frac{2}{A+\tilde{a}^2}} \leq \sqrt{\frac{2}{\tilde{a}^2+\alpha}}$, and property I holds for $0 \leq m < n\sqrt{\frac{2}{A+\tilde{a}^2}}$.

Condition I $\quad \varphi(x,y,y',e')=x^2+y^2+yy'+k^2(y'-0)^2$ where $e(x)=0$.

$$\varphi_y=2y+y'$$

$$\varphi_{y'}=y+2k^2y'$$

$$0=\frac{d}{dx}\varphi_{y'}\Big|_{y=0}=\varphi_y\Big|_{y=0}=0$$

Condition II'_N Let $Y'\neq y'$, (x,y,y') and (x,y,Y') be admissible points and $e(x)=0$.

$$E(x,y,y',Y')=\varphi(x,y,Y')-\varphi(x,y,y')-(Y'-y')\varphi_{y'}(x,y,y')$$

$$=x^2+y^2+yY'+k^2Y'^2-x^2-y^2-yy'-k^2y'^2-(Y'-y')(y+2k^2y')$$

$$=k^2(Y'^2-y'^2)-2k^2y'(Y'-y')$$

$$=k^2(Y'-y')(Y'-y')$$

$=k^2(Y'-y')^2>0$ for all (x,y,y'), (x,y,Y') which are admissible and $Y'\neq y'$. Therefore there exists a neighborhood N around $y=0$ such that II' holds for each admissible point in the neighborhood. Therefore II'_N holds.

Therefore by theorem 2.23 $y=0$ is a strong improper minimum for J.

The original purpose of this thesis was to expand Ewing's paper to n dimensional space. However due to a lack of time we were unable to look at this problem.

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