6-1-1970

On zeros of solution of self-adjoint differential equations.

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ON ZEROS OF SOLUTIONS OF SELF-ADJOINT DIFFERENTIAL EQUATIONS

A Thesis
Presented to the
Department of Mathematics
and the
Faculty of the Graduate College
University of Nebraska at Omaha

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts

by
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June, 1970

Accepted for the faculty of The Graduate College of the University of Nebraska at Omaha, in partial fulfillment of the requirements for the degree Master of Arts.

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CHAPTER I

INTRODUCTION

This study will deal with the zeros of solutions of self-adjoint linear differential equations of second order. In the following we consider some definitions and theorems that relate to ordinary differential equations.

Definition 1.1. A homogeneous linear differential equation of order n has the form $a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_ny = 0$ where $a_0 \neq 0$ and each $a_i = a_i(x)$ is continuous on an interval (a,b), $i = 1,2,\ldots,n$.

Definition 1.2. If $L(y)$ is a linear operator and $L(y) = a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x)$, then its adjoint, $\bar{L}(z)$, is denoted by $\left[a_0(x)z(x)\right]'' - \left[a_1(x)z(x)\right]' + a_2(x)z(x)$. If $L(y) = \bar{L}(y)$, then the differential equation, $L(y) = 0$, is self-adjoint of second order.

A number of theorems, some of them included without proof, were used as a basis for the study in this thesis.

Theorem 1.1. $L(y) = 0$ is self-adjoint if and only if $a_1(x) = a_0'(x)$. (5, p. 98)

Every equation of the form $a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0$, where $a(x)$, $b(x)$, and $c(x)$ are continuous on (a,b) and $a(x) > 0$, can be written in the form $\left[r(x)y'\right]' + p(x)y = 0$ by considering the integrating factor $\frac{1}{a(x)} \exp\left[\int \frac{b(x)}{a(x)}\, dx\right]$. Therefore, the self-adjoint equation $L(y) = 0$ can be expressed in the form $[r(x)y']' + p(x)y = 0$ where $r(x) = \exp \int [a_1(x)/a_0(x)]\, dx$ and $p(x) = \dfrac{a_2(x)}{a_0(x)} \exp \int \left[\dfrac{a_1(x)}{a_0(x)}\right] dx.$

Conversely, every differential equation of the form $[r(x)y']' + p(x)y = 0$, where $r(x) > 0$ and $r(x)$, $p(x)$ are continuous, is self-adjoint since $[r(x)y']' + p(x)y = r(x)y'' + r'(x)y' + p(x)y$, and the result follows from Theorem 1.1.

Let $u_1(x)$ and $u_2(x)$ be two solutions of the differential equation $a_0(x)u''(x) + a_1(x)u'(x) + a_2(x)u(x) = 0$.

<u>Definition 1.3</u>. Two solutions, $u_1(x)$ and $u_2(x)$, of a linear differential equation are said to be linearly dependent if there exist constants, c_1 and c_2, not both zero, such that $c_1u_1(x) + c_2u_2(x) = 0$ for $x \in (a,b)$. If $u_1(x)$ and $u_2(x)$ are not linearly dependent, they are said to be linearly independent.

<u>Definition 1.4</u>. The wronskian of two solutions of a linear differential equation of order two is the determinant:

$$w(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} = u_1(x)u_2'(x) - u_2(x)u_1'(x).$$

<u>Theorem 1.2</u>. Two solutions, $u_1(x)$ and $u_2(x)$, of a linear differential equation are linearly dependent if and only if their wronskian is zero. (5, p. 90)

<u>Theorem 1.3</u>. A formula credited to Abel states that if $u_1(x)$ and $u_2(x)$ are solutions of a self-adjoint differential

equation of the form

(1.1) $$[r(x)u']' + p(x)u = 0,$$

where $r(x)$ and $p(x)$ are continuous and $r(x) > 0$ on $[a,b]$, then $r(x)\left[u_1(x)u_2'(x) - u_1'(x)u_2(x)\right] = K$, a constant.

Proof:

Since u_1 and u_2 are solutions of (1.1), then we have $\left[r(x)u_1'(x)\right]' + p(x)u_1(x) = 0$ and $\left[r(x)u_2'(x)\right]' + p(x)u_2(x) = 0$. Multiplying the first equation by $-u_2$ and the second by u_1 and adding, we obtain

$$u_1\left[r(x)u_2'\right]' - u_2\left[r(x)u_1'\right]' = 0.$$

Integrating by parts from a to x, the equation becomes

$$\int_a^x u_1\left[r(x)u_2'\right]'dx - \int_a^x u_2\left[r(x)u_1'\right]'dx = 0 =$$

$$u_1u_2'r(x)\Big|_a^x - \int_a^x u_1'u_2'r(x)dx - u_2u_1'r(x)\Big|_a^x + \int_a^x u_2'u_1'r(x)dx.$$

Thus $r(x)\left[u_1(x)u_2'(x) - u_1'(x)u_2(x)\right] =$

$$r(a)\left[u_1(a)u_2'(a) - u_1'(a)u_2(a)\right],$$ a constant.

Using this formula, we can show that two solutions, $u_1(x)$ and $u_2(x)$, of (1.1) having a common zero are linearly dependent. To show this, we can employ Abel's formula $r(x)\left[u_1(x)u_2'(x) - u_1'(x)u_2(x)\right] = K$. Let the common zero be $x = x_0$. Then $K = 0$. But $u_1(x_0)u_2'(x_0) - u_1'(x_0)u_2(x_0)$ is the wronskian of the solutions u_1 and u_2. By Theorem 1.2, the solutions are linearly dependent.

The following theorem due to Sturm compares the zeros of solutions of a self-adjoint differential equation.

<u>Theorem 1.4</u>. If $u_1(x)$ and $u_2(x)$ are linearly independent solutions of (1.1), then between two consecutive zeros of $u_1(x)$ there will be one zero of $u_2(x)$.

<u>Proof</u>:

Let x_1 and x_2 be the two consecutive zeros of $u_1(x)$. For Case 1, let $u_1(x)$ be greater than zero for $x \in (x_1, x_2)$. Then $u_1'(x_1) > 0$ and $u_1'(x_2) < 0$. Suppose $u_2(x_1) > 0$. We know $u_2(x_1) \neq 0$ because u_1 and u_2 are linearly independent. Setting $x = x_1$ in Abel's formula we find

$$r(x_1)\left[u_1(x_1)u_2'(x_1) - u_1'(x_1)u_2(x_1)\right] = -\left[r(x_1)u_1'(x_1)u_2(x_1)\right],$$

so that $K < 0$. From this we see that

$$r(x_2)\left[u_1(x_2)u_2'(x_2) - u_1'(x_2)u_2(x_2)\right] < 0.$$

But this is true only when $u_2(x_2) < 0$. Since $u_2(x)$ is continuous, it must have a zero between x_1 and x_2. The same proof with the roles of u_1 and u_2 exchanged shows that there is only one zero of u_2 between the consecutive zeros of u_1. For Case 2 where $u_1(x) < 0$, the proof parallels that above since $-u_1(x)$ has the same zeros as $u_1(x)$.

Next we consider the zeros of solutions of pairs of self-adjoint differential equations. Consider a pair of equations

$$\left[r(x)u'\right]' + p(x)u = 0 \tag{1.2}$$

$$\left[r(x)u'\right]' + p_1(x)u = 0 \tag{1.3}$$

on an interval $[a,b]$ where $r(x) > 0$, $r(x)$, $p(x)$, $p_1(x)$ are continuous on $[a,b]$ and $p_1(x) \geq p(x)$, the strict inequality holding on at least one point of $[a,b]$. The well-known Sturm Comparison Theorem is stated next.

Theorem 1.5. Given the above equations and conditions, let $u_1(x)$ be a solution of (1.2) and let $u_2(x)$ be a solution of (1.3). Then, between each pair of zeros of u_1, there is at least one zero of u_2.

Proof:

Because u_1 and u_2 are solutions of (1.2) and (1.3), respectively, we have $[ru_1']' + pu_1 = 0$ and $[ru_2']' + p_1u_2 = 0$. If we multiply the first equation by u_2 and the second by $-u_1$ and add, we obtain $u_2[ru_1']' + u_2pu_1 - u_1[ru_2']' - u_1p_1u_2 = 0$, and hence

$$u_2[ru_1']' - u_1[ru_2']' = u_1u_2(p_1 - p). \tag{1.4}$$

Consider two points, a and b, that are consecutive zeros of u_1. Suppose u_2 has no zeros in (a,b). Let u_1 and u_2 both be positive in (a,b). This implies that $u_1'(a) > 0$ and $u_1'(b) < 0$. Integrating both members of (1.4) over $[a,b]$ we obtain $\int_a^b u_2[ru_1']'dx - \int_a^b u_1[ru_2']'dx =$

$u_2ru_1'\Big|_a^b - \int_a^b ru_1'u_2'dx - u_1ru_2'\Big|_a^b + \int_a^b ru_2'u_1'dx =$

$$r(u_2u_1' - u_1u_2')\Big|_a^b = \int_a^b (p_1 - p)u_1u_2\,dx.$$

The right hand side of this equation is positive. Therefore, $r(u_2u_1' - u_1u_2')\Big|_a^b = ru_2u_1'\Big|_a^b > 0$, since $u_1(a) = u_1(b) = 0$. Since we assumed $u_2 > 0$ on the interval (a,b), then $ru_2u_1'\Big|_a^b = r(b)u_2(b)u_1'(b) - r(a)u_2(a)u_1'(a) \not> 0$, and we have a contradiction. From this contradiction, we can infer the truth of the theorem.

An immediate consequence is seen in the following. If we consider solutions of the equation

$$(1.5) \qquad u'' + q(x)u = 0,$$

it can be seen that oscillations of these solutions depend on $q(x)$. If $q(x) \leqslant 0$, then no non-trivial solution of (1.5) can have more than one zero since, by the Comparison Theorem, a solution v, $\not\equiv 0$, of the differential equation $v'' = 0$ would have to vanish at least once between any two zeros of a solution of (1.5). However, $v = ax + b$ has only one zero. Thus, the equation (1.5) is disconjugate since every solution that is not identically zero has at most one zero on the defined interval.

If $q(x) \geqslant k^2 > 0$, then a comparison of (1.5) with the trigonometric differential equation, $u'' + k^2u = 0$, yields that any solution of (1.5) must vanish between two consecutive zeros of any solution $u(x) = A \cos k(x - x_1)$, of $u'' + k^2u = 0$; hence the solution vanishes in any interval of length π/k.

We can state, then, that if we have given the differential system $u'' + q(x)u = 0$, $u(a) = 0$, $q(x)$ continuous on $[a,b]$ and $0 < m \leqslant q(x)$, and if $b - a \geqslant \pi/\sqrt{m}$, where $m = k^2$ above, then $u(b) = 0$, or $u(x) = 0$ for $x \in (a,b)$.

We can prove, however, a theorem that is even more general. Consider the self-adjoint differential equation

$$(1.6) \qquad [r(x)u']' + p(x)u = 0,$$

where $r(x)$ and $p(x)$ are continuous and $r(x) > 0$ on (a,b). Consider also the functional $\int_a^b (ru'^2 - pu^2)dx$ on the interval $[a,b]$ with $r > 0$. If $u(x)$, $\not\equiv 0$, and $ru'(x)$ are functions of class C^1 on $[a,b]$ and if $u(a) = u(b) = 0$, then $u(x)$ is said to be an admissible function.

We will show that if $\exists$ an admissible function $u = u(x)$ along which $\int_a^b (ru'^2 - pu^2)dx \leq 0$, then a solution $y(x)$ of the system $[ru']' + pu = 0$, $y(a) = 0$, will have a zero on $(a,b]$.

The following definitions are used for the proof of the theorem.

<u>Definition 1.5</u>. A functional $F = J[y]$ has an extremal for $y = y_1$ if $J[y] - J[y_1]$ does not change sign in some neighborhood of the curve $y = y_1$.

<u>Definition 1.6</u>. The functional $F = \int_a^b (ru'^2 - pu^2)dx$ is said to be positive definite if it is greater than zero for all admissible $u(x) \not\equiv 0$.

The proof we will show is the contrapositive of the theorem just stated and is from the Calculus of Variations.

<u>Theorem 1.6</u>. If a solution $y = y(x)$ of $[ru']' + pu = 0$ on $[a,b]$, $y(a) = 0$, has no points conjugate to a on $[a,b]$, then the functional $\int_a^b (ru'^2 - pu^2)dx$ is positive definite for all admissible functions $u(x)$.

Proof:

The functional $\int_a^b (ru'^2 - pu^2)dx$ will be positive definite if it can be reduced to the form $\int_a^b r(x)\phi^2 dx$ where ϕ^2 is some expression which cannot be identically zero unless $u(x) = 0$. We will add a quantity of the form $\frac{d}{dx}(wu^2)$ to the integrand of the functional, where $w(x)$ is a differentiable function. Since $u(a) = u(b) = 0$, then $\int_a^b \frac{d}{dx}(wu^2)\, dx = 0$ and the value of the functional is not changed. Then we have $ru'^2 - pu^2 + (wu^2)' =$ $ru'^2 - pu^2 + w'u^2 + 2wuu' = ru'^2 + 2wuu' + (w' - p)u^2$. If $w(x)$ is selected to be a solution of the equation

$$r(w' - p) = w^2 \tag{1.7}$$

then we can express the preceding as

$$ru'^2 + 2wuu' + \left[\frac{w^2}{r}\right]u^2 = r(u' + \frac{w}{r}u)^2.$$

Thus if (1.7) has a solution defined on the whole interval $[a,b]$, then the functional $\int_a^b (ru'^2 - pu^2)dx$ can be expressed $\int_a^b r(u' + \frac{w}{r}u)^2 dx$ and is non-negative.

We must also consider the case where the new non-negative functional $\int_a^b r(u' + \frac{w}{r}u)^2 dx$ vanishes for some $y = y(x)$.

If this is the case, $y(x)$ is an extremal. A fundamental theorem of the Calculus of Variations says that if a functional $J[y]$ has an extremal for $y = y(x)$, then $y = y(x)$ satisfies its corresponding Euler equation. Thus $y(x)$ is a solution of

(1.9) $$[ru']' + pu = 0.$$

By hypothesis, $y(a) = 0$. Then, since $r > 0$,

$$\int_a^b r\left[y'(x) + \frac{w}{r}\, y(x)\right]^2 dx = 0 \text{ implies } y'(x) + \frac{w}{r}\, y(x) = 0.$$

At $x = a$, $y'(a) + \frac{w}{r}\, y(a) = 0$ and then $y'(a) = 0$.

But if $y(a) = y'(a) = 0$, then $y(x)$ must be identically zero. Therefore, $y = y(x)$ is not an admissible function. Thus the functional is positive definite.

We must show, then, that if there are no points in $[a,b]$ conjugate to a, then (1.7) has a solution defined on the whole interval $[a,b]$. If we let $w = -\frac{u'}{u}\, r$, where u is a new function, from (1.7) we obtain

$$r\left[\left(-\frac{u'}{u}\, r\right)' - p\right] = \left(-\frac{u'}{u}\, r\right)^2,$$

$$\left(-\frac{u'}{u}\, r\right)' - p = \left[\frac{u'}{u}\right]^2 r,$$

$$u'r\,\frac{u'}{u^2} - \frac{1}{u}\,(ru')' - p = \frac{u'^2}{u^2}\, r,$$

which is equivalent to (1.9).

Thus if there are no points conjugate to a in $[a,b]$, then (1.9) does not vanish anywhere in $(a,b]$ and $w = -\frac{u'}{u}\, r$ is a solution of (1.7) defined on the whole interval. Thus the theorem is proved.

If we consider the equation $3y'' + 3y = 0$, then $r = 3$ and $p = 3$. A solution of this equation is $y = \sin x$ on the interval $[a,b] = [0, 2\pi]$ where $y(0) = 0$. If we let $u(x) = \sin \frac{1}{2}x$, then we have an admissible function for which the functional $\int_a^b (ru'^2 - pu^2)dx$ is negative. Hence, by the contrapositive of Theorem 1.6, the solution $y = \sin x$ of $3y'' + 3y = 0$ will have a zero on $(0, 2\pi]$.

Conversely, on the interval $(0, \pi/2]$, $y = \sin x$ has no zero. From Theorem 1.6, for an admissible function such as $u = \sin 2x$, the function $\int_0^{\frac{\pi}{2}} (3u'^2 - 3u^2)dx$ is positive.

Theorem 1.6 will prove important in the theory developed in Chapter II.

CHAPTER II

MORE ON SELF-ADJOINT ORDINARY DIFFERENTIAL EQUATIONS

We saw in Theorem 1.6 that if the ordinary self-adjoint differential equation $(ry')' + py = 0$ on $[a,b]$ has a solution $y = y(x)$ with $y(a) = 0$ and if its corresponding functional $\int_a^b (ru'^2 - pu^2)dx$ is less than or equal to 0 for an admissible function $u(x)$, then $y(x)$ will have a zero on $(a,b]$.

But with the conditions and functional above, we can associate the functional $-\int_a^b u\left[(ru')' + pu\right]dx$ since the following is true. Integrating the first term of $\int_a^b (ru'^2 - pu^2)dx$ by parts, we obtain

$$ruu'\Big|_a^b - \int_a^b u(ru')'dx - \int_a^b pu^2dx.$$

For an admissible function $u(x)$, the first term is zero and we have $-\int_a^b u\left[(ru')' + pu\right]dx$. Thus it follows that if $u(x)$ is an admissible function and if

$$\int_a^b u(x)\left[(r(x)u'(x))' + p(x)u(x)\right]dx > 0,$$

then a solution $y(x)$ of $\left[r(x)y'\right]' + p(x)y = 0 \ni y(a) = 0$ vanishes on the interval $(a,b]$.

If we apply this idea to the previous example $u'' + a^2u = 0$, we see $r = 1$ and $p = a^2$. For an admissible function $u = \sin kx$ over $\left[0, \pi/k\right]$ where $a > k > 0$, we can apply the preceding,

$$\int_a^b u\left[(ru')' + pu\right]dx = \int_0^{\frac{\pi}{K}} (a^2 - k^2) \sin^2 kx \, dx > 0.$$

Hence, any solution $u(x)$ of $u'' + a^2u = 0 \ni u(0) = 0$ must vanish at least once on $0 < x \leqslant \pi/k$.

From the preceding results, we can continue with the following theorem which is a result of Leighton. (5, p. 604)

Theorem 2.1. Given $r(x)$ and $r_1(x) > 0$ and $r(x)$, $r_1(x)$, $p(x)$, and $p_1(x)$ continuous functions on (a,b), consider the equations

$$(2.1) \qquad \left[r(x)u'\right]' + p(x)u = 0$$

$$(2.2) \qquad \left[r_1(x)y'\right]' + p_1(x)y = 0.$$

If there exists an admissible function $u(x)$ such that

$$(2.3) \quad \int_a^b (r - r_1)u'^2 + (p_1 - p)u^2 \, dx > \int_a^b (ru'^2 - pu^2)dx$$

then, a solution $y(x)$ of (2.2), $y(a) = 0$, vanishes on the interval $(a,b]$.

Proof:

Inequality (2.3) can be expressed

$$\int_a^b ru'^2dx - \int_a^b r_1u'^2dx + \int_a^b p_1u^2dx - \int_a^b pu^2dx > \int_a^b ru'^2dx - \int_a^b pu^2dx.$$

This is true only if $-\int_a^b r_1u'^2dx + \int_a^b p_1u^2dx > 0$.

That is, $\int_a^b (r_1u'^2 - p_1u^2)dx < 0$. Hence, an admissible function $u(x)$ for which (2.3) is true will also meet the requirements for Theorem 1.6. Thus the theorem is proved.

If $u(x)$ is a solution of (2.1), $u(a) = u(b) = 0$, then the right side of the inequality (2.3) can be evaluated in the following way. Integrating by parts, we obtain

$$\int_a^b (ru'^2 - pu^2)dx = ruu'\Big|_a^b - \int_a^b u\left[(ru')' + pu\right]dx.$$

But both terms of the right side of the equality are zero. Hence, if $u(x)$ is a solution of (2.1) $\ni$ $u(a) = u(b) = 0$, and if $\int_a^b (r - r_1)u'^2 + (p_1 - p)u^2\, dx > 0$, then a solution $y = y(x)$ of (2.2), with $y(a) = 0$, will vanish on $(a,b]$.

The preceding can be seen to be a generalization of the Sturm-Picone conditions. These conditions state that if the equations (2.1) and (2.2) are considered, with $r(x)$, $r_1(x)$, $p(x)$, $p_1(x)$ continuous functions on (a,b); $r(x)$ and $r_1(x)$ greater than zero; and $r_1(x) \leq r(x)$ and $p_1(x) \geq p(x)$ on $[a,b]$, with strict inequality holding in at least one point of the interval $[a,b]$, then between two consecutive zeros of a solution of (2.1) there will be a zero of a solution of (2.2).

As an example (for which the Sturm-Picone conditions do not hold), we consider the equation $y'' + (2x + 1)y = 0$ on $[0,\pi]$ with a solution $y = y(x)$ such that $y(0) = 0$. We compare this equation with $u'' + u = 0$ which has a solution $u = \sin x$ on $[0,\pi]$. Using the previous data, we obtain

$$\int_0^\pi \left[(1 - 1)\cos^2 x + (2x + 1 - 1)\sin^2 x\right]dx = \left[\frac{x^2}{2} - x\,\frac{\sin 2x}{2} + \frac{\cos 2x}{2}\right]_0^\pi = \frac{\pi^2}{2} > 0.$$

Therefore, there exists a $c \in (0,\pi]$ for which $y(c) = 0$.

Next we will consider a result due to P. A. Haeder.

(4, p. 17) For the equation

$$(2.4) \qquad u'' + p(t)u = 0,$$

where $p(t)$ is continuous on (a,b), let $x \in (a,b)$ and let $\epsilon > 0$ imply $a < x-\epsilon < x+\epsilon < b$.

<u>Theorem 2.2</u>. If $p(t) > 0$ on $[x-\epsilon, x+\epsilon]$ and

$$(2.5) \qquad \int_{x-\epsilon}^{x+\epsilon} p(t)^{-1}\, dt \leqslant \frac{2\epsilon^3}{3},$$

then every solution of $u'' + p(t)u = 0$ vanishes at least once in $[x-\epsilon, x+\epsilon]$.

<u>Proof</u>:

Let $h(t) = (t-x+\epsilon)(t-x-\epsilon)$. Then $h'(t) = 2(t-x)$. Let $J[h] = \int_{x-\epsilon}^{x+\epsilon} [h'^2(t) - p(t)h^2(t)]dt$ be the functional that corresponds with $h(t)$. If we can show $J[h] \leqslant 0$, we can apply Theorem 1.6.

The functions $h(t)$ and $h'(t)$ are uniformly continuous on $[x-\epsilon, x+\epsilon]$ and $h(x-\epsilon) = h(x+\epsilon) = 0$.

Considering the first part of the integral, we have

$$\int_{x-\epsilon}^{x+\epsilon} h'^2 dt = 4\int_{-\epsilon}^{\epsilon} t^2\, dt = 4\, \frac{t^3}{3}\Big|_{-\epsilon}^{\epsilon} = \frac{8\epsilon^3}{3}.$$

In order to place bounds on the second part, we consider

$$\int_{x-\epsilon}^{x+\epsilon} |h|dt = \int_{x-\epsilon}^{x+\epsilon} \frac{\sqrt{p}}{\sqrt{p}}\, |h|dt \leqslant \left[\int_{x-\epsilon}^{x+\epsilon} ph^2\, dt\right]^{\frac{1}{2}} \left[\int_{x-\epsilon}^{x+\epsilon} p^{-1}\, dt\right]^{\frac{1}{2}}$$

by the Schwarz Inequality. Therefore,

$$(2.6) \qquad \left(\int_{x-\epsilon}^{x+\epsilon} |h|\, dt\right)^2 \left(\int_{x-\epsilon}^{x+\epsilon} p^{-1}\, dt\right)^{-1} \leqslant \int_{x-\epsilon}^{x+\epsilon} ph^2\, dt.$$

In the following evaluation, we see

$$\left(\int_{x-\epsilon}^{x+\epsilon} |h|dt\right)^2 = \left(\int_{x-\epsilon}^{x+\epsilon} |(t-x)^2 - \epsilon^2|dt\right)^2.$$

If we let T = (t-x), we have

$$\left(\int_{\epsilon}^{\epsilon} |T^2 - \epsilon^2| dT\right)^2 = \frac{16\epsilon^6}{9}$$

Substituting this value in (2.6), we obtain

$$\frac{16\epsilon^3}{9}\left(\int_{x-\epsilon}^{x+\epsilon} p^{-1}\,dt\right)^{-1} \leqslant \int_{x-\epsilon}^{x+\epsilon} ph^2\,dt.$$

But if we choose p(t) such that

$$\frac{8\epsilon^3}{3} \leqslant \frac{16\epsilon^6}{9}\left(\int_{x-\epsilon}^{x+\epsilon} p^{-1}\,dt\right)^{-1},$$

then

$$\int_{x-\epsilon}^{x+\epsilon} h'^2\,dt \leqslant \int_{x-\epsilon}^{x+\epsilon} ph^2\,dt$$

and hence, $J[h] \leqslant 0$. Thus, if $\int_{x-\epsilon}^{x+\epsilon} p^{-1}\,dt \leqslant \frac{2\epsilon^3}{3}$,

then $J[h] \leqslant 0$ and the conditions of Theorem 1.6 are met.

Along the same lines, Galbraith (2, p. 333) showed for the differential equation $y''(t) + p(t)y(t) = 0$ with $p(t) \geqslant 0$, monotone and concave on $[a,b]$, that if

$$(2.7) \qquad \int_a^b p(t)\,dt \geqslant \frac{9/8\ n^2\pi^2}{b-a},$$

where n is an integer, then every solution of $y'' + p(t)y = 0$ has at least n zeros in $[a,b]$.

We use the foregoing results in the following problem. Consider the equation $y''(x) + 12(x+1)y(x) = 0$ on $[0, 1]$. If it is true that $\int_0^1 12(x+1)\,dx \geqslant \frac{9/8\ n^2\pi^2}{1-0}$ for an integer n, then Galbraith's conditions are met.

$\int_0^1 12(x+1)\,dx = 12\left(\frac{x^2}{2} + x\right)\Big|_0^1 = 12(\frac{1}{2} + 1) = 18$ which is greater than $\frac{9/8\pi^2}{1} \approx 11$, when $n = 1$. Thus every solution of the differential equation $y'' + 12(x+1)y = 0$ has at least one zero on $[0, 1]$.

If we use Theorem 2.2 on the same problem, we obtain the following. Let $x \in (0,1) = \frac{1}{2}$. Then the interval we are considering is $[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$.

$$\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} \frac{1}{12(x+1)}\,dx = \frac{1}{12}\Big[\ln\,(x+1)\Big]_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} = \frac{1}{12}\ln\left(\frac{3/2+\epsilon}{3/2-\epsilon}\right).$$

We would like this to be less than or equal to $\frac{2\epsilon^3}{3}$. If we let $\epsilon = \frac{1}{2}$, we have $\frac{1}{12}\ln\frac{2}{1} \approx \frac{.69}{12} < \frac{2(\frac{1}{2})^3}{3} = \frac{1}{12}$.

Hence, a solution $y(x)$ of $y'' + 12(x+1)y = 0$ has a zero for some value of $x \in [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$ where ϵ is slightly less than $\frac{1}{2}$.

CHAPTER III

ZEROS OF SOLUTIONS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

We now consider the linear self-adjoint elliptic partial differential equation. This is defined by $Lu = 0$, where

(3.1) $$Lu = \sum_{i,j=1}^{n} D_j(a_{ij}D_i u) + bu, \quad a_{ij} = a_{ji},$$

on R, a bounded open set in n-dimensional Euclidean space, E^n, with boundary B having a piecewise continuous unit normal. For $x \in E^n$, $x = (x_1, x_2, \ldots, x_n)$ and D_i denotes differentiation with respect to x_i, $i = 1,2,\ldots,n$. We assume the following: (1.) a_{ij} and b are real and continuous on $\bar{R}$; (2.) the symmetric matrix (a_{ij}) is positive definite; (3.) a solution u of $Lu = 0$ is continuous on $\bar{R}$ and has uniformly continuous first partial derivatives in R; (4.) all derivatives in (3.1) exist, are continuous and satisfy $Lu = 0$, $\forall x \in R$. Henceforth, we shall use the symbol $\sum$ in place of the summation defined in (3.1), $\sum_{i,j=1}^{n}$.

The quadratic functional associated with (3.1) is

(3.2) $$J[u] = \int_R \left(\sum a_{ij} D_i u \, D_j u - bu^2 \right) dx.$$

The domain D of the functional J is defined to be the set of all real valued continuous functions on $\bar{R}$ which vanish on B and have uniformly continuous first partial derivatives on R.

The theorem following which is credited to Clark and

Swanson (1, p. 887) parallels Theorem 1.6 for the ordinary self-adjoint differential equation.

<u>Theorem 3.1</u>. Let L be the operator (3.1) and let $J[u]$ be the functional defined by (3.2). If $\exists\ u \in D$ not identically $0 \ni J[u] \leq 0$, then every solution v of $Lv = 0$ vanishes at some point of $\bar{R}$.

<u>Proof</u>:

Suppose $\exists$ a solution, v, of $Lv = 0 \ni v \neq 0$ at any point of $\bar{R}$. For $u \in D$, define

$$X^i = v\, D_i\left[\frac{u}{v}\right], \qquad Y^i = v^{-1} \sum_j a_{ij} D_j v, \qquad i = 1, \ldots, n.$$

$$\text{and} \quad E[u,v] = \sum a_{ij} X^i X^j + \sum_i D_i(u^2 Y^i).$$

Then $E[u,v] = \sum a_{ij} v D_i\left[\frac{u}{v}\right] v D_j\left[\frac{u}{v}\right] + \sum_i D_i(u^2 v^{-1} \sum_j a_{ij} D_j v) =$

$$\sum a_{ij} v^2 \left(\frac{vD_i u - uD_i v}{v^2}\right)\left(\frac{vD_j u - uD_j v}{v^2}\right) + \sum v^{-1} a_{ij} 2uD_i u D_j v +$$

$$\sum -v^{-2} u^2 a_{ij} D_i v D_j v + \sum u^2 v^{-1} D_i(a_{ij} D_j v) =$$

$$\sum a_{ij} v^{-2}(v^2 D_i u D_j u + u^2 D_i v D_j v - 2uv D_i v D_j u) +$$

$$\sum v^{-1} a_{ij} 2u D_i D_j v + \sum -v^{-2} u^2 a_{ij} D_i v D_j v +$$

$$\sum u^2 v^{-1} D_i(a_{ij} D_j v) =$$

$$\sum \left[a_{ij} D_i u D_j u + u^2 v^{-1} D_j(a_{ij} D_i v)\right] + bu^2 - bu^2 =$$

$$\sum a_{ij} D_i u D_j u - bu^2 + \frac{bu^2 v + \sum u^2 D_j(a_{ij} D_i v)}{v} =$$

$$\sum a_{ij}D_iuD_ju - bu^2 + u^2v^{-1}\left[\sum D_j(a_{ij}D_iv) + bv\right] =$$

$$\sum a_{ij}D_iuD_ju - bu^2 + u^2v^{-1}Lv.$$

But $Lv = 0$. Thus $J\left[u\right] = \int_R E\left[u,v\right]\, dx$, and we have

$$J\left[u\right] = \int_R \left[\sum a_{ij}X^iX^j + \sum_i D_i(u^2Y^i)\right] dx, \tag{3.3}$$

$u \in D$, $u = 0$ on B. By Green's formula $\int_R \sum_i D_i(u^2Y^i)\, dx$ of (3.3) is equal to zero. Hence

$$J\left[u\right] = \int_R \sum a_{ij}\, X^i\, X^j\, dx.$$

The matrix a_{ij} is positive definite and we have $J\left[u\right] \geqslant 0$, with equality holding iff $X^i = 0$, $i = 1, 2, \ldots, n$. However, if $0 = X^i = vD_i\left[\frac{u}{v}\right]$ and $v \neq 0$ in $\bar{R}$, then $D_i\left[\frac{u}{v}\right] = 0$ for every point in $\bar{R}$ and u is a constant multiple of v. But with $u = 0$ on B and $v \neq 0$ on B, u cannot be a constant multiple. Therefore, $J\left[u\right] > 0$ which is a contradiction. Thus the theorem is proved.

An extension of Theorem 2.1 can be seen in the following theorem credited to Clark and Swanson. (1, p. 888)

Consider the differential operators (3.1) and

$$L^*u = \sum D_j(a^*_{ij}D_iu) + b^*u, \tag{3.3}$$

where a^*_{ij} and b^* satisfy the same conditions as a_{ij} and b. The associated quadratic functional is

$$J^*\left[u\right] = \int_R \left(\sum a^*_{ij}D_iuD_ju - b^*u^2\right) dx. \tag{3.4}$$

Theorem 3.2. If $\exists$ a solution $u \not\equiv 0$ of $L^* u = 0$ in R $\ni u = 0$ on B and if $\int_R \left[\sum (a^*_{ij} - a_{ij}) D_i u\, D_j u + (b - b^*)\, u^2\right] dx$ is greater than or equal to zero, then every solution v of $Lv = 0$ vanishes at some point of $\bar{R}$.

Proof:

$$\int_R \left[\sum (a^*_{ij} - a_{ij}) D_i u\, D_j u + (b - b^*) u^2\right] dx \geq 0$$

implies $J[u] \leq J^*[u]$. But since $u = 0$ on B, by Green's formula, $J^*[u] = -\int_R u\, L^* u\, dx = 0$. Thus the conditions of Theorem 3.1 are met and v vanishes at some point of $\bar{R}$.

If we consider the self-adjoint elliptic partial differential equations defined by the differential operators (3.1) and (3.3) and the corresponding conditions previously stated, we have

$$\sum D_j (a_{ij}\, D_i u) + bu = 0 \tag{3.4}$$

$$\sum D_j (a^*_{ij}\, D_i u) + b^* u = 0 \tag{3.5}$$

Definition 3.1. Equation (3.4) is said to be a strict Sturmian majorant for (3.5) if

1.) $b \geq b^*$

2.) $(a^*_{ij}) \geq (a_{ij})$ i.e., the matrix $(a^*_{ij} - a_{ij})$ is non negative.

3.) either $b > b^*$ for some x_0 on R or if $b \equiv b^*$, then some x_0 at which $a^*_{ij} > a_{ij}$ and the common value of b and b^* at x_0 does not vanish.

If these conditions hold, then Theorem 3.2 follows.

That is, if $\exists$ a solution $u \not\equiv 0$ of (3.5) and if (3.4) is a strict Sturmian majorant of (3.5), then every solution v of (3.4) vanishes at some point of $\bar{R}$.

Next we will consider another proof credited to P. A. Haeder. We will let $\bar{I}_\epsilon$ be the interior and boundary of an n-dimensional cube formed in the following way. If $c = (c_1, c_2, \ldots, c_n)$ is a point in E^n and I_k is an interval along the x_k axis such that $I_k = (c_k - \epsilon, c_k + \epsilon)$ where $\epsilon > 0$, then $I_\epsilon = I_1 \times I_2 \times \ldots \times I_n$. We will define the operator Lu as in 3.1 and let a_{ij} and b be real and continuous on $\bar{I}_\epsilon$. Also the symmetric matrix (a_{ij}) will be positive definite on $\bar{I}_\epsilon$. The following functions are necessary also.

$$f(x_i) = (x_i - c_i + \epsilon)(x_i - c_i - \epsilon), \quad i = 1,2,\ldots,n.$$

$$h(x) = f(x_1)\, f(x_2) \ldots f(x_n) \text{ on } \bar{I}_\epsilon$$

The functional corresponding to (3.1) in this case is

$$(3.6) \qquad J[u] = \int_{I_\epsilon} \left[\sum a_{ij} D_i u D_j u - bu^2 \right] dx$$

which meets the same conditions as (3.2).

<u>Theorem 3.3</u>. Let Lu be defined by (3.1) and let the above conditions hold. Let P be defined to be the following.

$$P = \sup\left\{ \left(\sum a_{ij}^2 \right)^{\frac{1}{2}} : x \in \bar{I}_\epsilon \right\}.$$

If $b \geq \frac{5Pn}{2\epsilon^2}$ on $\bar{I}_\epsilon$, then every solution v of $Lv = 0$ vanishes at some point of $\bar{I}_\epsilon$.

<u>Proof</u>:

The function h(x) is real valued and continuous on $\bar{I}_\epsilon$, and it vanishes on $B(I_\epsilon)$. It is true that $\frac{\partial h}{\partial x_i} = h_i =$ $f(x_1)\, f(x_2) \ldots f(x_{i-1})\, 2(x_i - c_i)\, f(x_{i+1}) \ldots f(x_n)$, where h_i denotes $h_i(x)$, and that h_i is uniformly continuous on I_ϵ for $i = 1,2, \ldots, n$. We evaluate the following.

$$\int_{I_\epsilon} h^2\, dx = \int_{c_1-\epsilon}^{c_1+\epsilon} \cdots \int_{c_1-\epsilon}^{c_1+\epsilon} \prod_i \left[(x_i - c_i)^2 - \epsilon^2\right]^2 dx_i$$

If we let $t_i = (x_i - c_i)$, we have

$$\int_{-\epsilon}^{\epsilon} \prod_i (t_i^2 - \epsilon^2)^2\, dt_i = \left[\int_{-\epsilon}^{\epsilon} (t^2 - \epsilon^2)^2\, dt\right]^n =$$

$$\left[\int_{-\epsilon}^{\epsilon} (t^4 - 2t^2\epsilon^2 + \epsilon^4)\, dt\right]^n = \left(\frac{16\epsilon^5}{15}\right)^n .$$

$$\int_{I_\epsilon} h_i^2\, dx = 4\int_{c_1-\epsilon}^{c_1+\epsilon} \cdots \int_{c_{i-1}-\epsilon}^{c_{i-1}+\epsilon} \prod_{j\neq i} \int_{c_j-\epsilon}^{c_j+\epsilon}$$

$$\left[(x_j - c_j)^2 - \epsilon^2\right]^2 dx_j \int_{c_i-\epsilon}^{c_i+\epsilon} (x_i - c_i)^2 dx_i.$$

Letting $t_i = (x_i - c_i)$, we obtain

$$4\int_{-\epsilon}^{\epsilon} \prod_{j\neq i} (t_j^2 - \epsilon^2)^2\, dt_j \int_{-\epsilon}^{\epsilon} t_i^2\, dt_i =$$

$$4\left[\int_{-\epsilon}^{\epsilon} (t^4 - 2t^2\epsilon^2 + \epsilon^4)\, dt\right]^{n-1} \int_{-\epsilon}^{\epsilon} t^2\, dt =$$

$$4\left(\frac{16\epsilon^5}{15}\right)^{n-1} \left(\frac{2\epsilon^3}{3}\right).$$

Let $Q(x) = \sum a_{ij} D_i h D_j h$. From Schwarz' Inequality we have

$$Q(x) \leqslant \sqrt{\sum a_{ij}^2}\ \sqrt{\sum (h_i h_j)^2} = \sqrt{\sum a_{ij}^2}\ (\sum_i h_i^2).$$

We know that $P, = \sup\{(\sum a_{ij}^2)^{\frac{1}{2}} : x \in I_\epsilon\}$, exists because each a_{ij} is a continuous function on a closed and bounded set. Then

$$\int_{I_\epsilon} Q(x)\, dx \leqslant P \int_{I_\epsilon} \sum_i h_i^2 dx = \left(\frac{8P\epsilon^3 n}{3}\right)\left(\frac{16\epsilon^5}{15}\right)^{n-1}.$$

The proof of the theorem follows.

Since $h \in D$, the domain of $J[u]$, we can apply Theorem 3.1 if $J[u] \leqslant 0$. Now $J[h] = \int_{I_\epsilon} (Q(x) - bh^2)\, dx$ which is less than or equal to $\left(\frac{8P\epsilon^3 n}{3}\right)\left(\frac{16\epsilon^5}{15}\right)^{n-1} - \left(\frac{5Pn}{2\epsilon^2}\right)\left(\frac{16\epsilon^5}{15}\right)^{n}$ if $b \geqslant \frac{5Pn}{2\epsilon^2}$. But the difference above is equal to zero.

Hence, by Theorem 3.1, every solution v of $Lv = 0$ vanishes at some point on $\bar{I}_\epsilon$.

We now consider some examples. For the operator $Lv = v_{xx} + v_{yy} + 4v$, the partial differential equation $Lv = 0$ is elliptic. Let R be the interior of the square $[(0,0), (0,\pi), (\pi,\pi), (\pi,0)]$, and let $\bar{R}$ be R and its boundary. The corresponding functional is

$$\iint_R \left[\sum a_{ij} D_i u\, D_j u - bu^2\right] dx\, dy =$$

$$\iint_R \left[(u_x)^2 + (u_y)^2 - 4u^2\right] dx\, dy. \tag{3.7}$$

The function $u = \sin x \cos y$ is in D and since

$\cos^2 x \cos^2 y + \sin^2 x \sin^2 y = 1 - 2\sin^2 x \cos^2 y \leqslant 4\sin^2 x \cos^2 y$

for $x, y \in R$, then the functional (3.7) is less than or equal to 0. By Theorem 3.1, any solution v of $Lv = 0$ vanishes at some point of $\bar{R}$.

We can arrive at the same conclusion by using Theorem 3.2. If we compare $Lv = 0$ defined above with $L^*v = 0$, where $L^*v = v_{xx} + v_{yy} + 2v$, then there is a solution v of $L^*v = 0$, $v = \sin x \cos y$, that vanishes on the boundary of R. By Theorem 3.2, $\iint_R (4 - 2)(\sin x \cos y)^2 dx\, dy \geq 0$ and hence, every solution of $Lv = 0$ vanishes on $\bar{R}$.

Using Theorem 3.3 where $R = I_\epsilon$ when $c = (\frac{\pi}{2}, \frac{\pi}{2})$ and $0 < \epsilon = \frac{\pi}{2}$, we see that $b = 4 \geq \frac{5Pn}{2\epsilon^2} = \frac{5 \cdot 1 \cdot 2}{2(\frac{\pi}{2})^2} = \frac{5 \cdot 4}{\pi^2}$.

In this case $P = 1$. Thus by Theorem 3.3, we obtain the same result.

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V. ACKNOWLEDGMENTS

I wish to express my deep appreciation to Dr. P. A. Haeder whose patience, guidance, and assistance were invaluable in this study.

I also want to thank my husband, Bob, for his understanding and encouragement.