# A topology on the fundamental group.

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# A TOPOLOGY ON THE FUNDAMENTAL GROUP

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## A. Introduction and Background

For an arbitrary topological space algebraic topology prescribes a construction for a fundamental group. There is a natural way of imposing a topology on this set. We will examine this construction and the topological space which results.

We use the following notation and definitions:

1. If $X$ and $Y$ are topological spaces and $A \subset X$, $B \subset Y$ and if $f: X \to Y$ is a map, we write $f:(X,A) \to (Y,B)$ if $f(A) \subset B$.

2. $(Y,B)^{(X,A)} \equiv \{f:(X,A) \to (Y,B) \mid f \text{ is continuous}\}$.

3. $I \equiv [0,1]$, the closed unit interval; $\dot{I} \equiv \{0,1\}$, the points 0 and 1.

4. If $X$ is a topological space and $a \in X$, then $R$ is an equivalence relation on $(X,a)^{(I,\dot{I})}$ defined as follows:

If $f,g \in (X,a)^{(I,\dot{I})}$ then $fRg$ if and only if $f$ and $g$ are homotopic relative to $\dot{I}$, written $f \simeq g$ rel $\dot{I}$. That is, there exists a continuous function $F: I \times I \to X$ such that $F(0,t) = f(t)$, $F(1,t) = g(t)$, $F(x,0) = a$, $F(x,1) = a$ for all $x,t \in I$. $R$ is easily shown to be an equivalence relation, [2, p. 6]. We write $R_f \equiv \{g \in (X,a)^{(I,\dot{I})} \mid gRf\}$.

5. When we speak of a topology on $(Y,B)^{(X,A)}$ we use the "compact-open" topology. The compact-open topology has subbasic sets of the form $(C,U)$ where $C \subset X$ is compact, $U \subset Y$ is open, and $(C,U) \equiv \{f \in (Y,B)^{(X,A)} \mid f(C) \subset U\}$.

6. $\Omega(X,a) \equiv (X,a)^{(I,\dot{I})}$ with the compact-open topology is called the loop space of $X$ at $a$.

7. The fundamental group of a topological space X based at a point $a\varepsilon X$ is $\{R_f \mid f\varepsilon\Omega(X,a)\}$. We write $\pi(X,a)$ for the fundamental group of X based at a. Multiplication and inverse operations are defined as follows:

i. $R_f^{-1} = R_g$ where $g(t)=f(1-t)$

ii. $R_f \cdot R_g = R_{f\cdot g}$ where

$$f\cdot g(t) = \begin{cases} g(2t) & 0 \leqq t \leqq 1/2 \\ f(2t-1) & 1/2 \leqq t \leqq 1 \end{cases}$$

The proof that $\pi(X,a)$ is a group can be found in any text on algebraic topology, [2, pp. 6,7].

8. We say a space U is simply connected if it is path connected and if every $f:(I,\dot{I})\to(U,a)$, where $a\varepsilon U$, is homotopic to the constant map from I to a, relative to $\dot{I}$.

The set $\Omega(X,a)/R = \pi(X,a)$ may be given the identification topology by the map $p:\Omega(X,a)\to\Omega(X.a)/R$ defined by $p(f)=R_f$. In many cases this topology is the trivial or discrete topology in which every point is an open set. We wish to determine necessary conditions and sufficient conditions that this topology on the fundamental group be discrete. We make use of the following relationship between homotopy equivalence classes and the path components in $\Omega(X,a)$.

RESULT 1: The equivalence classes under homotopy rel $\dot{I}$ are the path components of $\Omega(X,a)$.

PROOF: Assume $f,g\varepsilon R_f$. Then $f\simeq g$ rel $\dot{I}$. Thus there exists a continuous $\Phi:I\times I\to X$ with $\Phi(0,t)=f(t)$, $\Phi(1,t)=g(t)$, $\Phi(t,0)=\Phi(t,1)=a$. Define $\hat{\Phi}:I\to(X,a)^{(I,\dot{I})}\equiv\Omega(X,a)$ by

$\hat{\Phi}(t)(t')=\Phi(t,t')$. Then $\hat{\Phi}(0)(t')=f(t')$, $\hat{\Phi}(1)(t')=g(t')$ and $\hat{\Phi}$ is continuous, [1, ch.XII,thm. 3.1(1)]. Thus $\hat{\Phi}$ is a path from f to g in $\Omega(X,a)$, and f and g are both in the same path component.

Assume f and g are in the same path component. Then there exists continuous $\hat{F}:I\to(X,a)^{(I,\dot{I})}$ with $\hat{F}(0)(t')=f(t')$ and $\hat{F}(1)(t')=g(t')$. Define $F(t,t')=\hat{F}(t)(t')$. F is continuous, [1, ch.XII, thm. 3.1(1)]. $F(0,t')=f(t')$ and $F(1,t')=g(t')$ by construction. Since $\hat{F}$ maps into $(X,a)^{(I,\dot{I})}$ we have $F(t,0)=F(t,1)=a$. Thus $F:f\simeq g$ rel $\dot{I}$, QED.

RESULT 2: The topology on $\Omega(X,a)/R$ is discrete if and only if the path components of $\Omega(X,a)$ are open.

PROOF: Let the topology on $\Omega(X,a)/R$ be the discrete topology. Let $p:\Omega(X,a)\to\Omega(X,a)/R$ be the identification map described above. Let P(f) be the path component of f in $\Omega(X,a)$. By result 1 P(f) is also the homotopy equivalence class of f. Thus P(f) is mapped by p to a single point in $\Omega(X,a)/R$, namely $R_f$. Since $\Omega(X,a)/R$ has the discrete topology $R_f$ is open. Thus $p^{-1}(R_f)$ is open in $\Omega(X,a)$. But $p^{-1}(R_f)$ is exactly the homotopy equivalence class of f which is exactly the path component P(f). Thus P(f) is open.

Assume the path components of $\Omega(X,a)$ are open. Let $x\in\Omega(X,a)/R$. $p^{-1}(x)$ is a path component in $\Omega(X,a)$ and thus open. Therefore x is open in $\Omega(X,a)/R$ so $\Omega(X,a)/R$ must be discrete, QED.

To simplify our work with open sets in the compact-open topology on $\Omega(X,a)$ we state without proof the following result. Its proof follows easily from [1, ch. XII, thm. 5.1b].

RESULT 3: Let $\mathcal{F}=\{C \mid C$ is a closed interval in $I\}$. Then the family $\{(C,U) \mid C\varepsilon\mathcal{F}$ and $U$ is open in $X\}$ is a subbasis for the compact-open topology on $\Omega(X,a)$.

B. Some Examples

The first example is the unit circle, $S^1$, centered at the origin in the complex plane, with the subspace topology. It is well known that the fundamental group of the unit circle is isomorphic to the group of integers, [2, pp. 13-16]. The element $n \epsilon \pi(S^1,1)$ is the equivalence class of paths whose net winding number about the origin is "n" in a direction determined by the sign of n. The topology on $\Omega(S^1,1)/R$ is discrete. To show this, we use result 2 and the following theorem.

RESULT 4: The path components of the loop space of the unit circle are open sets.

PROOF: Let $F \epsilon \Omega(S^1,1)$ and $P(F)$ be the path component of F. F may be expressed by $\exp(if(x))$ where f is a continuous real valued function with $\exp(if(0))=\exp(if(1))=1$ and $f(0)=0$. Then $f(1)=2\pi n$ for some $n \epsilon Z$ ( the integers ).

Let $h: I \to S^1$ be defined by $h(t)=\exp(2\pi i t)$. In $S^1$ define $A=h((1/3 - .01, 2/3 + .01))$, $B=h((2/3 - .01, 1] \cup [0, .01))$, and $C=h((.99, 1] \cup [0, 1/3+.01))$. Then A, B, and C are open in $S^1$.
Define:

$S \equiv (F^{-1} \cdot h([0,1/3]), C) \cap (F^{-1} \cdot h([1/3,2/3]), A) \cap (F^{-1} \cdot h([2/3,1]), B)$.

Clearly S is open in $\Omega(S^1,1)$ and $F \epsilon S$. It remains to be shown that $S \subset P(F)$.

Let $G \epsilon S$. Then G may be expressed by $\exp(ig(x))$ with g a continuous real valued function, and $g(0)=0$, $g(1)=2\pi m$ for some $m \epsilon Z$. Define $\Phi: I \times I \to S$ by $\Phi(t,x)=\exp(i(tg(x)+(1-t)f(x)))$.

Then $\Phi(t,0)=1$, $\Phi(0,x)=F(x)$, $\Phi(1,x)=G(x)$, and $\Phi$ is continuous. Note that $\Phi(t,1)=\exp(i(2\pi nt+2\pi m-2\pi mt))$, so if $m=n$, then $\Phi$ is a homotopy rel $\dot{I}$ from F to G and $G\varepsilon P(F)$.

Assume $m\neq n$. Define $\Psi(x)=f(x)-g(x)$. $\Psi$ is continuous. $\Psi(0)=f(0)-g(0)=0$ and $\Psi(1)=2\pi(m-n)\neq 0$. Assume $m-n>0$. By the intermediate value theorem there exists a point $x'\varepsilon I$ such that $\Psi(x')=\pi$. Then $f(x')-g(x')=\pi$. But this is not possible since $F,G\varepsilon S$ and $|f(x)-g(x)|<(1/3 + .02)2\pi$. Thus $m=n$ and F and G are in the same path component. Therefore $F\varepsilon S\subset P(F)$ and P(F) is open, QED.

A second example is commonly referred to as the "Hawaiian Earring" space. For every $n=1,2,...$ construct a circle of radius $1/n$ tangent to the Y axis at the origin, 0, in the right half of the Euclidean plane. The Hawaiian earring, H, takes the subspace topology from the Euclidean plane. While the unit circle will allow a path to wind only a finite number of times, H will allow, for example, a path which loops around each circle in succession. The topology on $\Omega(H,0)/R$ is not discrete, as the following result shows.

RESULT 5: The path components of $\Omega(H,0)$ are not open.

PROOF: Assume the path components of $\Omega(H,0)$ are open. Let $f\varepsilon\Omega(H,0)$. There exists a basic open set, S, about f in P(f). That is, $f\varepsilon S\subset P(f)$. S is a finite intersection of n sets of the form $(C_i,U_i)$, where $C_i$ is a closed interval in I (by result 3) and $U_i$ is open in H, for all $i=1,2,...,n$. Note that if the $C_i$'s do not cover I, then S will not be a subset of P(f). This assures us that 0 is in at least one $U_i$ and possibly several. The intersection of these open sets contains 0 and all but a finite number of loops in H. Let $U=\bigcap\{U_i \mid 0\varepsilon U_i\}$. Note that a set of the form [0,a] is in $\bigcap\{C_i \mid 0\varepsilon U_i\}$. Let $g:(I,\dot{I})\to(U,0)$ be a path which travels once about some loop in U. Define $f_1$, $f_2$ by $f_1(t)=g(2t/a)$, $f_2(t)=g(a/2 - 2t/a)$ on $[0,a/2]$ and $f_1(t)=f_2(t)=f(2(t-a/2))$ on $[a/2,a]$, and $f_1(t)=f_2(t)=f(t)$ on $[a,1]$. Since $f_1([0,a])\subset U$ and $f_2([0,a])\subset U$, $f_1$ and $f_2$ are in S. But clearly they are not homotopic. This is a contradiction since by assumption $S\subset P(f)$. Thus the path components of $\Omega(H,0)$ are not open, QED.

There exist many other examples of spaces whose fundamental group topologies are not discrete. Two simple cases follow. Proofs that their fundamental group topologies are not discrete would be similar to that of result 5.

a. The real plane less points with both coordinates rational, with the subspace topology from the real plane.

b. The real plane less points with both coordinates irrational, with the subspace topology from the real plane.

## C. Sufficient Conditions that the Fundamental Group Topology be Discrete

In searching for sufficient conditions for the fundamental group topology to be discrete, we must, for an arbitrary function f, construct an open set $S \subset \Omega(X,a)$ such that $f \varepsilon S \subset P(f)$. We will impose conditions on X, construct S, and then show that any $g \varepsilon S$ is homotopic to f relative to $\dot{I}$. The construction of this homotopy requires a lemma which depends on the next result, stated here without proof, [2, p. 12].

RESULT 6: If X is a path connected topological space, the following are equivalent:

a. X is simply connected;

b. Every $f:S^1 \to X$ extends to $E^2$ (the closed unit ball).

RESULT 7: (Lemma) Let $f,g:I \to U$ where U is simply connected, $p_0:I \to U$ is a path from g(0) to f(0) and $p_1:I \to U$ is a path from g(1) to f(1). There exists a homotopy $H:g \simeq f$ such that $H(s,0)=p_0(s)$, $H(s,1)=p_1(s)$.

PROOF: Define $H:Fr(I \times I) \to U$ (Fr indicates the boundary) by:

$$H(s,0)=p_0(s)$$
$$H(1,t)=f(t)$$
$$H(s,1)=p_1(s)$$
$$H(0,t)=g(t).$$

There exists a homeomorphism $G:E^2 \cong I \times I$ with $G|S^1:S^1 \cong Fr(I \times I)$.

The map $H \cdot G: S^1 \to U$ extends to $R: E^2 \to U$ by result 6 since U is simply connected. Thus $R \cdot G^{-1}: I \times I \to U$ is continuous and $R \cdot G^{-1} | Fr(I \times I) = H$. Thus $R \cdot G^{-1}: g \simeq f$ with $R \cdot G^{-1}(s,0) = p_0(s)$ and $R \cdot G^{-1}(s,1) = p_1(s)$, QED.

We now establish the first sufficient condition.

RESULT 8: Let X be a path connected topological space and $a \varepsilon X$. Let X have a cover of open sets $\{U_\alpha | \alpha \varepsilon A\}$ such that:

a. Each $U_\alpha$ is simply connected;

b. For each $(\alpha, \beta) \varepsilon A \times A$, $U_\alpha \cap U_\beta$ is path connected.

Then the path components of $\Omega(X,a)$ are open.

PROOF: Let $f \varepsilon \Omega(X,a)$. For each $x \varepsilon f(I)$ there is an open simply connected neighborhood $U_x$. Then $f^{-1}(U_x)$ is open in I and equal to a union of intervals open in I. That is, $f^{-1}(U_x) = \cap \{I_\alpha | \alpha \varepsilon A_x\}$ where $A_x$ is an indexing set and each $I_\alpha$ is an interval open in I. The union over all $x \varepsilon f(I)$ and all $\alpha \varepsilon A_x$ of $I_\alpha$ is an open cover for I. Since I is compact there is a finite subcover. In fact, there exists a minimal subcover in the sense that no interval is a subset of any other interval. Rename these intervals so that $I_0 = [0, b_1)$, $I_m = (a_m, b_{m+1})$ for $0 < m < n$, and $I_n = (a_n, 1]$ where $0 < a_1 < b_1 < \ldots < a_n < b_n < 1$, and $f(I_i) \subset U_i$. For each $i = 1, \ldots, n$ pick $c_i$ so that $a_i < c_i < b_i$ and let $c_0 = 0$ and $c_{n+1} = 1$. Define $V_i = U_{i-1} \cap U_i$ for $i = 1, \ldots, n$ and note that each $V_i$ is open and path connected. Define an open set $S \subset \Omega(X,a)$ as the intersection over $i = 0, \ldots, n$ of the sets $([c_i, c_{i+1}], U_i)$. By construction $f \varepsilon S$ and S is open. It remains to be shown that $S \subset P(f)$.

Let $g\varepsilon S$. For all $i=1,\ldots,n-1$ $g(c_i), f(c_i)\varepsilon V_i\subset U_i$ and $g(c_{i+1}), f(c_{i+1})\varepsilon V_{i+1}\subset U_i$. Since for all i, $V_i$ is path connected, there exists a path $p_i: I\to V_i\subset U_i$ such that $p_i(0)=g(c_i)$ and $p_i(1)=f(c_i)$. The lemma guarantees there exists a homotopy $H_i: I\times[c_i, c_{i+1}]\to U_i$ with $H_i(t,c_i)=p_i(t)$, $H_i(t,c_{i+1})=p_{i+1}(t)$. For $[c_0,c_1]$, $f(c_0)=g(c_0)=a$ so we define $p_0(t)=a$ for all t. Similarly for $[c_n,c_{n+1}]$, $f(c_{n+1})=g(c_{n+1})=a$ so we define $p_{n+1}(t)=a$ for all t. Using the same lemma homotopies $H_0$ and $H_n$ are produced as above. Thus on each $[c_i,c_{i+1}]$ we have $H_i: f|[c_i,c_{i+1}]\simeq g|[c_i,c_{i+1}]$ and $H_i$ and $H_{i+1}$ agree on $I\times\{c_{i+1}\}$. Define $H=H_i$ on $I\times[c_i,c_{i+1}]$ for each i. H is well defined and continuous. Thus $H: f\simeq g$ rel $\dot{I}$. Therefore $S\subset P(f)$, so P(f) must be open, QED.

We have a second sufficient condition.

RESULT 9: Let X be a path connected topological space and $a\varepsilon X$. If X has a basis of simply connected sets, then the path components of $\Omega(X,a)$ are open.

PROOF: Let $f\varepsilon\Omega(X,a)$. Each $x\varepsilon f(I)$ has a simply connected neighborhood $U_x$. We proceed as in the proof of result 8 to define the closed sets in I, $[c_i,c_{i+1}]$, and the open sets in X, $U_i$. We have $f(c_i)\varepsilon U_i\cap U_{i+1}$, an open set. Thus there exists a simply connected open set $V_{i+1}$ such that $f(c_i)\varepsilon V_{i+1}\subset U_i\cap U_{i+1}$. The open set S in $\Omega(X,a)$ is defined to be the intersection over $i=0,\ldots,n$ of $([c_i,c_{i+1}],U_i)$ and over $i=1,\ldots,n$ of $(\{c_i\},V_i)$. By construction S is open and $f\varepsilon S$. Again we proceed as in the proof of result 8 to show $S\subset P(f)$. Thus P(f) is open, QED.

These two results provide sufficient conditions that the fundamental group topology be discrete. Either may be used to show that the topology on $\Omega(S^1,1)/R$ is discrete.

## D. Necessary Conditions that $\Omega(X,a)/R$ be Discrete

An examination of the examples of spaces with non-discrete fundamental group topologies indicates that such spaces possess a point or points that have no simply connected open neighborhoods. This intuitive idea is made more concrete by the following definition.

DEFINITION: If X is a topological space and $a \varepsilon B \subset X$, then B is said to have "property A" with respect to a if any continuous $f:(S^1,1) \to (B,a)$ can be extended to $f^*:E^2 \to X$. ($E^2$ is the closed unit disc.)

This leads to the following result.

RESLUT 10: Let X be a path connected topological space and let the path components of $\Omega(X,a)$ be open. Then each $x \varepsilon X$ has a neighborhood that has property A with respect to x.

PROOF: Assume not. Assume $x \varepsilon X$ has no neighborhood that has property A. Since X is path connected there is a path from a to x, call it $\sigma'$. Let $\sigma = \sigma' \cdot (\sigma'^{-1})$. We have $\sigma(t) = \sigma(1-t)$, or specifically $\sigma(0) = \sigma(1) = a$, and $\sigma(1/2) = x$. By hypothesis $P(\sigma)$ is open in $\Omega(X,a)$, so there exists an open set S such that $\sigma \varepsilon S \subset P(\sigma)$. S may be taken to be a finite intersection of n sets of the form $(C_i, U_i)$, where we may assume (by result 3) that each $C_i$ is a closed interval in I. Then $U = \bigcap \{U_i \mid i=1,\ldots,n \text{ and } x \varepsilon U_i\}$ is open. U cannot have property A with respect to x. That is, there exists a map $\tau:(S^1,1) \to (U,x)$ which does not extend to $E^2$. We define $h:I \to S^1$ by $h(t) = \exp(2\pi i t)$. Then

$\tau\cdot h:(I,\dot{I})\to(U,x)$. Let $J=\{i|x\varepsilon U_i\}$. Choose $c<a<1/2<b<d$ so that:

i. $[c,d]\cap(\cap\{C_i\,|\,i\notin J\})=\emptyset$

ii. $[c,d]\subset\sigma^{-1}(\cap\{U_i\,|\,i\varepsilon J\})=\sigma^{-1}(U)$.

We define a map, $\rho$, by $\rho=\sigma$ on $I-(c,d)$, $\rho$ maps on $[c,a]$ as $\sigma$ maps on $[c,1/2]$, $\rho$ maps on $[a,b]$ as $\tau\cdot h$ maps on $I$, and $\rho$ maps on $[b,d]$ as $\sigma$ maps on $[1/2,d]$. More precisely:

$$\rho(t) = \begin{cases} \sigma(t) & 0\leqq t\leqq c \\ \sigma(c+(t-c)(c-1/2)/(c-a)) & c\leqq t\leqq a \\ \tau\cdot h((t-a)/(b-a)) & a\leqq t\leqq b \\ \sigma(1/2 + (t-b)(d-1/2)/(d-b)) & b\leqq t\leqq 1 \\ \sigma(t) & \end{cases}$$

Define another map $\rho':(I,\dot{I})\to(X,a)$ by $\rho'(t)=x$ if $t\varepsilon[a,b]$ and $\rho'(t)=\rho(t)$ if $t\varepsilon I-[a,b]$. $\rho$ and $\rho'$ are both in S since they both map $[c,d]$ inside U and agree with $\sigma$ on $I-[c,d]$. But they are not homotopic. To see this assume $\rho\simeq\rho'$ rel $\dot{I}$. Note that $R_{\tau\cdot h}$ and $R_{1_x}$ (where $1_x$ is the constant map to x) are elements of $\pi(X,x)$. But $R_\rho=R_{\sigma'^{-1}\cdot\tau\cdot h\cdot\sigma'}$ and $R_{\rho'}=R_{\sigma'^{-1}\cdot 1_x\cdot\sigma'}$ are elements of $\pi(X,a)$. $\pi(X,a)$ and $\pi(X,x)$ are isomorphic and that isomorphism maps $R_\rho$ to $R_{\tau\cdot h}$ and $R_{\rho'}$ to $R_{1_x}$, [2, pp. 7,8]. Thus $\tau\cdot h\simeq 1_x$ rel $\dot{I}$. Assume $H:\tau\cdot h\simeq 1_x$. Let $J:I\times I\to E^2$ be defined by $J(s,t)=(1-s)\exp(2\pi it)$. J is an identification, H is continuous and constant on fibers, so $H\cdot J^{-1}:E^2\to U$ is continuous. And $H\cdot J^{-1}|S^1=\tau$. Thus $H\cdot J^{-1}$ is an extension of $\tau$ from $S^1$ to $E^2$. But by hypothesis such an extension doesn't exist. Thus $\rho$ and $\rho'$ cannot be homotopic and S is not a subset of $P(\sigma)$. This contradiction proves our result.

E. Further Properties of $\Omega(X,a)/R$

To this point we have shown that there are some spaces X for which $\Omega(X,a)/R$ is an interesting space and we have given sufficient conditions and a necessary condition that this space be interesting; that is, not discrete. We can borrow familiar results from topology and algebraic topology to show some properties of $\Omega(X,a)/R$.

RESULT 11: Let $h:(X,a)\to(Y,b)$ be a continuous function. Define $\tilde{h}:\Omega(X,a)\to\Omega(Y,b)$ by $\tilde{h}(f)=h\cdot f$. Then $\tilde{h}$ is continuous.

PROOF: It is sufficient to show that the inverse image under $\tilde{h}$ of a subbasic set in $\Omega(Y,b)$ is open in $\Omega(X,a)$, [1, chap. III, thm 8.3(3)]. A subbasic set in $\Omega(Y,b)$ has the form $(A,V)$ where $A\subset I$ is compact and $V\subset Y$ is open. Thus

$$\begin{aligned}h^{-1}(A,V) &= \{g\varepsilon\Omega(X,a)\,|\,\tilde{h}(g)(A)\subset V\}\\ &= \{g\varepsilon\Omega(X,a)\,|\,h\cdot g(A)\subset V\}.\end{aligned}$$

But $h\cdot g(A)\subset V$ and $g(A)\subset h^{-1}(V)$ are equivalent, so we have

$$\begin{aligned}h^{-1}(A,V) &= \{g\varepsilon\Omega(X,a)\,|\,g(A)\subset h^{-1}(V)\}\\ &= (A,h^{-1}(V)).\end{aligned}$$

Since A is compact, and h is continuous, $h^{-1}(V)$ is open, so $\tilde{h}^{-1}(A,V)$ is open in $\Omega(X,a)$. Thus $\tilde{h}$ is continuous, QED.

RESULT 12: Let h be as in result 11. Let R and S be equivalence relations on $\Omega(X,a)$ and $\Omega(Y,b)$ respectively defined as homotopic equivalence rel $\dot{I}$. Then $\tilde{h}$, defined as in result 11, is a relation preserving map.

PROOF: Let $f,g\varepsilon\Omega(X,a)$ with $fRg$. Then there exists $F:I\times I\to X$ such that $F(0,x)=g(x)$, $F(1,x)=f(x)$, and $F$ is continuous. Consider $\tilde{h}(f)$, $\tilde{h}(g)$. We have $\tilde{h}(f)=h\cdot f$, $\tilde{h}(g)=h\cdot g$. Let $H=h\cdot F$. Then $H:I\times I\to Y$ is continuous and $H(0,x)=h\cdot g(x)=\tilde{h}(g)(x)$, $H(1,x)=h\ f(x)=\tilde{h}(f)(x)$. Thus $\tilde{h}(f)S\tilde{h}(g)$. Therefore $\tilde{h}$ is a relation preserving map, QED.

RESULT 13: $\tilde{h}:\Omega(X,a)/R\to\Omega(Y,b)/S$, defined as the map induced by passing $h:\Omega(X,a)\to\Omega(Y,b)$ to the quotient, is continuous.

PROOF: This proof follows easily from the previous results and [1, p. 126, thm. 4.3].

We have on our set $\Omega(X,a)/R$ now, both a group structure and a topological structure. It is natural to ask if $\Omega(X,a)/R$ is a topological group. It remains only to show that inverse taking and multiplication are continuous functions. However an attempt to show that multiplication is continuous runs into problems. We hope that such a proof might be found and that $\Omega(X,a)/R$ is indeed a topological group, but no proof has been found.

## F. Conclusion

We have exhibited a natural way to impose a topology on the fundamental group. We have shown by example and by results that spaces exist for which this topology in non-trivial. We have shown some elementary properties of this topology.

Further investigations might determine a single necessary and sufficient condition on X for $\Omega(X,a)/R$ to be non-trivial. Also it might be proved that $\Omega(X,a)/R$ is a topological group, perhaps by imposing restrictions on X. We hope that this construction might be linked to existing results in algebraic topology; perhaps to simplify the computation of complicated fundamental groups.

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