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WC spaces: Between T[subscript 1] and T[subscript 2].

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WC SPACES: BETWEEN T_1 AND T_2

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CHAPTER ONE: INTRODUCTION

In this paper a separation property is introduced which is a consequence of the Hausdorff separation axiom T_2, but not conversely. This property is characterized by a convergence property of neighborhood filterbases of points in a topological space which is defined and shown to be intermediate to standard convergence and accumulation of neighborhood filterbases.

In Chapter Two it is shown that spaces that have the separation property are T_1 spaces but not necessarily Hausdorff. Several properties of these spaces are examined in Chapter Three. Chapter Four deals with the relationship between these intermediate spaces and other intermediate spaces found in the literature. In Chapter Five topics for further study are posed.

The following standard definitions will be of value.

Separation Axioms

DEFINITION 1.1: A topological space is called a T_1 space if, given any two points in the space, there is an open neighborhood of each which does not contain the other.

DEFINITION 1.2: A topological space is called a T_2 or Hausdorff space if any two distinct points have disjoint open neighborhoods.

Filterbases and Convergence

DEFINITION 1.3: (1) Let Y be a topological space. A filterbase $\mathcal{A}$ in Y is a family, $\mathcal{A} = \{A_\alpha \mid \alpha \in A\}$, of subsets of Y having the two properties:

(a) for each $\alpha \in A$: $A_\alpha \neq \phi$

(b) for each α, for each β, there exists a γ: $A_\gamma \subset (A_\alpha \cap A_\beta)$

(2) Let Y be a topological space and $y_\circ \varepsilon Y$. The family $\mathbb{A}(y_\circ)$ of all open neighborhoods of $y_\circ$ is called the neighborhood filterbase of $y_\circ$.

DEFINITION 1.4: Let $\mathbb{A}=\{A_\alpha | \alpha \varepsilon A\}$ be a filterbase in the topological space Y and $y_\circ \varepsilon Y$. Then: (1) $\mathbb{A}$ converges to $y_\circ$, written $\mathbb{A} \rightarrow y_\circ$, if for any neighborhood $U(y_\circ)$ there exists a $A_\alpha \varepsilon \mathbb{A}$ such that $A_\alpha \subset U(y_\circ)$.

(2) $\mathbb{A}$ accumulates at $y_\circ$, written $\mathbb{A} \succ y_\circ$, if for any neighborhood $U(y_\circ)$, for each $A_\alpha \varepsilon \mathbb{A}$: $A_\alpha \cap U(y_\circ) \neq \phi$.

DEFINITION 1.5: Let $\mathbb{A}=\{A_\alpha | \alpha \varepsilon A\}$ and $\mathbb{B}=\{B_\beta | \beta \varepsilon B\}$ be two filterbases in Y. $\mathbb{B}$ is subordinate to $\mathbb{A}$, written $\mathbb{B} \vdash \mathbb{A}$, if for each $A_\alpha \varepsilon \mathbb{A}$ there exists a $B_\beta \varepsilon \mathbb{B}$: $B_\beta \subset A_\alpha$.

NOTATION: In this paper $\mathbb{R}$ and $\mathbb{R}^+$ will denote the real line and the subset $\{y \varepsilon \mathbb{R} \mid y>0\}$ respectively. Neighborhood will mean open neighborhood and will be abbreviated nbd.

The following characterizations of T_1 and Hausdorff spaces in terms of convergence properties of nbd filterbases in the spaces will be useful later.

An easy consequence of well-known results [1, p. 213, 214] is: A space is Hausdorff iff each nbd filterbase in the space accumulates to a unique point.

Before stating the characterization for T_1 spaces, the following trivial lemma, without proof, is necessary.

LEMMA 1.1: For space Y, let $y_\circ \varepsilon Y$ and let $\mathbb{A}(y_\circ)$ be the nbd filterbase of $y_\circ$. Then $\mathbb{A} \rightarrow y_\circ$ and $\mathbb{A} \succ y_\circ$.

THEOREM 1.1: A space Y is T_1 iff each nbd filterbase in Y converges to a unique point.

Proof. Suppose Y is not T_1. Then for some pair of distinct points $y, y' \varepsilon Y$ and any nbd U of y, $y' \varepsilon U$. If V be any nbd of y', $U \cap V$ is a nbd of y' and hence $U \cap V \varepsilon \mathbb{A}(y')$. By Lemma 1.1 $\mathbb{A}(y') \rightarrow y'$ and since $U \cap V \subset U$, $\mathbb{A}(y') \rightarrow y$ also.

Conversely, suppose for some pair of distinct points $y, y' \varepsilon Y$, the nbd filterbase $\mathbb{A}(y') \rightarrow y$. Then for any nbd U of y there exist some $A_\alpha \varepsilon \mathbb{A}(y')$ such that $A_\alpha \subset U$. Since $y' \varepsilon A_\alpha$ this implies that for any nbd $U(y)$, $y' \varepsilon U(y)$. Thus Y is not T_1.

Next is defined a nbd filterbase convergence property intermediate to convergence and accumulation.

DEFINITION 1.6: Let Y be a space and let $y, y' \varepsilon Y$ be distinct points. The nbd filterbase $\mathbb{A}(y')$ weakly converges to y, written $\mathbb{A} \rightsquigarrow y$, if for any nbd $U(y)$ there exist some $A_\alpha \varepsilon \mathbb{A}(y')$ such that $A_\alpha \subset \overline{U}$.

The following theorem shows that weak convergence is an intermediate property.

THEOREM 1.2: For the nbd filterbase $\mathbb{A}(y')$, $\mathbb{A} \longrightarrow y$ implies $\mathbb{A} \rightsquigarrow y$ implies $\mathbb{A} \succ y$, but no converse implication holds.

Proof. Suppose the nbd filberbase $\mathbb{A}(y')$ converges to y. Then for any nbd $U(y)$ there exist some $A_\alpha \varepsilon \mathbb{A}(y')$ such that $A_\alpha \subset U$. Thus $A_\alpha \subset \overline{U}$ and $\mathbb{A}$ weakly converges to y.

Suppose $\mathbb{A}(y')$ weakly converges to y. Then for any nbd $U(y)$ there exists a $A_\alpha \varepsilon \mathbb{A}(y')$ such that $A_\alpha \subset \overline{U}$. Since $y' \varepsilon A_\alpha$, then $y' \varepsilon \overline{U}$ and for every nbd of y', that is each $A_\alpha \varepsilon \mathbb{A}(y')$, $A_\alpha \cap U \neq \phi$, so that $\mathbb{A}(y')$ accumulates at y.

That the implications are not reversible will be shown later in spaces that are counterexamples to the converses.

CHAPTER TWO: $\mathcal{WC}$ SPACES: BETWEEN T_1 AND T_2

In this chapter weak convergence is used to define the $\mathcal{WC}$ separation property which is between the T_1 and T_2 separation axioms. Several equivalent characterizations of the $\mathcal{WC}$ property are given and spaces satisfying this intermediate separation property are shown to be T_1 spaces but not necessarily T_2 spaces.

DEFINITION 2.1: A space Y has property $\mathcal{WC}$ if every nbd filterbase in Y weakly converges to a unique point.

Spaces with the $\mathcal{WC}$ property can be characterized by other topological properties as shown in the following theorems.

THEOREM 2.1: A space Y is $\mathcal{WC}$ iff for any pair of distinct points $y,y' \varepsilon Y$, there exists a nbd $U(y)$, such that no nbd of y' is contained in $\overline{U}$.

Proof. Trivial

THEOREM 2.2: A space Y is $\mathcal{WC}$ iff for any pair of distinct points $y,y' \varepsilon Y$, there exists a nbd $U(y)$, such that every nbd of y' has a point not in $\overline{U}$.

Proof. Trivial

THEOREM 2.3: A space Y is $\mathcal{WC}$ iff for any pair of distinct points $y,y' \varepsilon Y$, there exists a nbd $U(y)$, such that for any nbd $V(y')$, there exists a $y'' \varepsilon V(y')$ such that for some nbd $W(y'')$, $W \cap U = \phi$.

Proof. This follows from Theorem 2.2

THEOREM 2.4: A space Y is $\mathcal{WC}$ iff for any pair of distinct points $y,y' \varepsilon Y$, there exists a nbd $U(y)$, such that for any nbd $V(y')$, $U \cap V$ is not dense in V.

Proof.

If Y is WC then for any distinct points y and y' in Y there exists a nbd $U(y)$ such that for each nbd $V(y')$, $V \not\subset \overline{U}$. Let $\overline{V\cap U}_V$ be the closure of $V\cap U$ in V. Then $\overline{V\cap U}_V = V\cap\overline{V\cap U} \subset \overline{V\cap U} \subset \overline{V}\cap\overline{U} \subset \overline{U} \neq V$.

Conversely, suppose the condition is true. Then for any distinct y and y' there exists a nbd $U(y)$ such that for each nbd $V(y')$, $V \neq \overline{V\cap U}_V = V\cap\overline{V\cap U}$. Thus, there exists a $y''\in V$ such that $y''\notin\overline{U}$ and hence $V\not\subset\overline{U}$.

THEOREM 2.5: A space is WC iff for any pair of distinct points $y, y'\in Y$, there exists a nbd $U(y)$, such that $y'\in\overline{\mathcal{C}\overline{U}}$.

Proof. $y'\in\overline{\mathcal{C}\overline{U}}$ iff every nbd of y' nontrivially intersects $\mathcal{C}\overline{U}$.

The following theorem characterizes spaces which are strictly WC, that is WC but not Hausdorff.

THEOREM 2.6: A space Y is strictly WC iff Y is WC and there exists a pair of distinct points $y, y'\in Y$ such that:

a) for any nbd V of $y, y'\in\overline{V}$, and

b) there exists a nbd U of y: $y'\in Fr\overline{U}$.

Proof. If Y is strictly WC then there exists a pair of distinct points $y, y'\in Y$ such that for any nbd V of y and for any nbd W of y', $W\cap V\neq\phi$. Thus $y'\in\overline{V}$. By Theorem 2.5 there exists a nbd U of y such that $y'\in\overline{\mathcal{C}\overline{U}}$. Thus for this nbd $y'\in\overline{\mathcal{C}\overline{U}}\cap\overline{U} = Fr\overline{U}$.

Conversely, if the condition holds, Y is not T_2 since y and y' do not have disjoint nbds.

The next theorems show that Hausdorff spaces are WC spaces and that WC spaces are T_1 spaces. Counterexamples are given to show that the converses are not true.

THEOREM 2.7: If Y is WC, then Y is T_1, but not conversely.

Proof. Suppose space Y is WC, then every nbd filterbase in Y weakly converges to a unique point. By Lemma 1.1 every nbd filterbase in Y converges to at least one point and by Theorem 1.2 no nbd filterbase can converge without weakly converging. Thus every nbd filterbase in Y converges to a unique point. By Theorem 1.1 Y is T_1.

EXAMPLE 2.7.1: A counterexample to the converse is an infinite set with the cofinite topology in which a subset is open if and only if its complement is finite. In this T_1 space the closure of any nbd is the entire space and thus every nbd filterbase in the space weakly converges to every point of the space.

THEOREM 2.8: If Y is Hausdorff then Y is WC, but not conversely.

Proof. Suppose space Y is Hausdorff. Then every nbd filterbase in Y accumulates to a unique point. By Lemma 1.1 every nbd filterbase in Y weakly converges to at least one point and by Theorem 1.2 no nbd filterbase can weakly converge without accumulating. Thus every nbd filterbase in Y weakly converges to a unique point. By Definition 2.1 Y is WC.

EXAMPLE 2.8.1: For a counterexample to the converse consider this construction.

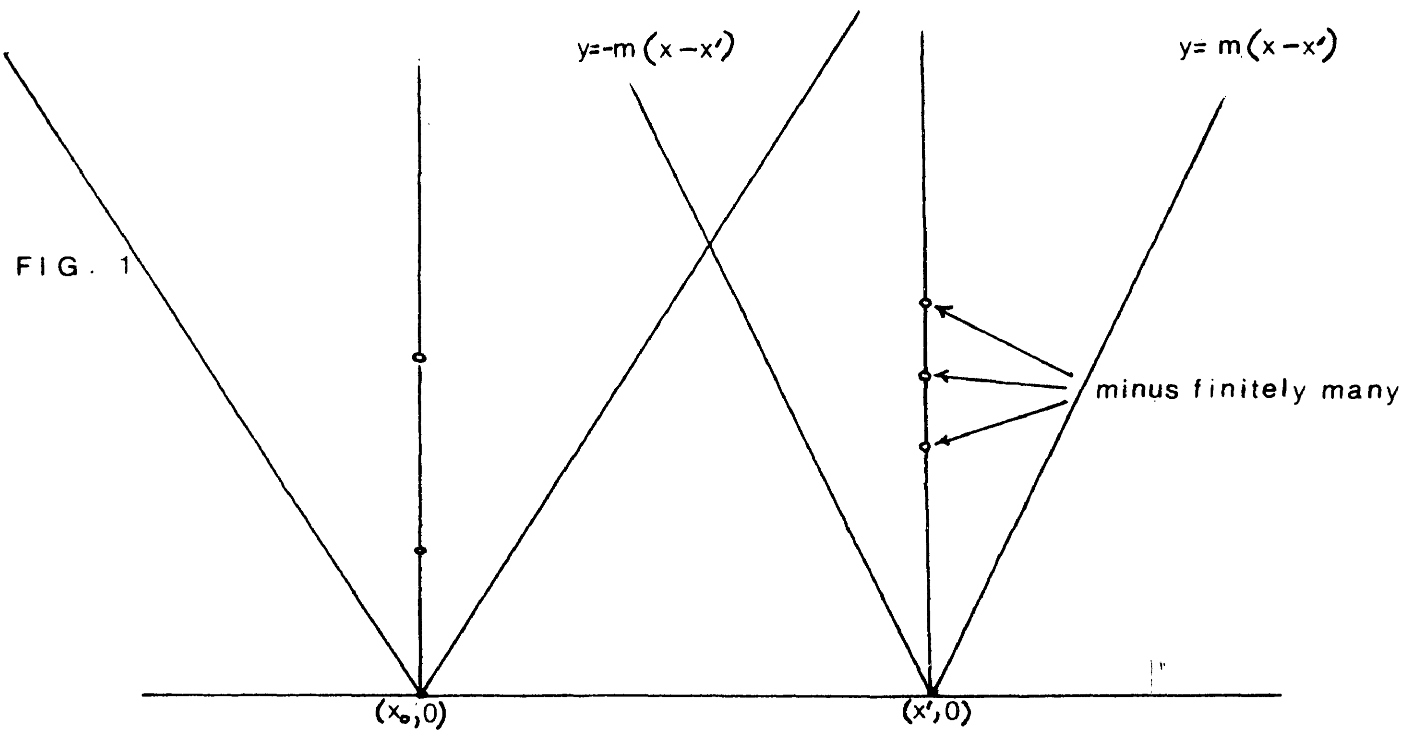
Let set Y be the upper half of the real plane bounded by the x-axis, $Y=\mathbb{R}\times(\mathbb{R}^{+}\cup\{0\})$. For the points $\{(x,y)\in Y \mid y>0\}$ use the discrete topology in which each point is both open and closed. For any point $(x_o,0)\varepsilon Y$ of the x-axis, let basic nbds be described by

$$\{(x,y)\varepsilon Y|y\geq m|x-x_o|\} - \{(x_o,y_1),\cdots,\{(x_o,y_n)\}$$

where $m>0$ and each $y_i>0$.

These are sets in Y in the form of wedges with the apex on the x-axis at $(x_o,0)$ and bisected by the line $\{(x_o,y)|y\varepsilon Y\}$ minus finitely many of the points of the bisecting line. See Figure 1.

FIG. 1



This space is not Hausdorff since for any pair of distinct points of the x-axis, every nbd of one point intersects every nbd of the other point.

To show that the space is WC, if $(x',y')\varepsilon Y$ is a point <u>not</u> on the x-axis, $\{(x',y')\}$ is an open and closed nbd of (x',y') disjoint from any other distinct point. Let $(x_o,0)$ be any point of the x-axis and let U be the basic nbd of $(x_o,0)$ of the form $U=\{(x,y)\varepsilon Y|y\geq\frac{1}{2}|x-x_o|\}$.

If $(x_1,0)$ is any other point of the x-axis, then $(x_1,0)\varepsilon\overline{U}$, but any nbd of $(x_1,0)$ has an open point not on the x-axis not contained in U and hence not in $\overline{U}$. Thus by Theorem 2.2 Y is WC, and also T_1 by Theorem 2.7, but Y is not T_2.

CHAPTER THREE: SOME PROPERTIES OF WC SPACES

In this chapter the subspace and invariance properties of WC spaces are investigated.

THEOREM 3.1: An open subspace of a WC space is WC.

<u>Proof</u>. Let Y be a WC space and let $A \subset Y$ be an open subspace of Y. Suppose A is not WC, then there exists a pair of distinct points $a, a' \varepsilon A$, such that for the rel A nbd filterbase $\mathbb{A}'(a')$, for any rel A nbd $U_A(a)$ there exists a $A'_\alpha \varepsilon \mathbb{A}'$ such that $A'_\alpha \subset \overline{U}_A(a)$. Now since A is open in Y, each set in $\mathbb{A}'(a')$ is open in Y and in fact $\mathbb{A}' \subset \mathbb{A}(a')$, the nbd filterbase of a' in Y. For any nbd $U(a) \subset Y$ there exists a rel A nbd $U_A \subset U$. Thus for any nbd $U(a)$ there exists $A'_\alpha \varepsilon \mathbb{A}' \subset \mathbb{A}(a')$ such that $A'_\alpha \subset \overline{U}_A \subset \overline{U}$. This means that Y is not WC which proves the counterpositive.

Although open subspaces of WC spaces are WC, this is not true in general for non-open subspaces as the following counterexamples show.

EXAMPLE 3.1.1: A closed subspace of a WC space need not be WC. Let set Y be the upper half of the real plane bounded by the x-axis with the topology described in <u>Example 2.8.1</u>. Let $(x_o,0)$ be a fixed point of the x-axis. Let $A \subset Y$ be the subset consisting of the x-axis and the perpendicular line $\{(x_o,y)\varepsilon Y | y>0\}$. A is closed since $\mathcal{C}A$ has the discrete topology and each element of $\mathcal{C}A$ is open. The subspace $A \subset Y$ is not WC. Consider any rel A basic nbd of $(x_o,0)$. These consist of the perpendicular line $\{(x_o,y)\varepsilon Y | y \geqslant 0\}$ minus finitely many points of the subset $\{(x_o,y)\varepsilon Y | y>0\}$. Let $(x',0)$ be any point of the x-axis distinct from $(x_o,0)$. The rel A basic nbds of the point consist of sets of the form $\{(x',0)\} \cup \{(x_o,y)\varepsilon Y | y>q\}$ where $q \varepsilon \mathbb{R}^+$. The closure of any nbd of $(x_o,0)$ consists of the nbd and the x-axis. If (x_o,y_n) is the greatest element of $\{(x_o,y)\varepsilon Y | y>0\}$ not contained in the nbd, then any nbd of $(x',0)$ of the form $\{(x',0)\}\cup\{(x_o,y)\in Y \mid y>y_n\}$ is a nbd contained in the closure of the nbd of $(x_o,0)$. Thus the closed subspace A is not WC.

EXAMPLE 3.1.2: A non-open subspace of a WC space need not be WC. Let Y be the space of the previous example. Let $A\subset Y$ be the subspace consisting of the x-axis minus the fixed point $(x_o,0)$ union with the perpendicular line $\{(x_o,y)\in Y \mid y>0\}$. This subspace is not open since no nbd of any point of the x-axis minus $(x_o,0)$ is contained in A. This subspace is not closed because $(x_o,0)$ is in the closure of A but not in A. This subspace is not WC as can be seen by using any two points of the x-axis.

THEOREM 3.2: The homeomorphic image of a WC space is WC.

<u>Proof</u>. Let X be WC and let $f:X\rightarrow Y$ be a homeomorphism. Let $y,y'\in Y$ be any distinct points. Since X is WC there exists a nbd U of $f^{-1}y$ such that $f^{-1}y'\in\overline{\mathcal{C}\overline{U}}$ by <u>Theorem 2.5</u>. Thus $y'\in f\overline{\mathcal{C}\overline{U}}$. Because f is bijective, continuous and closed, $f\overline{\mathcal{C}\overline{U}}=\overline{\mathcal{C}\overline{fU}}$. So $y'\in\overline{\mathcal{C}\overline{fU}}$ and fU is a nbd of y, and by <u>Theorem 2.5</u> Y is WC.

The continuous bijective image of a WC space need not be WC. Let X be the real line with the discrete topology and Y the real line with the cocountable topology in which a set is open if and only if its complement is countable. The map $f:X\rightarrow Y$, where $f(x)=x$, is a continuous bijection and X is WC, but $f(X)=Y$ is not WC.

Since the open bijective image of a T_1 space is T_1, the open bijective image of a WC space is T_1.

Before concluding this chapter, consider the position of product spaces of WC spaces.

THEOREM 3.3: If $\{Y_\alpha \mid \alpha\in A\}$ is a family of spaces each WC, then the product space $\prod_\alpha Y_\alpha$ is WC.

Proof. Suppose $\prod_\alpha Y_\alpha$ is not WC. Then there exists a pair of distinct points $\{y_\alpha\},\{z_\alpha\}\varepsilon\Pi Y_\alpha$ such that for any nbd U' of $\{y_\alpha\}$ there exists a basic nbd $V'=<V_{\alpha_1},\cdots,V_{\alpha_n}>$ of $\{z_\alpha\}$ contained in $\overline{U'}$.

Since $\{y_\alpha\}$ and $\{z_\alpha\}$ are distinct there is a $\beta\varepsilon A$ such that $y_\beta\neq z_\beta$. Now $<U_\beta>$ is a nbd of $\{y_\alpha\}$ and there exists a basic nbd of $\{z_\alpha\}$ of the form $<V_{\alpha 1},\cdots,V_{\alpha n}>$ contained in $\overline{<U_\beta>}$. The projection $p_\beta:\Pi Y_\alpha\longrightarrow Y_\beta$ is a continuous open surjection and thus $p_\beta<V_{\alpha 1},\cdots,V_{\alpha n}>\subset p_\beta\overline{<U_\beta>}\subset\overline{p_\beta<U_\beta>}$. But this means that for the factor Y_β there exist distinct points $p_\beta\{y_\alpha\},p_\beta\{z_\alpha\}$ in Y_β such that for any nbd $U=p_\beta<U_\beta>$ of $p_\beta\{y_\alpha\}$ there exists a nbd $V=p_\beta<V_{\alpha_1},\cdots,V_{\alpha n}>$ of $p_\beta\{z_\alpha\}$ such that $V\subset\overline{U}$ and hence Y_β is not WC.

Since closed subspaces of WC spaces need not be WC it remains to be proven that if a product space is WC then each factor is WC.

CHAPTER FOUR: *WC* SPACES AND OTHER SPACES BETWEEN T_1 AND T_2

In his paper [2], Wilansky introduced two intermediate separation axioms.

DEFINITION 4.1: A space is *US* if every convergent sequence in the space converges to a unique limit.

DEFINITION 4.2: A space is *KC* if every compact set in the space is closed.

LEMMA 4.1: Hausdorff implies *KC* implies *US* implies T_1, but no converse implication holds. [2]

The following examples show the relationship between these intermediate spaces and *WC* spaces.

EXAMPLE 4.1: A space that is *US* and *KC* but not *WC*.
Let Y be an uncountable set with the cocountable topology. This space is *US* and *KC* [2], but not *WC* since the closure of any non-empty open set is the entire space.

EXAMPLE 4.2: A space that is *WC* but not *KC* or *US*.
Let set Y be the upper half plane bounded by the x-axis with the topology described in <u>Example 2.8.1</u>. This space is *WC* but not *US*, and hence not *KC*. Let $(x_o,0),(x',0)\varepsilon Y$ be any two distinct points of the x-axis. The sequence $n\rightarrow(\frac{1}{2}(x_o+x'),n)$ converges to both $(x_o,0)$ and $(x',0)$.
(In fact, the sequence converges to all the points of the x-axis.)

The author is indebted to Dr. J. Scott Downing for his assistance in the construction of the spaces used in the next three examples.

EXAMPLE 4.3: A space that is *US* but not *KC* or *WC*.
Let set Y be $\{y\varepsilon\mathbb{R}\,|0<y\leq 1\}$. Give the subset $\{y\varepsilon Y|0<y<1\}$ the discrete topology in which each point is both open and closed. Let nbds of 0 be

$\{0\}$ union all but countably many points $y_i \in Y$ such that $0<y_i<1$ and let nbds of 1 be $\{1\}$ union all but finitely many points $y_i \in Y$ such that $0<y_i<1$.

Y is *US*. If a sequence converges to any point $y' \in Y$ such that $0<y'<1$, the sequence must have all but finitely many values equal to y' since $\{y'\}$ is an open nbd of y'. The sequence cannot converge to any other point of Y since each point distinct from y' has a nbd not containing y'. Any sequence converging to 0 must have all but finitely many values equal to 0 since if infinitely many sequence values did not equal 0, there is a nbd of 0 not containing these values. Since each point in Y distinct from 0 has a nbd not containing 0, a sequence converging to 0 cannot converge to any other point. Thus no convergent sequence in Y can converge to more than one point and Y is *US*.

Y is not *KC*. The set $B=\{y \in Y | 0<y\leq 1\}$ is compact since any open cover of B contains a nbd U of 1. This nbd U together with one member of the cover for each of the finitely many points of B not in U is a finite subcover. However, B is not closed since 0 is in $\overline{B}$ but not in B.

Y is not *WC*. Let U be any nbd of 1. $\overline{U}-\{1\}$ is a nbd of 0. Thus the closure of any nbd of 1 contains a nbd of 0 and the nbd filterbase of 0 weakly converges to both 0 and 1.

EXAMPLE 4.4: A space that is *WC* and *US* but not *KC*.
Let set $Y=\mathbb{R}$, the real line. Let each point of Y except 0 and 1 be both open and closed. Let basic nbds of 0 be intervals of the form $(0-\delta,1)$, δ a positive number, minus countably many $y_i \in Y$ such that $0<y_i<1$. Let basic nbds of 1 be intervals of the form $(0,1+\delta)$, δ a positive number, minus finitely many $y_i \in Y$ such that $0<y_i<1$.

This space is *WC*. If $y' \in Y$ is any point not equal to 0 or 1, $\{y'\}$ is an open and closed nbd of y' disjoint from any other point of Y. Thus for any pair of distinct points in Y in which one of the points is not 0 or 1, each point has a nbd whose closure does not contain the other point. There exists a nbd of 0, namely $U=(0-\frac{1}{2},1)$, such that each nbd of 1 fails to be in $\overline{U}=U\cup\{1\}$. Similarly, there exists a nbd of 1, namely $V-(0,1+\frac{1}{2})$, such that each nbd of 0 fails to be in $\overline{V}=V\cup\{0\}$.

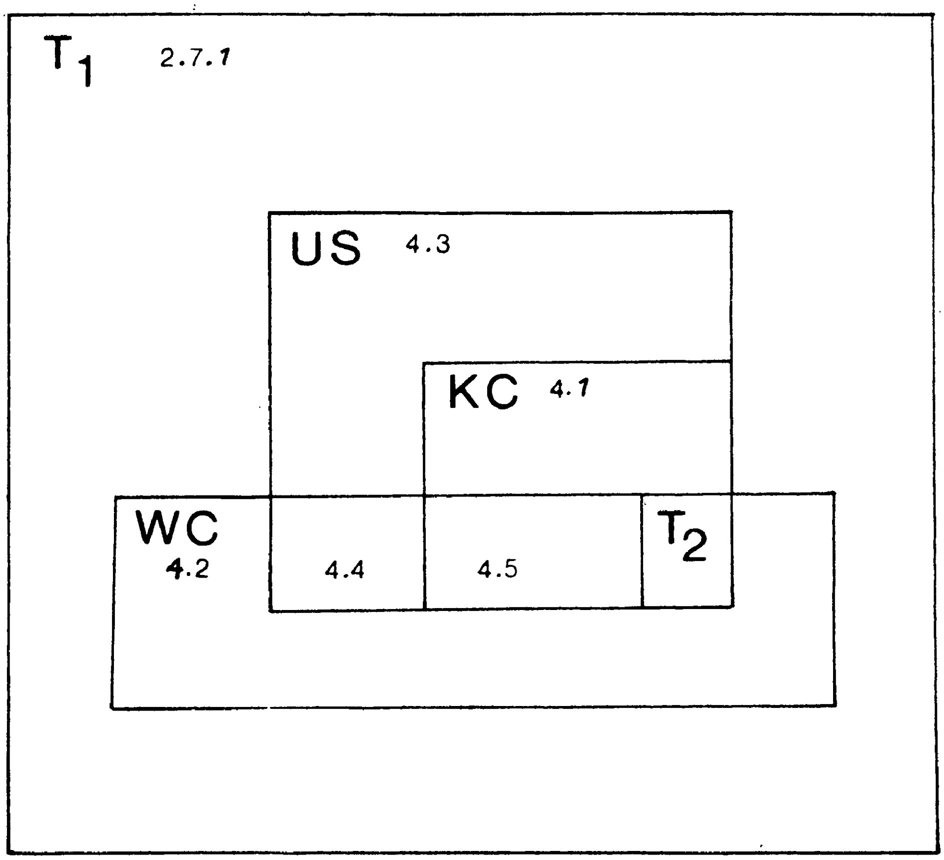
To show that this space is US, let ψ be a convergent sequence in Y. If ψ converges to a point $y'\varepsilon Y$ distinct from 0 and 1, all but finitely many values of ψ must be contained in the open nbd $\{y'\}$. Since each point distinct from y' has a nbd not containing y', ψ cannot converge to any other point. If ψ converges to 1, the sequence must have an infinitely countable set of values in the basic nbd $U=(0,1+\delta)$, δ some positive number. Let I^+ be the set of positive integers and J the subset of I^+ such that $\psi(J)\subset(0,1)\subset U$. Since the set $(0-\delta,1)-\psi(J)$ is a basic nbd of 0, the sequence cannot converge to 0. Thus no convergent sequence in Y converges to more than one point.

Y is not KC. The set $B=\{y\varepsilon Y|0<y\leqslant 1\}$ is compact but not closed since 0 is in $\overline{B}$ but not in B.

EXAMPLE 4.5: A space that is WC, US, and KC but not Hausdorff. Let Y be the space in Example 4.4 but let basic nbds of 1 be the intervals of the form $(0,1+\delta)$ minus countably many y_i such that $0<y_i<1$. This space is WC and US and also KC since any compact set in Y is finite and hence closed. Y is not Hausdorff since 0 and 1 do not have disjoint nbds.

Figure 2 summarizes the relationship between T_1, WC, US, KC, and T_2 spaces.

FIG. 2



CHAPTER FIVE: FURTHER TOPICS

In this final chapter topics are presented for further study of WC spaces.

QUESTION 1: What properties must a WC space have to insure that it is Hausdorff? Note that Example 2.8.1 is a 1° countable WC space that is not Hausdorff.

QUESTION 2: Are there necessary properties for a WC space to be US or KC and conversely?

QUESTION 3: How many consequences of Hausdorff spaces are true of WC spaces?

QUESTION 4: Are WC spaces invariant under maps weaker than homeomorphisms?

QUESTION 5: What can be said of compact WC spaces?

QUESTION 6: What can be said of quotient spaces of WC spaces?

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